BIOMETRIKA

A JOURNAL FOR THE STATISTICAL STUDY OF
BIOLOGICAL PROBLEMS

FOUNDED BY
W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

EDITED BY
KARL PEARSON
ASSISTED BY
EGON SHARPE PEARSON

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May, 1933

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Normal Males (top row), Normal Females (middle row) and Juveniles (bottom row) Naga Crania.

VOLUME XXV MAY, 1933 PARTS I AND II

BIOMETRIKA

A STUDY OF THE NAGA SKULL.

BY ELISABETH KITSON, B.A., M.Sc.

With the assistance of G. M. MORANT, D.Sc.

IN 1926 an expedition organised by the Government of Burma was sent into the Naga Hills for the purpose of suppressing the practice of human sacrifice there. A collection of skulls and other bones of the victims of sacrificial rites was made and these remains were despatched to the Indian Museum, Calcutta, in 1928. They were studied there by B. S. Guha and P. C. Basu and their report has recently been published*. It is said (p. 4) that: "Including small fragments the total number of bones was 217, of which 21 were whole or portions of arm and leg bones, 31 small pieces and 117 and 43 frontal and occipital parts respectively of skulls. Only five of the skulls had the cranial vaults complete, though even in these the infra-occipital region and the greater part of the basis cranii have been removed." Of these cranial specimens 65 were loaned to the Biometric Laboratory by the courtesy of the India Office and the Government of Burma, and they form the subject of the present paper†. On account of the very incomplete nature of nearly all the specimens, the numbers for which the majority of the usual measurements can be found are considerably less than the complement of 65. Owing to the kindness of Sir Arthur Keith, I was able to supplement my original measurements of Naga skulls by those of seven specimens in the Museum of the Royal College of Surgeons. In calculating means it was also possible to use some of the individual measurements given by Guha and Basu for the crania collected in 1927 which were not sent to the Biometric Laboratory and some of their measurements of three Naga skulls in the Indian Museum, Calcutta (Table XV of the *Report*). Measurements provided in the following sources were also included:

(*a*) Sir William Turner: "Contributions to the Craniology of the People of the Empire of India. Part I. The Hill Tribes of the North-East Frontier and the People of Burma." *Transactions of the Royal Society of Edinburgh*, Vol. XXXIX, Part III, No. 28 (1899), pp. 703—747. Individual measurements are given on p. 720 of one female and seven male Naga skulls.

* "A Report on the Human Relics recovered by the Naga Hills (Burma) Expedition for the Abolition of Human Sacrifice during 1926—27." *Anthropological Bulletins from the Zoological Survey of India*, No. 1 (July, 1931).

† A male incomplete calvaria—consisting of complete frontal, right and left parietal and left temporal bones together with the greater part of the occipital—was sent to the Biometric Laboratory with the Naga remains. This bore no number or inscription on arrival and it was subsequently numbered 18. It does not appear in the report cited and it is of a different type, and presumably of a different race, from the crania described there.

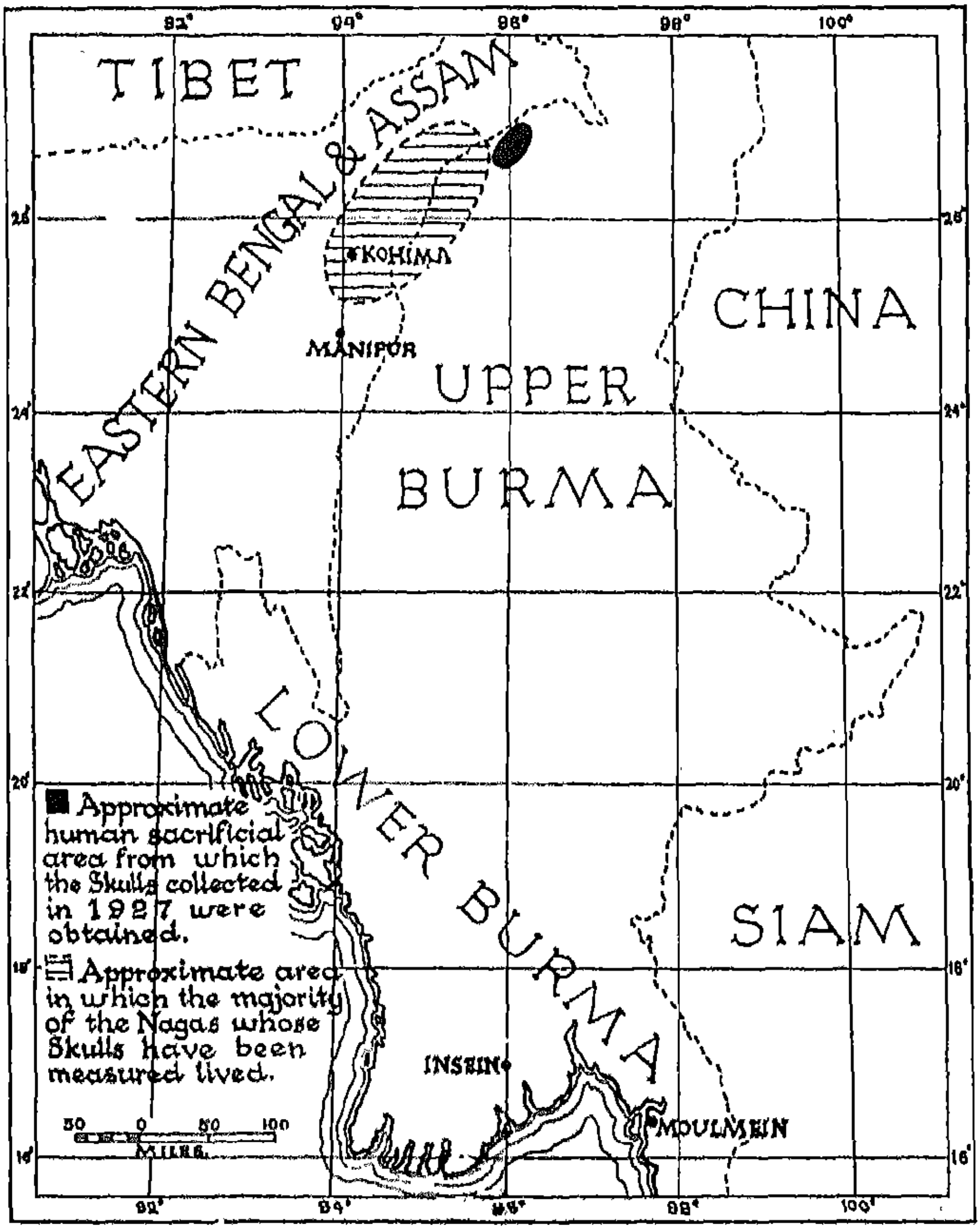
(*b*) Prof. (afterwards Sir) George D. Thane: "On some Naga Skulls." *The Journal of the Anthropological Institute*, Vol. XI (1882), pp. 215—219. Individual measurements are given of four Naga skulls in the Museum of the Royal College of Surgeons which I have re-measured and of one other specimen not in that museum.

The Naga Hills are in the province of Eastern Bengal and Assam close to the north-west frontier of Burma. All the crania collected in 1927 were taken from villages in Burma lying in an area, known as the Triangle, within which human sacrifice was practised before that date. This is shown on the map in the *Report* and its position is indicated on the map (Fig. 1) on p. 3. Little reliable information regarding the origin of the victims could be obtained. It was ascertained that they did not belong to the tribes living within the sacrificial area, but that they probably came from adjoining districts to the west and south-west. According to one chief the victims belonged to the Singpa, Wakka, Himhku, Nukpa, Yaugngaw and Kyetsan tribes. It is safe to assume that the majority of the remains represent head-hunting Naga tribesmen, but those of a few captives or stray foreigners from other parts may be included. It is said in the *Report* that one or more holes had been drilled through each of the fragmentary skulls so that a string might be passed through in order to suspend the relic (see Plate I). At least two of the specimens (numbered in the Biometric Laboratory 44, the anterior half of a cranium, and 66, a supra-occipital) have no such holes, however. The three skulls in the Indian Museum, for which measurements are provided by Guha and Basu, are supposed to be those of Angami Nagas and these were collected near Kohima which is one hundred and thirty miles south-west of the Triangle.

I am indebted to Miss M. L. Tildesley, Curator of the Department of Human Osteology, for the following particulars of the Naga skulls in the Museum of the Royal College of Surgeons. No. 6·621 (Flower's 1907 *Catalogue* No. 652[1]) was taken from Ninu (95° 18′ E., 26° 47′ N.), a Konyak Naga village thirty miles to the west of the Triangle. It was decorated with twelve rings of wire attached to the zygomatic arches, orbits and nasal cavity. It is complete with the lower jaw and it has not been pierced for suspension. This is probably the skull of a member of the Konyak tribe and not that of a sacrificial victim*. Nos. 6·6221 and 6·6222 were taken from the Konyak Naga village of Chongvi (94° 49′ E., 26° 32′ N.) fifty miles to the west of the Triangle. They were decorated with horns and tassels and the former still has these ornaments attached. The posterior parts of the calvariae are missing and No. 6·6222 has a hole pierced through the frontal bone and its mandible is fastened on with strips of bamboo. The two individuals represented had certainly been sacrificial victims and it is believed that they were probably Konyak Nagas. No. 6·6231 (Red No. 798 = No. 773 in Barnard Davis's *Thesaurus Craniorum*†) is the complete skull of a freebooter who was shot on a plundering expedition about the middle of the nineteenth century. The tribe to which the individual belonged is

* Dr J. H. Hutton has given his opinion regarding the origin of the Naga skulls in the Royal College of Surgeons and several of the remarks quoted here are on his authority.

† It is said in this catalogue that the "occipital, atlas and dentata" are all ossified together. However, the atlas of this specimen, as Professor Thane has previously noted, was never fused to the occiput.



BURMA & ADJOINING COUNTRIES SHOWING THE NAGA TERRITORY.

FIG. 1.

not known, but he is most likely to have been an Angami Naga. No. 6·6232 (Red No. 794 = No. 774 in the *Thesaurus Craniorum*) is the complete cranium (without the mandible) of a youth who had been a servant to Colonel Hanney. He is said to have been about eighteen years of age at the time of his death and the tribe to which he belonged is not known. No. 6·6233 (Red No. 795 = No. 1750 in the *Supplement to Thesaurus Craniorum*) is the complete cranium without the mandible of a Naga named Lentee. "He was murdered, it was supposed, by his woman." The tribe to which this individual belonged is not known but Lentee is said to be a common name among the Ao Nagas, while it is seldom, if ever, met with among other tribes. The Ao Nagas live to the south-west of the Konyak Nagas. Nos. 6·6234 and 6·6235 are two almost complete crania, without mandibles, presented by Dr J. H. Hutton. They were taken from the cemetery of the Ao Naga village of Mongsemyimti (*ca.* 94° 45′ E., 26° 27′ N.) in 1928. This is approximately 60 miles west of the southern corner of the Triangle.

The skull E in Professor Thane's paper is the only one he describes which is not in the Royal College of Surgeons, and it is said to have been obtained from the same neighbourhood as No 6·621 in the museum there. Hence it is probably that of a Konyak Naga. The eight skulls dealt with by Sir William Turner were taken from the house of a Tonkal Naga in the upper village of Hwining which is some forty miles north-east of Manipur. This is approximately one hundred and fifty miles south-west of the Triangle. The custom there is to bury the dead, so these specimens were evidently trophies although they are almost complete crania. They are believed to be those of Tonkal Nagas from other villages. It is probable that the vast majority of these crania, for which measurements are now available, are those of Nagas of various tribes who lived in an area some one hundred and fifty miles long and eighty broad extending north-east from a point twenty-five miles north of Manipur and lying parallel to, and possibly across, the frontier between Burma and Assam (see map, Fig. 1). A few individuals from outside this area, and possibly some who were not Nagas, may have been included. The sample is too small, and the particulars regarding the origin of the specimens are too indefinite, to make any comparisons between different groups of Nagas possible. As far as I could judge from the skulls I was able to handle, the group is a racially homogeneous one and the measurements of the total sample appear to suggest the same conclusion. All the material was pooled for the purpose of providing mean measurements.

The selected group of the skulls collected in 1927 which was sent to the Biometric Laboratory bore no numbers or inscriptions by which they could be identified on arrival. They were numbered serially (1—66) there, and nothing further was done with No. 18 for the reasons stated above (p. 1, footnote). From the photographs provided in Guha and Basu's *Report* I was able to identify twelve of the specimens with certainty*, and what may be called the corresponding London and Calcutta

* Viz. the skulls numbered 2, 8, 8, 14, 29, 34, 35, 36, 37, 47, 51 and 50 in the Biometric Laboratory. Our No. 51 corresponds to No. N. 4 of Guha and Basu and it does not appear in our table of individual measurements as it is an unsexed fragment. They also give photographs of skulls which they numbered N. 95, N. 170 and N. 182, and these were not sent to the Biometric Laboratory.

numbers for these will be found in Table VII of individual measurements given at the end of this memoir. By comparing measurements it was also possible to identify twenty additional specimens with a sufficient degree of probability. The majority of the remaining thirty-three crania sent to the Biometric Laboratory, for which Guha and Basu's numbers cannot be found, are those of juvenile individuals and measurements of these are not given in the *Report*. The technique of measurement followed there is that of the Monaco Congress*, with a few modifications and additions, and the majority of the definitions used may be supposed identically the same, for practical purposes, as those of the biometric scheme which I followed. We are thus given an opportunity of comparing the measurements taken by different observers on the same thirty† skulls, though nearly all these are, unfortunately, very incomplete. The distributions of the differences between Guha and Basu's measurements and ours are given in Table I for fourteen characters and it can be seen at once that there is a deplorably bad correspondence in nearly every case. It is known from laboratory practice that if measurements on the same skulls are repeated by a single observer, or if they are taken by two different observers, then the maximum differences found should never exceed 2 mms. in the case of the characters considered. This will only be so, of course, if the workers have been adequately trained and if they interpret the definitions of the measurements in precisely the same ways. It is evident that these conditions do not hold in the present instance since only three of the distributions of differences lie within the limits $-2{\cdot}05$ and $+2{\cdot}05$. My queried measurements were omitted in compiling the table and there are none such given by Guha and Basu. Where my reading differed from theirs by more than 1·5 mms. it was taken independently by Dr G. M. Morant and these comparisons confirmed the fact that my measurements had been taken in accordance with the customary biometric technique. Some of the larger differences in Table I, particularly in the case of LB and S_1, are almost certainly due to errors of 5 or 10 mms. in reading a scale, but other large ones can only be attributed to the fact that Guha and Basu had radically different conceptions of the ways in which the measurements were to be taken from ours. The discordance is most marked in the case of the palatal length and we cannot imagine what definition can have been applied which would give consistently smaller readings than the Monaco *longueur de la voûte palatine* which is our G_1' and Martin's chord from staphylion to orale. It would clearly be unsafe to accept all Guha and Basu's measurements of the Naga crania collected in 1927 which were not sent to the Biometric Laboratory. For the purposes of computing means I used the values of B', LB, J, S_1, $G'H$, NB, $O_1'L$, L, B, S_2, S_1', S_2', fml and fmb given for those skulls and their values for the first seven of these measure-

* G. Papillault: "Entente Internationale pour l'Unification des Mesures craniométriques et céphalométriques." *Congrès International d'Anthropologie et d'Archéologie Préhistorique, Compte Rendu de la treizième Session, Monaco 1906*. Tome II (1908), pp. 377—394.

† There are thirty-two skulls sent to the Biometric Laboratory for which the numbers in Guha and Basu's *Report* can be found. One of these (London No. 51) is an occipital fragment on which no sufficiently accurate measurements can be taken although S_2 and S_2' are given in the *Report*: another (No. 3) is a juvenile specimen for which no measurements are given there. Another juvenile (No. 14 = Calcutta N. 176) is included in the tables of female adult measurements in the *Report*.

TABLE I.

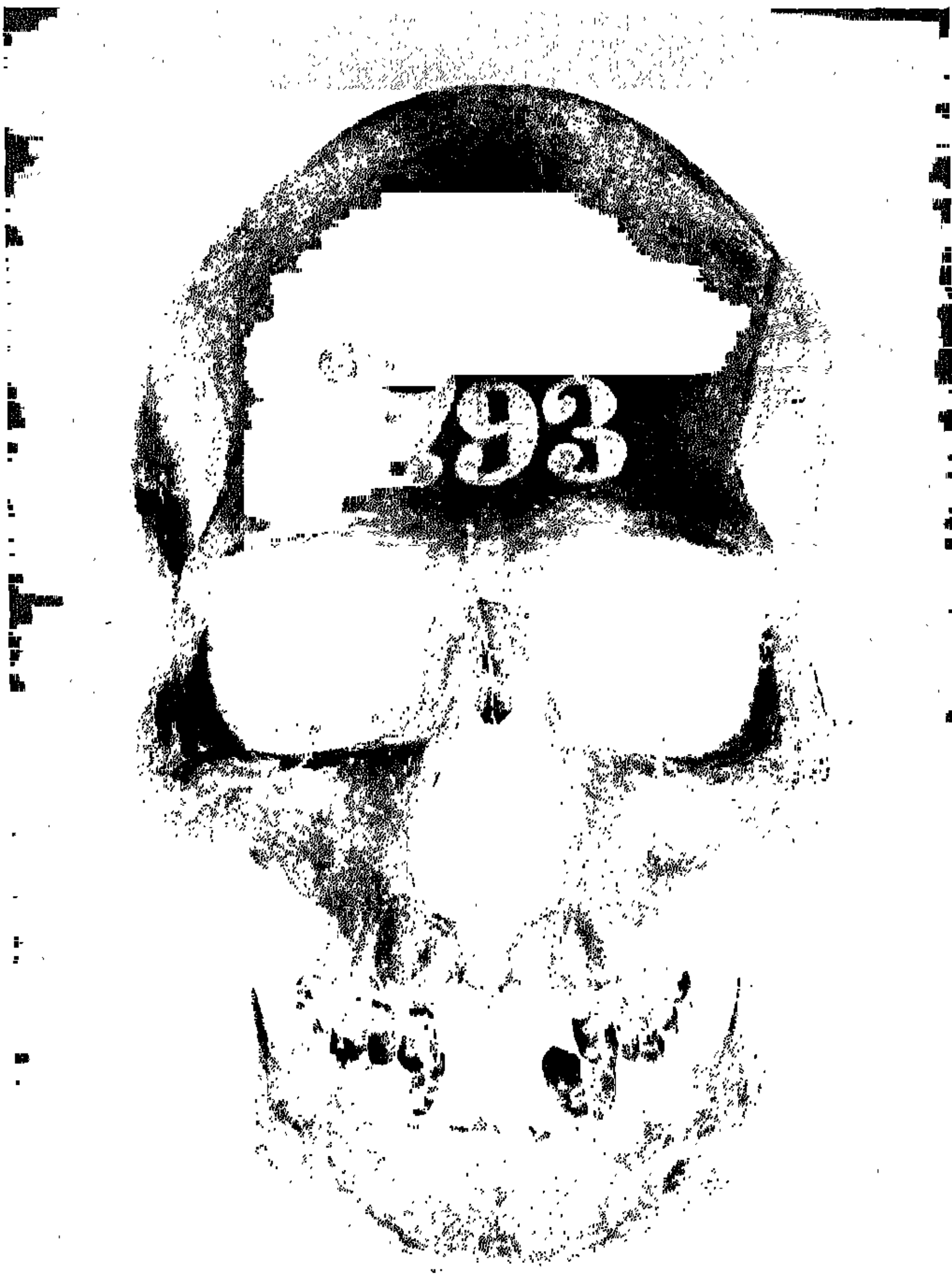
Differences (in mms.) between Measurements of the same Naga Skulls taken by Kitson (K.) and Guha and Basu (G.B.).

Differences in mms. (K.—G.B.)	B'	H'	B''	LB	J	S_1	S_1'	$G'H$	NB	$O_1'L$	O_1L	G_1'	G_2	NH'
−10·05—−9·05	—	—	—	—	—	1	—	—	—	—	—	—	—	—
−9·05—−8·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−8·05—−7·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−7·05—−6·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—
−6·05—−5·05	—	1	—	—	—	—	—	—	—	—	—	—	—	—
−5·05—−4·05	—	2	—	1	—	—	—	—	—	—	—	—	—	—
−4·05—−3·05	—	1	—	—	—	—	—	1	—	—	—	—	—	—
−3·05—−2·05	3	—	—	—	2	—	—	1	—	—	2	—	—	7
−2·05—−1·05	6	1	1	—	—	2	1	—	2	—	3	—	—	4
−1·05—−0·05	16	1	3	—	1	4	7	4	12	3	14	—	—	7
0	2	—	2	3	6	2	1	2	3	1	—	—	—	1
0·05— 1·05	—	—	3	2	10	6	4	8	9	3	3	2	1	2
1·05— 2·05	—	—	2	1	1	2	1	2	—	—	1	2	7	1
2·05— 3·05	—	—	2	—	—	1	—	2	—	—	—	5	4	—
3·05— 4·05	—	—	3	—	—	—	—	—	—	—	—	4	6	—
4·05— 5·05	—	—	1	—	—	—	—	—	—	—	—	1	2	—
5·05— 6·05	—	—	—	—	—	—	—	—	—	—	—	3	1	—
6·05— 7·05	—	—	1	—	—	1	—	—	—	—	—	2	—	—
7·05— 8·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—
8·05— 9·05	—	—	—	—	—	—	—	—	—	—	—	—	—	—
9·05—10·05	—	—	1	—	—	—	—	—	—	—	—	—	—	—
10·05—11·05	—	—	—	—	—	—	—	—	—	—	—	1	—	—
Totals	27	6	19	7	20	19	14	20	26	7	23	20	21	22

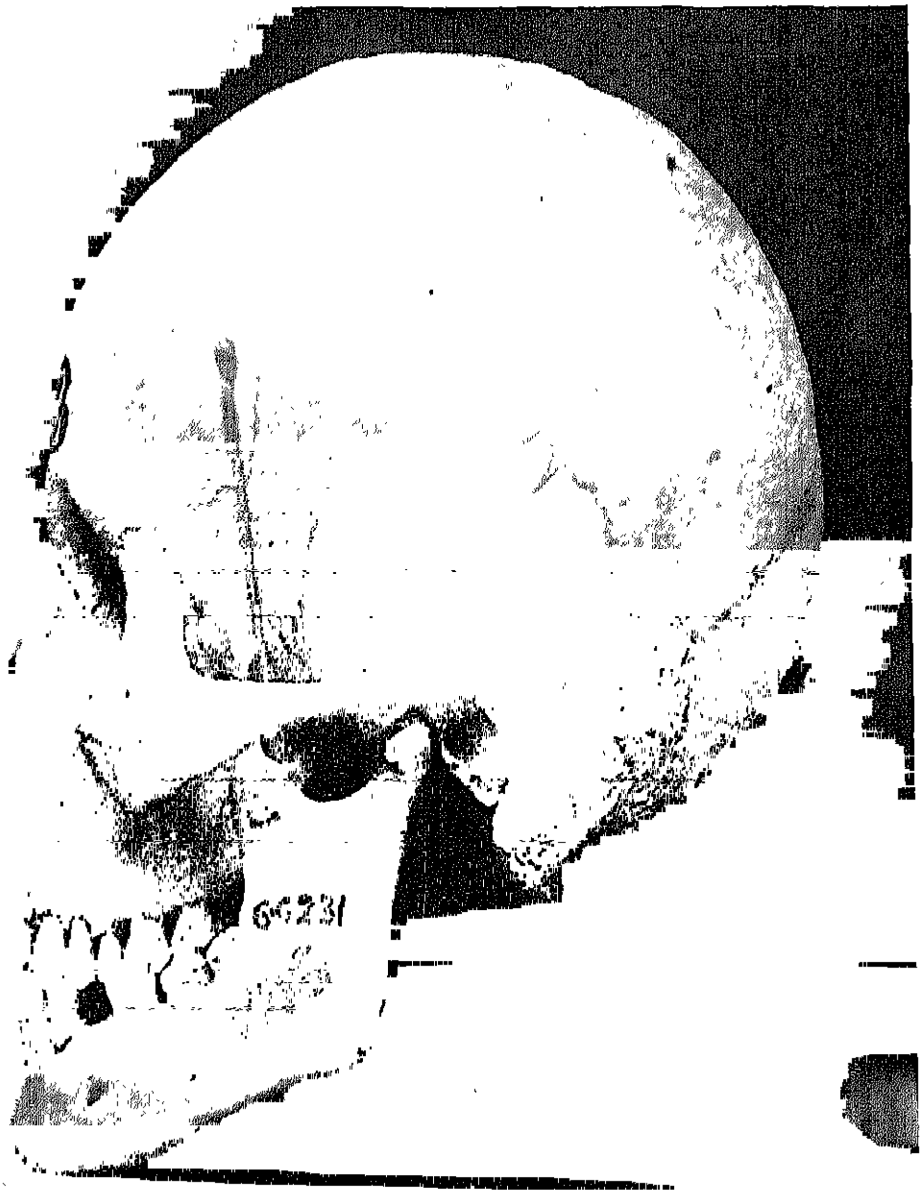
ments were also used in computing standard deviations and coefficients of variation*. The comparisons between Guha and Basu's readings for B', LB and $G'H$ are not altogether satisfactory, but it may be assumed that the inclusion of their values of these measurements for the additional skulls will not affect the constants appreciably.

The sexing of the Naga skulls collected in 1927 is particularly difficult owing to the fact that most of them are very defective. The sexual characters appear to be as well marked as for most races, however. Among the sample sent to the Biometric Laboratory we distinguished nineteen male adults: sixteen of these were supposed male by Guha and Basu and they give no sex for the other three which are occipital fragments. Of our fourteen female adults five were supposed male and five female by Guha and Basu, while the remaining four cannot be identified with their numbers. They distinguished sixty-one male and twenty-four females in their total sample though there was probably no such clear preponderance of

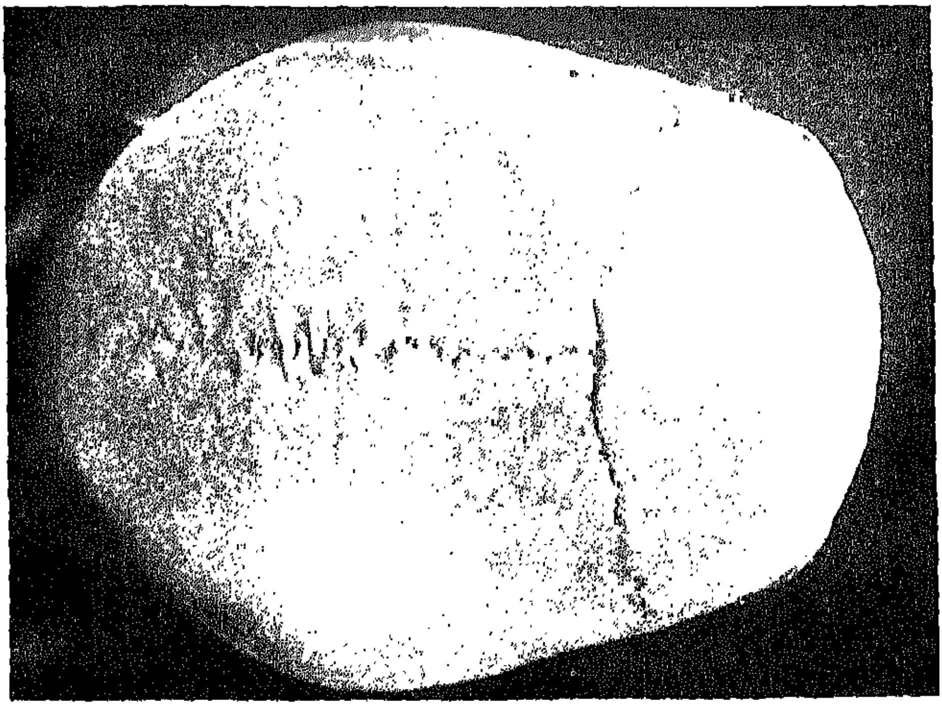
* These are also the only measurements taken by Guha and Basu on the three Naga skulls in the Indian Museum which I have used.



Typical Male Naga Skull. *Norma facialis.* (R.C.S. 6·6231.)

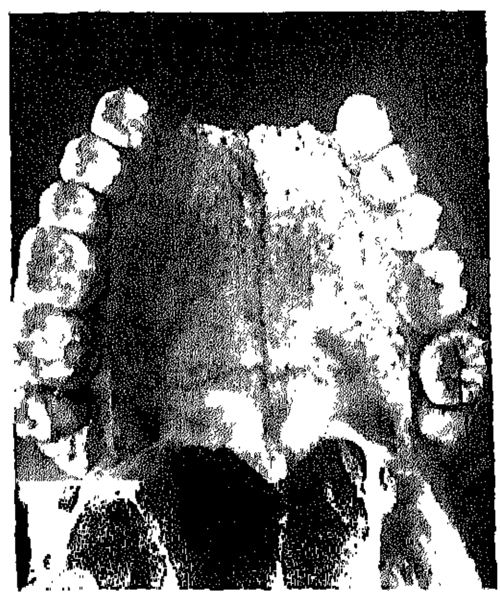


Typical Male Naga Skull. *Norma lateralis.* (R.C.S. 6·6231.)



A. Typical Male Naga Skull. *Norma verticalis*. (R.C.S. 6·6231.)

B. Male Naga Skull (R.C.S. 6·6232) with wormian bones in place of nasal bones.



C. Male Naga Skull (1927 series, B.L. No. 44) showing the unerupted third left molar horizontal and preventing the second molar from erupting.

one sex over the other if our sexing is more correct. The remainder of the sample of the 1927 skulls sent to the Biometric Laboratory is made up by twenty-one specimens representing immature individuals and eleven fragments which probably represent adults and for which no sexes can be given. The immature character of a few of the skulls placed in the juvenile group is somewhat uncertain owing to their incomplete nature, but one (London No. 14 = Calcutta No. N. 176) with a complete palate which was undoubtedly juvenile is included in Guha and Basu's tables of female adult measurements. Our individual measurements of all these skulls which were sent to London, with the exception of the adult unsexed fragments, are given in Table VII at the end of this paper together with those of all the Naga skulls in the Royal College of Surgeons. One of the last is immature; the sexes of five others were either known, or they were estimated by Barnard Davis, Flower and Thane and their consistent determinations were accepted; the remaining two were collected in 1928 and I have supposed that one of these (No. 6·6235) is male and the other (No. 6·6234) female. I accepted the sexes given for the skulls for which I have used previously published measurements and these include those in Guha and Basu's *Report* which were not sent to London.

Remarks on sutures, for adult skulls, and on teeth for all I was able to examine are given in Table VII. Unless otherwise stated, the coronal, sagittal and lambdoid sutures of the adult specimens are all open. There appears to be a clear preponderance of young adults over fully mature individuals in the case of both sexes and only one (No. 6·6235)*, which is a male, can be called ageing. Turner found one aged Tonkal Naga skull among the eight sacrificial specimens he examined. No. 6·6235 came from a cemetery so there is a suggestion that young adults and children were preferred as victims by the Naga head-hunters. The metopic specimens are three (Nos. 31, 6·6232 and 6·6233) out of twenty possible male crania, three (Nos. 4, 40 and 47) out of sixteen possible female and two (Nos. 43 and 49) out of twenty possible juvenile specimens. Guha and Basu say for their total sample that the metopic suture is present "in fifteen specimens in various proportions." The number which they could examine for this anomaly is not stated, but it was probably about one hundred. It seems reasonable to conclude that, judging from the Naga crania known, the metopic suture occurs in more than ten per cent. of cases without regard to sex or age, no specimens so young that the suture would not normally have closed being included. The frequency is unusually high for a non-European race. It is possible that it is inaccurate owing to the fact that the collectors of the crania favoured those which were metopic. One male cranium with interparietal bones was found among five male, four female and two juvenile specimens which it was possible to examine for this anomaly: this (No. 50) has the *ossa triangularia* only separate†. Wormian bones were found in more

* The numbers beginning 6· are those of the skulls in the Royal College of Surgeons. Others from 1 to 66 given in the text are the Biometric Laboratory numbers of the skulls collected in 1927 which were sent there.

† Guha and Basu give a photograph of an unsexed occipital fragment (No. N. 95) having the complete form of interparietal bones (Plate IV, Fig. 7): this was not sent to the Biometric Laboratory.

than fifty per cent. of each sex and of the juveniles, while epipteric bones occurred with the like percentage. The region of the pterion could be examined on both sides in the case of seventeen male skulls and there is no example of fronto-temporal articulation, though one specimen (No. 31) shows a close approach to this condition on both sides: the same region could be examined for seventeen females on the right and sixteen on the left, while there is one case (No. 40) of fronto-temporal articulation on the left and the right pterion of the same skull is normal: for the juveniles the crania which it was possible to examine number fourteen on the right and seventeen on the left, while there is one case (No. 17) of fronto-temporal articulation on the left and the right pterion of the same skull is defective. Turner found two cases of fronto-temporal articulation on the left side only among the eight Naga skulls he examined. There are examples of single or multiple tympanic perforation in all the groups. A female specimen (No. 6·6234) shows traces of the sutures between the ex- and supra-occipitals on both sides. Most of the teeth of the adults preserved are considerably worn but in a good state of preservation. There are eighteen male skulls on which the palates are almost complete and for fifteen of these no teeth had been lost before death. Molars were the only teeth lost in the case of the other three and no carious teeth other than molars were found for the same group. Owing to their failure to erupt, two of the eighteen specimens had no third molars on either side and one other had no third molar on the right. There are several examples of 'shovel-shaped' incisors (see Figs. 3 and 4, Plate V, in Guha and Basu's *Report*). One male (No. 44) has the unerupted third molar on the left side horizontal and its crown is in contact with the second molar which has thus been prevented from erupting (see Plate IV C below). A male skull (No. 6·6232) has wormian bones in place of the nasal bones and the ethmoid is exposed below them (see Plate IV B). The absence of the nasal bones appears to be a very rare anomaly. Three examples of it were described by the present writer in a paper on a series of Teita skulls from Kenya Colony*. A few healed wounds were found on the Naga skulls. None of these is large except that on a strong male specimen (No. 37) which had a severe sword-cut on the right side of the frontal bone (see Plate VI, Fig. 1 in Guha and Basu's *Report* and Plate I below).

No geographical divisions of the Naga crania for which measurements are available can be made, but it is of interest to compare the means of three groups obtained in different ways. This comparison is made in Table II for the characters which can be measured on the largest numbers of specimens. The groups are:

(*a*) the crania collected in 1927 measured by the present writer,

(*b*) the crania collected in 1927 measured by Guha and Basu but not by the present writer,

(*c*) the crania measured by Turner (seven male and one female), the female measured by Thane, two male and one female crania in the Indian Museum,

* "A Study of the Negro Skull with special Reference to the Crania from Kenya Colony." *Biometrika*, Vol. XXIII (1931), pp. 271—314. See pp. 284—285 and Plate IV A, B and C.

Calcutta, for which measurements are provided by Guha and Basu, and the crania in the Royal College of Surgeons measured by the present writer.

These three groups are made up of small numbers and considerable differences between the means may be expected if only on account of random sampling. There is, nevertheless, a fairly good correspondence throughout while there is a remarkably close one between groups (*a*) and (*c*). The means for the (*b*) group solely differ appreciably from those for the other two by having smaller male values of B', S_1, $G'H$ and J. Only a small part, if any, of these differences can be attributed to the fact that differences would be found between Guha and Basu's measurements for these characters and ours taken on the same crania. The male means computed from their measurements for the crania which we measured are: $B' = 95{\cdot}8$ (16), $S_1 = 128{\cdot}1$ (12), $G'H = 70{\cdot}5$ (13) and $J = 133{\cdot}7$ (14). All these differ by less than one mm. from the corresponding means of ours given in Table II. It is probable

TABLE II.

Mean Measurements for Groups of Naga Skulls.

	Male			Female		
	(*a*) Kitson (1927 series)	(*b*) Guha and Basu (remainder of 1927 series)	(*c*) Turner, Guha and Basu (Indian Museum), Kitson (R.C.S.)	(*a*) Kitson (1927 series)	(*b*) Guha and Basu (remainder of 1927 series)	(*c*) Turner, Thane, Guha and Basu (Indian Museum), Kitson (R.C.S.)
B'	94·9 (16)	91·5 (38)	93·9 (13)	84·1 (13)	88·6 (18)	85·0 (6)
LB	99·9 (7)	99·9 (15)	99·1 (10)	90·3 (3)	93·3 (3)	93·3 (3)
S_1	128·6 (12)	124·7 (19)	127·3 (13)	122·1 (11)	121·8 (4)	121·1 (5)
$G'H$	70·4 (13)	68·6 (36)	68·1 (11)	64·6 (11)	62·7 (16)	66·6 (4)
J	134·1 (14)	128·0 (21)	133·8 (10)	127·9 (7)	124·1 (10)	123·3 (5)
NB	27·1 (16)	26·8 (36)	26·5 (12)	26·1 (11)	25·4 (16)	25·7 (5)
$O_1'L$	39·1 (5)	38·4 (36)	38·4 (4)	38·1 (8)	37·4 (17)	40·1 (2)

that the appreciable differences found between the male means of the two samples of skulls collected in 1927 are due to sexing. It has been seen that Guha and Basu were inclined to assign a larger proportion to the male series than we were and hence their male means would tend to be smaller than ours. The sexes of three of the skulls in Group (*c*) were known and those of the others were assigned by Turner, Flower, Guha and Basu (three) and Kitson (two). The close agreement of the means for this group with ours for the 1927 series encourages the belief that our sexing of the 1927 series is substantially correct and, if this is so, Guha and Basu's is probably in error. It can be shown that the sex ratios given by their means are particularly small. The samples are too small to make it possible to decide these questions at all definitely. The means for the three groups were pooled to give the best mean values which it is possible to obtain for Naga skulls at present and it is unlikely that these figures differ appreciably from the true means of the

TABLE III.

*Constants of Variation for Naga and Egyptian Skulls**.

	Standard deviations		Coefficients of variation			
	Nagas		Nagas		Egyptians *E*	
	Male	Female	Male	Female	Male	Female
B'	4·97 ± ·29	4·27 ± ·33	6·30 ± ·31	4·80 ± ·38	4·28 ± ·07	4·11 ± ·08
LB	4·01 ± ·34	—	4·02 ± ·34	—	3·99 ± ·06	3·65 ± ·07
S_1	5·26 ± ·38	—	4·16 ± ·30	—	4·88 ± ·08	4·50 ± ·09
$G'H$	4·78 ± ·29	4·18 ± ·36	7·23 ± ·45	6·54 ± ·50	5·90 ± ·10	5·64 ± ·11
NH'	2·55 ± ·23	—	4·90 ± ·44	—	5·65 ± ·09†	5·31 ± ·10†
NB	1·64 ± ·10	1·59 ± ·13	6·12 ± ·37	6·19 ± ·52	7·27 ± ·12	6·98 ± ·14
J	5·83 ± ·41	—	4·43 ± ·32	—	3·55 ± ·06	3·02 ± ·08
$O_1'L$	1·79 ± ·13	1·96 ± ·18	4·04 ± ·33	5·10 ± ·48	4·06 ± ·07‡	3·97 ± ·08‡
O_2L	2·13 ± ·19	—	6·27 ± ·57	—	5·50 ± ·09	5·62 ± ·11

* The numbers of skulls on which the Naga constants are based can be seen from Table IV: the smallest is 27.

† These are for the Frankfurt nasal height NH, L in place of NH'.

‡ These are for orbital breadths found by Fawcett's curvature method in place of dacryal breadths.

available sample owing to errors of sexing. The pooled means for all the characters considered are in Table IV. There is a very satisfactory agreement between the corresponding male and female Naga indices and angles, allowing for the fact that the samples are very small. It is generally found, as in the present case, that the female occipital (*Oc. I.*) and orbital indices are greater than the male while the male simotic ($100\ SS/SC$) and palatal height-breadth indices ($100\ EH/G_2$) are greater than the female. The measurements suggest that the series for the two sexes represent the same racial type and a direct comparison of the skulls we were able to examine had suggested the same conclusion. The following mean indices are found for the immature crania: $100\ G'H/GB = 67{\cdot}6$ (16), $100\ NB/NH, R = 54{\cdot}2$ (18), $100\ O_2/O_1,\ L = 85{\cdot}5$ (20) and $100\ SS/SC = 24{\cdot}2$ (18). As far as can be seen the juvenile specimens represent the same racial type as the adult.

The standard deviations and coefficients of variation are given in Table III for all the characters which can be measured on 27 or more crania. No significant differences in variability are found between the two sexes. Comparison is made in the table with the coefficients of variation given for the long Egyptian *E* series of 26th—30th Dynasty skulls§. It is known that this series, which came from a single cemetery, is less variable than nearly all that have been obtained from European sites. The constants for nine characters can be compared in the case of the males, the Naga value being greater for six and less for the other three. The difference

§ Karl Pearson and Adelaide G. Davin: "On the Biometric Constants of the Human Skull." *Biometrika*, Vol. XVI (1924), pp. 328—363.

TABLE IV. *Mean Measurements of the Naga and a Burmese Series.*

Characters*	Male: Nagas (pooled)	Male: Burmese from Insein Prison (Turner)†	Female: Nagas (pooled)
F	179·8 (6)	—	170·9 (4)
L	181·3 (14)	173·1 (29)	173·1 (8)
B	137·9 (18)	142·1 (29)	132·9 (10)
H	92·8 (67)	93·1 (29)	86·4 (37)
H'	113·6 (10)	—	110·6 (14)
H'	136·0 (16)	135·6 (29)	127·6 (8)
LB	99·7 (32)	98·6 (29)	92·3 (9)
S_1'	111·8 (19)	—	107·1 (16)
S_2'	115·7 (7)	—	111·3 (9)
S_3'	99·0 (8)	—	99·5 (2)
S_1	123·5 (44)	127·9 (29)	121·8 (20)
S_2	129·7 (15)	123·5 (29)	123·3 (9)
S_3	115·3 (15)	111·3 (29)	106·1 (4)
S	370·3 (13)	362·7 (29)	348·5 (1)
U	504·1 (7)	—	484·5 (6)
Bregmatic Q'	306·9 (11)	—	289·0 (8)
J	131·0 (45)	133·8 (29)	125·1 (22)
$G'H$	69·1 (60)	69·7 (28)	63·9 (31)
GL	95·9 (15)	97·2 (28)	89·3 (6)
GB	99·8 (16)	—	94·0 (14)
NH, R	51·5 (20)	—	48·3 (16)
NH, L	51·9 (20)	—	48·4 (15)
NH' ‡	52·1 (28)	52·1 (29)	47·9 (21)
NB	26·8 (64)	25·3 (29)	25·7 (32)
O_1, R	43·0 (20)	—	41·4 (16)
O_1, L	42·3 (20)	—	41·5 (15)
O_1', R	39·7 (9)	—	37·8 (11)
O_1', L	38·5 (45)	—	37·8 (27)
O_2, R	33·5 (20)	33·7 (29)	32·8 (17)
O_2, L	33·9 (28)		32·4 (17)
Dd	28·1 (9)	—	26·9 (9)
DC	20·9 (9)	—	21·1 (9)
DS	6·3 (9)	—	6·6 (9)
SC	8·7 (17)	—	8·0 (15)
SS	2·9 (16)	—	2·2 (14)
G_1	50·9 (18)	—	46·8 (10)
G_1'	45·8 (20)	—	42·2 (10)
G_2	42·8 (19)	—	39·8 (12)
RH	12·7 (18)	—	10·9 (12)
fml	37·0 (11)	34·4 (29)	34·2 (6)
fmb	31·1 (7)	—	29·4 (6)
$100\ B/L$	76·9 (14)	82·2 (29)	76·7 (8)
$100\ H'/L$	70·7 (11)	78·4 (29)	70·6 (5)
$100\ B/H'$	102·2 (6)	104·0 (29)	101·2 (4)
Oc. I.	60·7 (7)	—	64·0 (2)
$100\ fmb/fml$	84·0 (7)	—	87·3 (5)
$100\ G'H/GB$	68·8 (11)	—	69·2 (12)
$100\ NB/NH, R$	52·3 (20)	—	54·3 (14)
$100\ NB/NH, L$	51·9 (20)	—	54·2 (14)
$100\ NB/NH'$	51·4 (28)	48·7 (29)	54·4 (19)
$100\ O_2/O_1, R$	77·7 (20)	—	79·1 (16)
$100\ O_2/O_1, L$	78·9 (20)	—	78·5 (15)
$100\ O_2/O_1', R$	83·4 (9)	—	85·9 (11)
$100\ O_2/O_1', L$	84·9 (8)	—	84·5 (10)
$100\ DS/DC$	40·3 (9)	—	31·8 (9)
$100\ SS/SC$	35·0 (16)	—	25·6 (14)
$100\ G_2/G_1$	85·5 (17)	—	84·9 (9)
$100\ G_2/G_1'$	94·9 (19)	—	94·1 (9)
$100\ RH/G_2$	29·7 (18)	—	26·2 (12)
$N\angle$	65°·8 (15)	68°·3 (28)	67°·2 (5)
$A\angle$	72°·4 (15)	69°·9 (28)	71°·3 (5)
$B\angle$	41°·8 (15)	41°·7 (28)	41°·6 (5)

* Definitions of the measurements denoted by the index-letters will be found in *Biometrika*, Vol. xxi (1929), pp. 89—94.

† The following additional means can be given for the Burmese from Insein Prison: C = 1404·0 (29), Glabellar U = 500·6 (28), Broca's Q' = 304·8 (29), Biasterionic B = 107·0 (29), Lacrymal O_1 = 39·3 (29), 100 O_2/Lacrymal O_1 = 85·9 (29).

‡ This is Broca's nasal height from the nasion to the "base" of the anterior nasal spine.

between the coefficients of variation exceeds 2·5 times its probable error in the case of B' (Δ/p.e. $\Delta = 3·4$) $G'H$ (2·9), J (2·7) and NB (3·0), while the Naga value is greater than the Egyptian for the first three of these characters. Three of the four female constants which can be compared are greater for the Nagas than for the Egyptians, but the difference is only significant in the case of $O_1'L$ (2·5) The Naga series is rather more variable than the Egyptian, but, as far as can be seen, it is not more heterogeneous than most which have to be accepted as representing a single racial type.

Comparisons between the male Naga and other Asiatic cranial series were made by the method of the coefficient of racial likeness. These coefficients have recently been given between all pairs of 26 Asiatic series* and we selected nine from these either on account of the fact that they represent neighbouring peoples, or because a rough comparison of the means suggested that they might be allied to the Nagas. The series chosen represent Tagals, a supposed non-negrito people from the Philippine Islands, Dayaks from Borneo (both measured by von Bonin), Tibetans of the *A* type from the south-west of Tibet and Nepalese (Morant), Chinese from the province of Fukien (Harrower), late prehistoric Chinese from Kansu and Honan (Black), Burmese of the *A* type from Moulmein (Tildesley), Ainos (Koganei) and Hindus from Bengal, Orissa and Southern India (Danielli, Turner and Mantegazza) Comparison was also made with a series of 29 crania of Burmese men who died in Insein Prison, Lower Burma. Measurements of these were given by Sir William Turner† and the previously unpublished means are in Table IV. The coefficients of racial likeness between the Naga and these ten series are given in Table V and the relationships suggested by these criteria are shown in Fig. 2 which is based on Fig. 3 in the paper by Woo and Morant cited. They concluded that the most reasonable classification was given if only the lowest orders of reduced coefficients were considered and for this purpose they ignored all greater than 19. There are four with the Nagas below this value. The lowest is that of 4·87 with the Tagals and there are several Asiatic series for which the lowest reduced coefficient which has yet been found is greater than this. The connection with the Dayaks is nearly as close though it is less intimate than that between the Tagals and Dayaks. The Tibetans *A* and the Nepalese are further removed from the Nagas, but their resemblance to them is still far closer than that between the Naga and any of the Asiatic types with the exception of those just mentioned. The Nagas, Tagals, Dayaks and Tibetans *A* may be considered to form a closely interrelated group since their relationships to one another are more intimate than any between them and any other Asiatic types for which craniological data are available. As far as can be seen at present it is only by tracing relationships through this group that

* T. L. Woo and G. M. Morant: "A Preliminary Classification of Asiatic Races based on Cranial Measurements." *Biometrika*, Vol. xxiv (1932), pp. 108—154. The means themselves or references to sources in which they may be found are provided in the above paper: all have been published in *Biometrika*.

† "Contributions to the Craniology of the People of the Empire of India, Part I. The Hill Tribes of the North-East Frontier and the People of Burma." *Transactions of the Royal Society of Edinburgh*, Vol. xxxix. Part III. No. 23 (1899). The measurements of the crania of the Burmese from Insein Prison are in Tables III and IV.

TABLE V.

Male Coefficients of Racial Likeness between the Naga and other Asiatic Series.*

Series	Crude Coefficients		Reduced Coefficients	
	All characters	Indices and angles	All characters	Indices and angles
Tagals ...	1·19 ± ·19 (24) [24·1, 25·0]	−0·56 ± ·34 (8) [15·4, 23·1]	4·87 ± ·77	−3·02 ± 1·84
Dayaks	1·69 ± ·19 (24) [24·1, 44·1]	0·25 ± ·34 (8) [15·4, 42·7]	5·43 ± ·61	1·09 ± 1·52
Tibetans *A* ...	3·24 ± ·20 (23) [24·7, 39·0]	1·61 ± ·34 (8) [15·4, 35·4]	11·05 ± ·68	7·50 ± 1·58
Nepalese ...	1·98 ± ·20 (23) [24·7, 45·7]	0·14 ± ·34 (8) [15·4, 44·4]	12·74 ± ·62	0·62 ± 1·49
Chinese Fukien	5·20 ± ·20 (22) [20·8, 36·0]	3·24 ± ·36 (7) [15·0, 36·0]	20·20 ± ·66	14·71 ± 1·64
Burmese (Tavoy)	6·78 ± ·23 (17) [28·2, 28·8]	10·49 ± ·43 (5) [16·6, 28·6]	23·80 ± ·81	49·97 ± 2·03
Burmese *A* ...	7·19 ± ·20 (23) [24·7, 40·2]	6·80 ± ·34 (8) [15·4, 38·7]	23·09 ± ·65	31·09 ± 1·54
Prehistoric Chinese	7·28 ± ·21 (21) [25·0, 36·6]	3·48 ± ·39 (6) [15·5, 34·7]	24·10 ± ·69	10·27 ± 1·82
Ainus	12·32 ± ·24 (16) [27·9, 78·0]	4·03 ± ·43 (5) [15·0, 73·8]	29·93 ± ·58	16·16 ± 1·71
Hindus	13·77 ± ·23 (17) [28·2, 76·7]	1·51 ± ·43 (5) [16·6, 63·2]	33·30 ± ·56	5·74 ± 1·62

* The number in round brackets following the coefficient is the number of characters on which it is based. The numbers in square brackets below the coefficient are the mean numbers of skulls available for the characters used in computing it; the first is for the Naga and the second for the other series in the comparison.

the Indian races can be linked up with the Northern Oriental races, the Chinese and Japanese, on the one hand, or with the Southern Oriental races, the Southern Burmese, Javanese and Aëtas, on the other hand. The members of the aforementioned closely interrelated group of intermediate types also stand between the Northern and Southern Oriental groups. This state of affairs accords reasonably well with geographical considerations. The Nagas and Tibetans *A* are more or less centrally placed with regard to the peoples of India and the Orient. The Tagals and Dayaks inhabit islands to the south-east to-day, but there is some evidence, considered below, which suggests that they came from the same inland area. If this is so it is

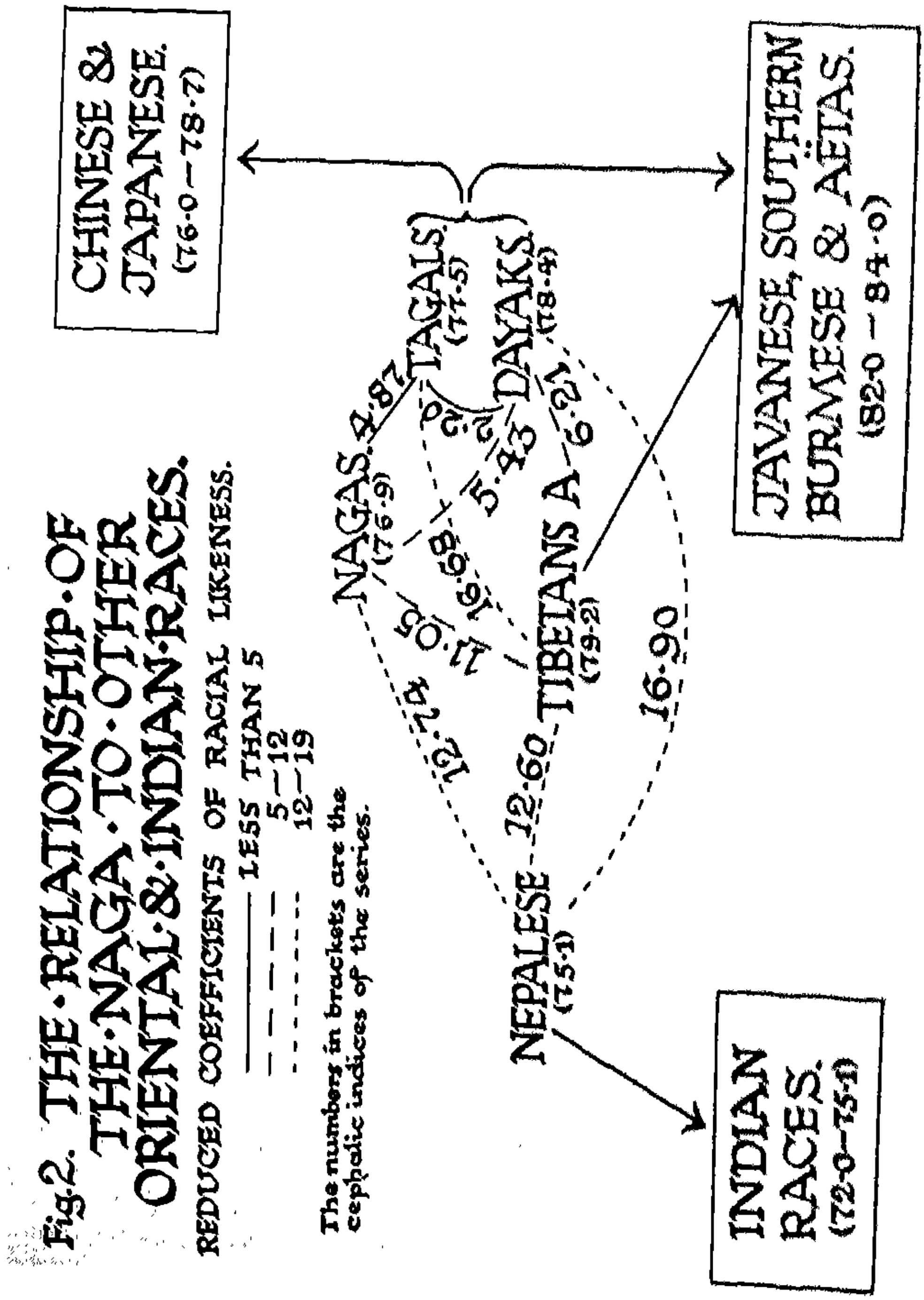


Fig. 2. THE · RELATIONSHIP · OF THE · NAGA · TO · OTHER ORIENTAL · & · INDIAN · RACES.
REDUCED COEFFICIENTS OF RACIAL LIKENESS.
LESS THAN 5
5—12
12—19
The numbers in brackets are the cephalic indices of the series.
CHINESE & JAPANESE. (76·0—78·7)
JAVANESE, SOUTHERN BURMESE & AËTAS. (82·0—84·0)
INDIAN RACES. (72·0—75·1)
NAGAS (76·9)
TAGALS (77·5)
DAYAKS (78·4)
TIBETANS A (79·2)
NEPALESE (75·1)
4·87
7·20
5·43
6·21
16·68
11·05
12·74
12·60
16·90

less surprising to find that the Nagas, who are quite dissimilar to the Southern Burmese, can be linked up with them by the Tagals and Dayaks. The Nagas have a reduced coefficient of 23·80 with the Burmese series from Insein Prison and the almost identical value of 23·99 with the Burmese *A* from Moulmein. These two series from neighbouring towns have a crude coefficient of 3·57 ± ·23 for 17 characters and the reduced value is 10·64 ± ·69.

Dr J. H. Hutton deals with the question of the origin of the Naga tribes in general in his book on the Angami tribe*. It is said that the weight of tradition points to migration from districts immediately to the south of the region occupied by the people to-day. "Where the Nagas came from before they reached the country near Manipur is a much more difficult problem.... All sorts of origins have been ascribed to the race. They have been connected with the head-hunters of Malay and the races of the Southern Seas on the one hand, and traced back to China on the other....On the basis of language their origin is assigned by Sir G. Grierson to the second wave of emigration, that of the Tibeto-Burmans, from the traditional cradle of the Indo-Chinese race in North-Western China...." In a footnote Dr Hutton accepts the view that "the Nagas have very strong cultural affinities with the natives of the Asiatic Islands, notably Borneo and the Philippine Islands, and perhaps physical affinities with some of them." The fact that there are strong affinities between some aspects of the cultures of these peoples is one which has been stated by Mr Henry Balfour†. Dr Charles Hose refers to the same cultural connection in a recent book‡. He writes: "Dr Hutton, Colonel L. W. Shakespear, Mr T. P. Mills and others have written very valuable and interesting books on the Naga tribes...and there seems to me to be a very close similarity in the legends, superstitions, customs, habits and arts of these tribes and [those of] the adjacent highlands of the remainder of the Brahmaputra basin, which is characteristic of one or other of the ruder lank-haired tribes of Borneo, Sumatra, the Philippines, and the other islands of the Malay Archipelago." The coefficients of racial likeness between the slender cranial series available suggest forcibly that there is a close physical, as well as cultural, relationship between the Nagas, the Dayaks and the "non-negrito" inhabitants of the Philippine Islands. If the arrangement shown in Fig. 2 illustrates the true connections of the Oriental and Indian races then it is probable that the Dayaks and Tagals came from an inland area which may well have been close to the region occupied by the Nagas to-day. It may be noted that the living people belonging to these races and the living Tibetans are generally said to possess Mongolian traits though they are not pure Mongolians. They are frequently said to have resulted from a blend of Caucasian and Mongoloid elements.

The numbers of individuals making up the Naga cranial series are so small that it would not be profitable to make any detailed comparision between the means for single characters and those given for other Asiatic series. The mean cephalic index

* *The Angami Nagas* (1921), p. 8.

† *The Journal of the Royal Anthropological Institute*, Vol. XLIV (1914), p. 67.

‡ *Natural Man, A Record from Borneo* (1926), p. 12.

for 14 male skulls is 76·9 and for 8 females it is 76·7. Measurements given by Dr Hutton* for 237 Naga men belonging to various tribes lead to a mean cephalic index of 78·4. Remembering that the living index is expected to be about two points greater than the cranial, we have good reason to believe that our values are very close to the true ones for Naga skulls in general. The coefficient of racial likeness between the Nagas and Tagals could be based on 24 characters and only one of these shows a significant difference, i.e. an α greater than 10, this is for $G'H$ ($\alpha = 16{\cdot}03$). The same thing is found in the comparison with the Dayaks (α for $G'H = 19{\cdot}36$) and the Naga upper facial height is shorter than the other two. The coefficient with the Tibetans A could be based on 23 characters and only one of these is found to indicate a significant difference, viz. H' ($\alpha = 14{\cdot}10$), and the upper facial heights are not differentiated in this case. Three or more characters are found to differ significantly in the case of every one of the other comparisons. The Nagas differ most markedly from the Nepalese and Hindus on account of their greater bizygomatic, calvarial and nasal breadths, from the Chinese on account of their lesser upper facial height and from the Southern Burmese on account of their lower cephalic index, lesser upper facial height and calvarial and nasal breadths, but greater calvarial length. The Aino cephalic index of 76·5 is very close to the Naga value of 76·9 and the coefficient of racial likeness was calculated in this case although no close connection was expected. Of the 16 characters compared six are found to differ significantly: these are LB ($\alpha = 48{\cdot}25$), J (43·51), B' (26·82), $G'H$ (26·21), $N\angle$ (21·76) and NB (16·17). The Naga type has a smaller facial skeleton than the Aino.

Of the adult Naga crania available in the Biometric Laboratory there were only six male and four female complete enough to be used in the ordinary way for the purpose of providing contours. No transverse or horizontal contours were drawn. The sagittal drawings were made in the usual way for all the adult specimens on which both the nasion and bregma could be located. Where possible the cranium was orientated by making the nasion, bregma and lambda the same height above the drawing board, but if the lambda was missing the anterior nasal spine, alveolar point or basion was used in its place. A new method of constructing the type had to be devised in order that use might be made of the outlines of the incomplete specimens for which the $N\gamma$ base line is not available. Figs. 3 and 4 are the male and female types and the mean measurements from which they were constructed are given in Table VI. The base line is, as usual, $N\gamma$ given by six male and four female crania only. For these the means of the ordinates VI—IX and the x's and y's of the lambda, inion, opisthion, auricular point and sub-orbital point were found in the usual way. They also provide the mean values of the angle $\beta N\gamma$. The position of the bregma is then given by the means of the $N\beta$ chords available for 15 male and 12 female crania. This line—extended where necessary—is used as an accessory base line and it is divided into the tenths and other divisions indicated. The tip of the anterior nasal spine (NS), the alveolar point, the anterior (p) and

* *Op. cit.*, Appendix XI.

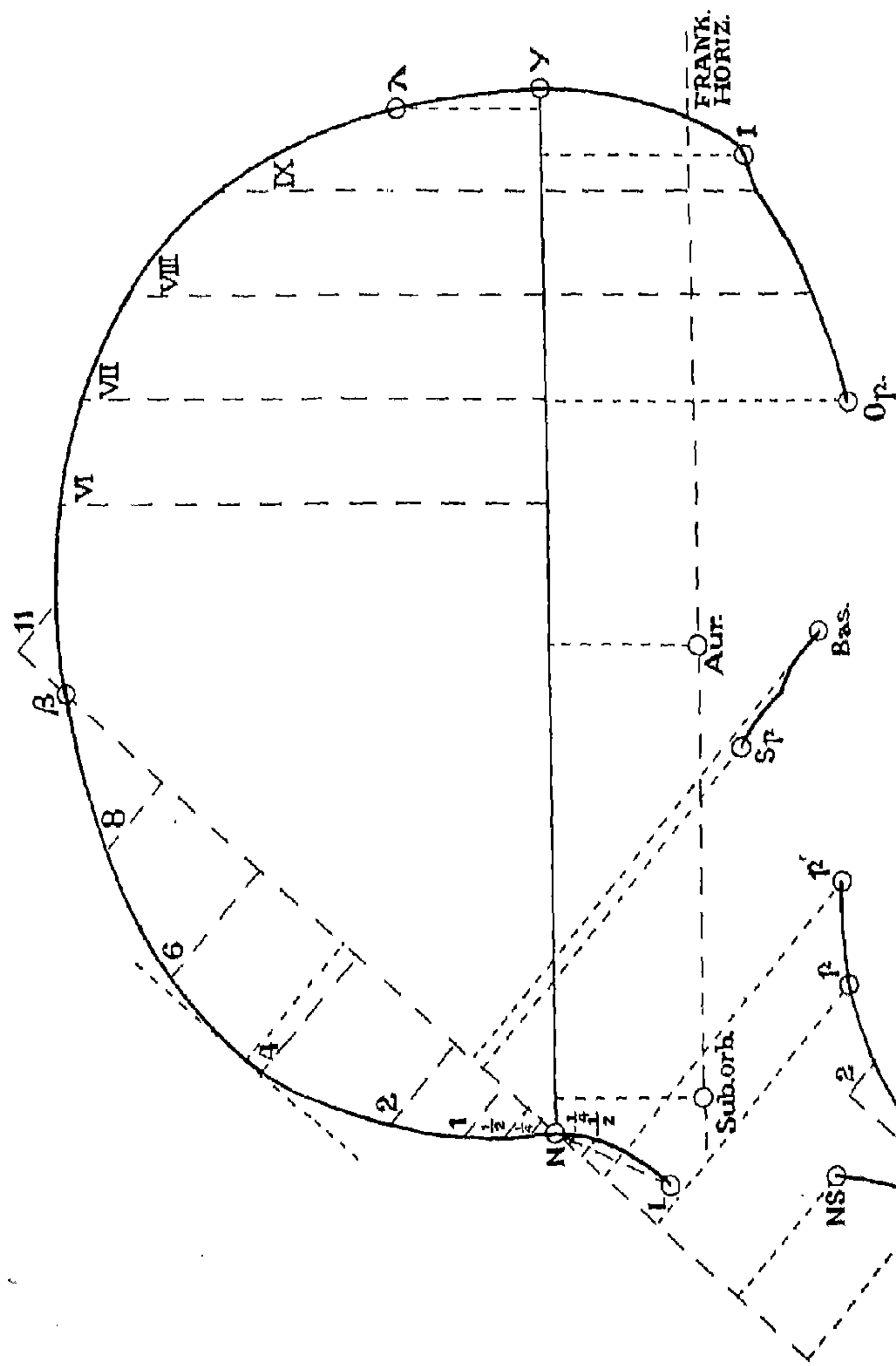


FIG. III. Sagittal Type Contour of 15 Male Naga Skulls.

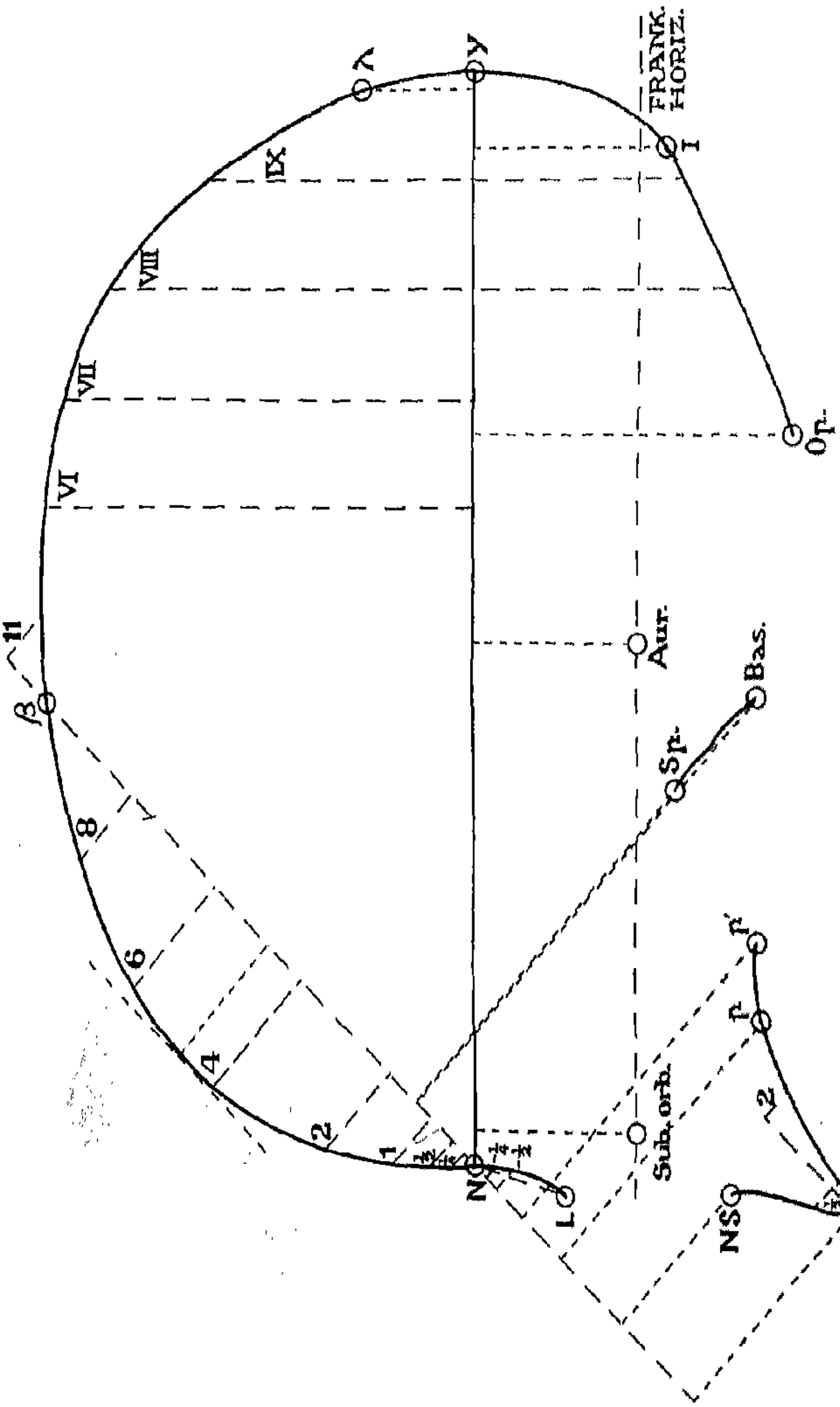


Fig. IV. Sagittal Type Contour of 12 Female Naga Skulls.

TABLE VI.

Naga Sagittal Contours. Mean Values.

	$N\gamma$	Ordinates above $N\gamma$				Ordinates below $N\gamma$		λ		Inion		Opisthion	
		6	7	8	9	8	9	x from γ	y	x from γ	y	x from γ	y
♂	174·8 (6)	85·1 (6)	81·3 (6)	72·6 (6)	57·1 (6)	47·1 (4)	37·8 (4)	3·6 (6)	25·1 (6)	11·3 (4)	35·8 (4)	52·7 (4)	53·4 (4)
♀	172·4 (4)	77·6 (4)	74·0 (4)	65·9 (4)	48·0 (4)	45·3 (2)	36·7 (3)	3·5 (4)	20·3 (4)	12·4 (3)	33·5 (3)	57·3 (2)	55·7 (2)

	Sub-orb.		Aur. Pt.		$N\beta$	$\angle\beta N\gamma$	Ordinates above $N\beta$						
	x from N	y	x from γ	y			$N\frac{1}{4}$	$N\frac{1}{2}$	1	2	4	6	8
♂	5·4 (6)	26·3 (6)	93·2 (6)	26·3 (6)	112·3 (15)	49°·2 (6)	3·9 (15)	6·9 (15)	11·3 (15)	18·3 (15)	25·8 (15)	23·8 (15)	14·9 (15)
♀	5·6 (4)	23·9 (4)	90·0 (4)	28·9 (4)	106·6 (12)	47°·0 (4)	3·2 (12)	5·9 (12)	10·3 (12)	17·2 (12)	23·8 (12)	22·0 (12)	14·5 (12)

	Ordinates below $N\beta$			Max. Sub. from $N\beta$		Nose		N. S.		Alv. Pt.		Basion	
	11	$-N\frac{1}{4}$	$-N\frac{1}{2}$	x from N	y	NL	$\angle\beta NL$	x from N	y	x from N	y	x from N	y
♂	10·5 (12)	1·9 (15)	3·5 (15)	48·3 (15)	26·1 (15)	21·9 (4)	17°·3 (4)	41·8 (7)	26·9 (7)	57·5 (12)	38·6 (12)	18·8 (10)	94·9 (10)
♀	9·6 (9)	2·0 (12)	4·0 (12)	50·4 (12)	24·9 (12)	17·2 (6)	27°·7 (6)	35·8 (10)	28·6 (10)	52·0 (11)	41·2 (11)	14·2 (4)	88·5 (4)

	p		p'		Sp.		Ordinates from base line through Alv. parallel to $N\beta$			Mid. Sub. from Bas.-Sp. Chord
	x from N	y	x from N	y	x from N	y	$\frac{1}{2}$ above	$\frac{1}{2}$ below	2 below	
♂	23·4 (13)	52·6 (13)	11·5 (11)	65·5 (11)	16·5 (11)	70·8 (11)	6·6 (12)	1·8 (12)	6·9 (12)	+0·7 (10)
♀	21·1 (10)	52·1 (10)	12·0 (9)	60·6 (9)	14·4 (5)	68·1 (5)	4·5 (11)	1·5 (10)	6·1 (11)	−0·2 (4)

posterior (p') extremities of the palate bones, the sphenobasion and basion all have their x-co-ordinates measured from N along $N\beta$ and their y-co-ordinates perpendicular to this line. In order to obtain additional points in the pre-maxillary region it is also necessary to draw a line through the alveolar point parallel to $N\beta$, ordinates being taken from it at distances of $\frac{1}{20}$th and $\frac{1}{8}$th of $N\beta$ from the alveolar point.

The type sagittal contours constructed in this way are based on very small numbers of crania and no detailed comparisons with other types would be justified. Their most peculiar characteristics are the smooth and almost vertical sections of the frontal bones above the nasion,—the male figure having no distinguishable glabella prominence,—and the lack of projection of the sections of the nasal bones. These are characters which we should expect to find in the case of an Oriental type. A comparison with the type contours provided for other Oriental series shows that *all their frontal and nasal sections are remarkably alike*. The Tibetans *A* and Nepalese differ from the Nagas, Chinese, Burmese, Javanese, Dayaks and Tagals in having more projecting nasal bones.

DESCRIPTION OF PLATES.

I. Normal males (top row), normal females (middle row) and juveniles (bottom row) from the Naga crania collected in 1927. The Biometric Laboratory numbers of these specimens, reading from left to right, are: top row 37, 38, 44 and 18, middle row 4, 28, 12 and 46, bottom row 5, 16, 6 and 48. No. 37 has a large healed wound on the right side of the frontal bone.

II. Typical male Naga skull (R.C.S. 0·6281). *Norma facialis* (ca. 0·8 natural size).

III. Typical male Naga skull (R.C.S. 0·6281). *Norma lateralis* (ca. 0·65 natural size).

IV. A. Typical male Naga skull (R.C.S. 0·6281). *Norma verticalis* (ca. 0·7 natural size).

B. The nasal bridge of a male Naga skull (R.C.S. 0·6282) showing wormian bones in place of the nasal bones (ca. 2·0 natural size). The nasal bridges of two African negro (Teita) crania having the same anomaly are shown in *A Study of the Negro Skull* (*Biometrika*, Vol. XXIII (1931)), Plate IV B and C.

C. The palate of a male Naga skull (1927 series, B.L. No. 44) showing the unerupted third left molar horizontal and preventing the second molar from erupting (ca. natural size).

I	*G'H*	*GL*	*GB*	*NH, R*	*NH, L*	*NH'*	*NB*	O_1R
2·5	68·3	—	101·9	50·0	49·9	49·2	27·7	42·8
0·0	69·0	95·3	102·7	54·0	53·3	53·0	27·7	42·8
8·5	74·9	—	—	51·3	52·1	51·3	28·9	43·0
6·5	72·0	93·1	100·8 ?	52·1	51·0	51·6	23·8	43·4
3·0	70·7	—	107·4 ?	53·3	52·9	52·5 ?	30·0	43·8
8·5	70·3	97·7	—	50·0	50·7	50·7	24·8	45·7
0·5	68·5 ?	98·2 ?	91·8 ?	49·3	50·0	—	28·0	42·9
2·0	70·5	96·5	109·5	53·9	53·2	52·8	27·7	44·4
3·5	—	—	94·4	49·6	49·2	49·2	23·9	41·5
2·5	70·3	—	104·9	53·2	54·0	53·0	25·9	44·9
4·5	70·6	—	—	53·6	54·5	53·6	27·1	44·1
—	—	—	93·3	54·0	56·4	55·5	27·0	41·0
6·0	68·3	98·7	—	49·4	49·5	49·5	26·0	42·0
—	71·0	107·0	—	50·1	51·8	51·1	27·0	45·1
1·0	70·9	—	100·3 ?	50·9	51·8	51·0	27·0	43·0
4·0	—	—	103·8	52·0	53·2	52·9 ?	29·0	44·0
—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—
R.C 8·0	68·2	97·1	103·4	53·8	54·2	54·2	25·1	43·0
—	—	—	96·1	43·2	43·9	43·3	26·0	42·2
" 5·0	64·2 ?	87·7 ?	96·1	51·4	51·2	51·1	26·2	39·2
" 9·0	77·8	94·0	99·0	53·8	53·1	52·9 ?	26·6	42·0
—	—	—	91·3	47·3	46·3	45·7	26·9	38·1
—	67·9	—	—	50·2	51·0	49·8	29·0	—
8·0	62·1	—	96·8	49·3	49·0	48·1	25·2	41·1
8·5	—	—	94·1	—	—	—	—	40·9
1·0	64·8	92·0	92·2	45·0	44·1	43·7	24·8	38·2
—	63·3	—	94·0 ?	46·6	47·1	46·0	27·0	43·8
—	66·8	—	87·7	49·8	49·8	49·8	23·5	43·0
5·5	68·1 ?	—	96·8	50·7	50·2	50·1	25·1	41·7
6·5	67·7	—	98·0	48·0	48·5	48·1	25·7	41·8
3·0	59·0	82·4	97·0	47·1	47·9	47·4	25·8	41·2
2·5	62·4	—	99·0	46·7	47·2	47·1	28·9	44·3
—	61·5	—	94·7 ?	44·6	45·0	44·8	25·2	39·0
—	67·3	—	—	50·0	51·2	50·5	—	43·0
—	—	—	—	46·2	—	46·2 ?	—	38·7
R.C 7·5	68·0	90·8	92·3	48·8	49·3	49·2	28·2	41·0
0·0 ?	66·9	88·0 ?	88·3	50·3	50·4	50·1	23·9	41·9
" 6·0	69·4	91·0	94·0	49·7	49·1	49·1	26·2	44·5
9·5	58·2	—	93·7 ?	44·9	44·2	44·1	25·1	39·0
4·5	57·5	—	81·2	40·2	40·8	40·6	21·8	36·9
8·0	50·0	71·0	72·8 ?	35·0	35·3	35·3	20·2	33·8
1·0	51·0	—	74·3 ?	35·8	36·0	35·9	20·2	36·9
—	—	—	—	—	—	—	—	36·0
—	61·1	—	84·0	46·1	46·3	46·3	23·3	40·0
—	54·2	—	81·3 ?	39·0	39·0	39·0	22·8	35·0
8·0	—	—	76·0	37·8	37·9	37·4	20·6	36·8
—	58·9	—	—	44·0	44·9	44·6	28·0	41·8
—	56·8	81·7	92·4	43·7	43·0	42·9	23·0	34·7
—	—	—	—	—	—	—	—	—
—	59·5	—	87·2	44·0	44·1	44·1	22·8	40·1
—	59·0 ?	—	82·0	—	44·0 ?	43·9 ?	21·5 ?	39·2
—	57·9	—	81·7	41·8 ?	42·0 ?	41·7 ?	22·8	40·7
—	60·2	—	90·6	44·0	43·8	43·5	23·0	39·5
—	61·8 ?	—	91·3	47·8	47·8	47·4	24·2	40·0
3·5	57·2	—	87·2	44·8	45·1	45·0	21·7	40·2
—	60·8	—	92·0	46·0	45·5	45·0	25·0	38·0
6·0	53·8	68·1	84·8	38·8	38·6	38·2	21·6	36·3
—	—	—	92·0	43·0	44·0	43·5	25·0	39·0
—	—	—	—	—	—	—	—	—
R.C 5·5	69·8 ?	91·3 ?	99·0 ?	52·1	51·3 ?	51·1	23·9	44·8

THE ALBANIANS OF THE NORTH AND SOUTH

(1) INTRODUCTORY ACCOUNT OF MEASUREMENTS AND PHOTOGRAPHS TAKEN IN 1929.

By MIRIAM L. TILDESLEY.

The Albanian people, whose tenancy of the Balkans is said to extend farther back into the mists of antiquity than that of any other Balkan race, and whose language—save for borrowings—has no affinities with other European tongues, is divided into two groups, Gegë in the North, Toskë in the South. The question at once suggested to the physical anthropologist is whether the difference of name covers also a difference of type. The measurements and profile photographs which form the basis of the statistical study which follows this paper were collected in an attempt to furnish some sort of answer to this question, the subjects being therefore chosen in approximately equal numbers from North and South.

Light upon the physical characters and relationships of the Albanian people was however not the only, nor even the original object in making these investigations. They were prompted also by a quite different purpose. And since the circumstances which gave rise to this other purpose affect the records themselves, it will be necessary to explain them briefly here.

In 1928, impressed by the enormous waste of effort and opportunity involved continually in the making of anthropological records which are to a large extent non-comparable among themselves, the writer published a paper entitled "Racial Anthropometry—A plan to obtain International Standardization of Method*." In the course of it reasons were given for supposing that the technique elaborated by Professor Rudolf Martin probably formed the most hopeful starting-point from which to build up an internationally acceptable technique. The reasons given—the chief of which was the wide following already obtained by the Martin school—were general: they included none based upon first-hand experience of the technique by the writer herself. As such experience would clearly be useful both in subsequent discussions of the problem and in estimating the reliability of records made by others in the field, she took an early opportunity of obtaining some instruction in the technique and then of practising it under field conditions. Both the amount of time spent in study and that available for field work were unavoidably restricted owing to various circumstances.

* *Journ. Roy. Anthrop. Inst.*, Vol. LVIII (1928), p. 351. [The Editor feels bound to state that in publishing this account he has not overlooked the humour of this explanation of the writer's purposes.]

The limited amount of training must be clearly realised: it would be most unfair to her instructors to pose as a fully trained anthropometrist of the Martin school. The anthropological training given at the Anthropologisches Institut in Munich extends over several years and includes a far longer and more thorough practice of anthropometric technique than it was possible to crowd into the three brief weeks of her stay there. She has to record her deep gratitude to Professor Theodor Mollison and Dr Wilhelm Giesler for the pains they took to make this short stay yield the maximum of profit. Intensive anthropometric instruction was given by Dr Giesler to a class of two—the other student being Mr R. H. Post, whose visit to Europe has enabled him to record in detail some of the differences between the techniques employed by various teachers of anthropometry in Europe*. Practical instruction in photography was given by Professor Mollison; and advice as to instruments and equipment by both. The writer cannot forbear to express once more her grateful acknowledgment of the time and effort so generously expended. At best, however, it could be but three weeks of preparation, and the question arises as to how far one can rely upon the observations subsequently taken. It was part of the plan to get at any rate a measure of the observer's own variability, by making the observations on a series of individuals twice over on different occasions; also, if possible, to test her personal equation against that of some other user of Martin's technique by independent measurement of the same series. Unfortunately neither of these projects could be carried out on the return to England owing to the demands of work accumulated in her absence; and it would have been useless to do so after some months' interval in which memory had lost its sharp outlines. There is no certainty that the personal equation established during a few weeks' application of a technique will be found unchanged months afterwards.

In this particular, therefore, execution fell short of the plan. And unfortunately it must be not merely admitted but emphasized that this failure robs her records of a good deal of their possible value. The fact that most other published measurements on the living body suffer from the same defect does not help matters. No measurement on the living can be taken at its face value as representing absolutely the size of a given character in a given person, for this will have a certain range of inherent variation as well as variation over which the observer has some (but not absolute) control, namely the posture of the subject and the observer's own identification of terminals, his accuracy of reading and his ability to perform the same movements and pressures repeatedly in exactly the same way. From the standard deviation of the difference between pairs of observations by himself on a series one could work out the value of the standard observational error for any of his constants; similarly one could evaluate and allow for the effect of difference of personal equation between pairs of workers. Given enough of these records, one could fall back upon the most probable value forecast by them in cases where no direct measure of the difference was available for the pair whose measurements it was desired to pool or compare. It is strongly suspected that the combination of standard error, standard

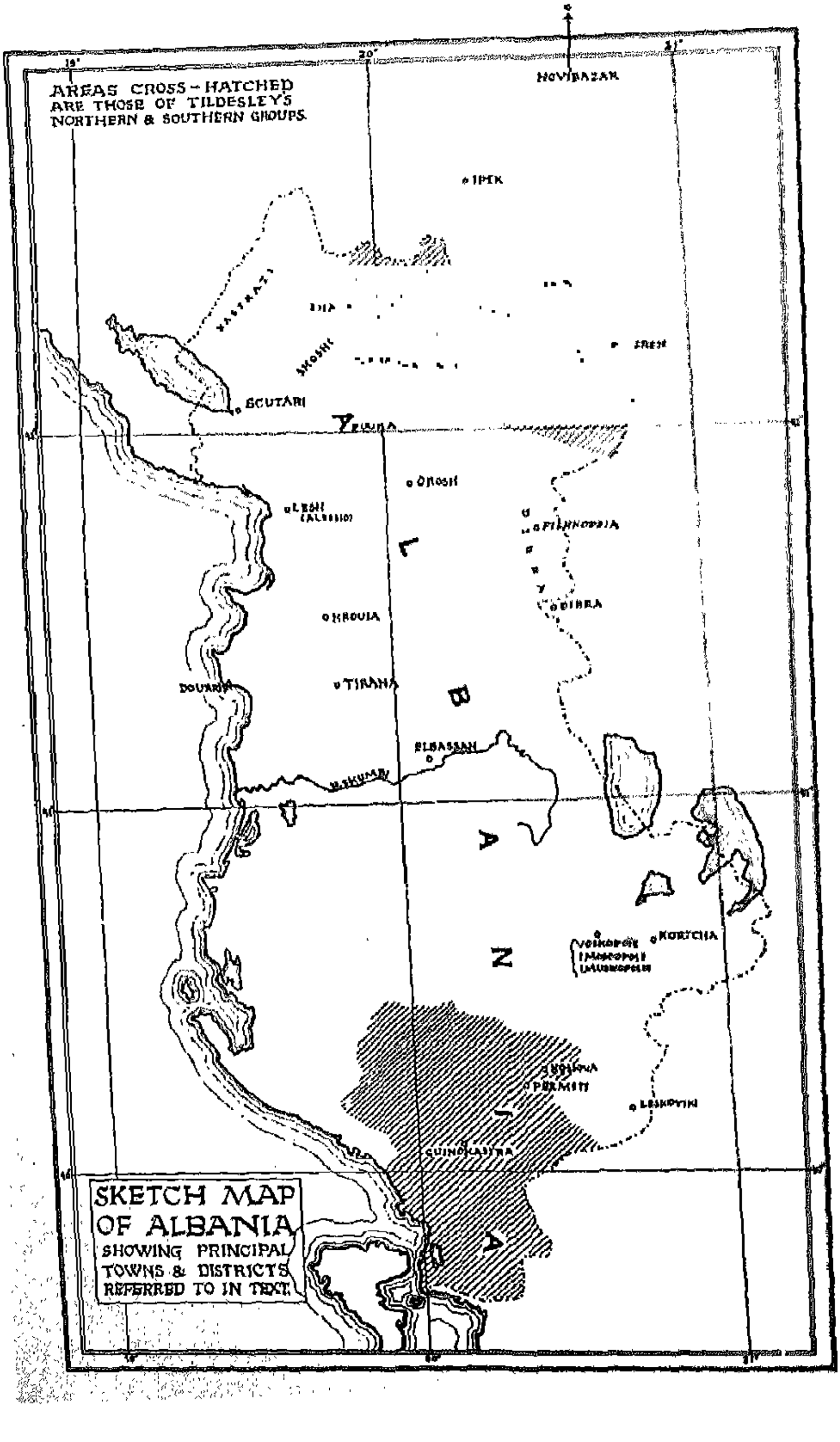
* Anthropologische Messungen am lebenden Menschen, *Handbuch d. biolog. Arbeitsmethoden*, Bd. VII (19[illegible]), S. 161.

observational error, and personal-equation difference would frequently be found to mask actual inter-racial differences for many characters that anthropologists now measure; and when to these factors we add a difference in the actual definitions given in the different techniques employed, the unreliability of face-value differences in the results obtained is of course greatly increased. If the systematic recording and publishing of standard observational errors and personal-equation differences should result, as it probably would, in the abandonment of a good number of observations now taken on the living, the elimination of these futile and time-wasting measurements need not be regretted, even if they exposed the worthlessness of results obtained by much hard work in the past. The past cannot be recalled, but the future is ours.

The actual measurements recorded by the writer, therefore, have only an uncertain value. Since it is now impossible to give any exact measure of their reliability the best she can do is to record certain impressions and experiences. Also, she is not the only worker who has used a technique without adequate training, and done so under the difficulties experienced in the field. Some in fact do so having had none at all; and others who have had as much as she may not have put it into practice fresh from instruction, as she did; also their field conditions may easily have been more difficult. It is probable therefore that her comments will apply in part at any rate to other published figures than merely her own. Before detailing these, however, she must explain what the field conditions were, and what subjects were measured.

The subjects on which observations were made in 1929 were soldiers in Albania's conscript army. Their nominal ages were 21 and 22 years, but these figures are among those that cannot be taken at their face value, for two reasons. One is that even many of the more educated Albanian townsfolk have not up till now been accustomed to keep any record of their ages; still less the uneducated villagers whom she measured. Their exact ages, therefore, were not known, and were only imputed to them in the army records. Secondly, conscription having been only very recently instituted, many young men were drafted into the army who would have been taken some years sooner if a regular army had come earlier into existence: it was quite obvious that in some cases the men were nearer thirty than twenty, and a few of them probably on the far side of thirty. They were drawn, as has been said, in roughly equal numbers from North and South. The actual districts were largely determined by the number of men from each present at that time in the regiment at Shkodër (Scutari). It was hoped at first to measure the Gegës of Dibra. This choice had been suggested by a great authority on Albania, Miss Edith Durham; also, whenever the subject of stature or physique came up in Albania itself, the remark was generally made "You should see the men of Dibra. They are tall, fine Albanians." As however there were not enough men of Dibra in the regiment, the Albanian prefecture of Kukës* was chosen, lying to the north and north-west of

* This prefecture is part of the extensive district extending much to the north of the present boundary of Albania which under Turkish dominion was called Kossovë. It has nothing to do with the town of Kossova in the southern area. The name is still in use though not marked on modern maps.



SKETCH MAP OF ALBANIA SHOWING PRINCIPAL TOWNS & DISTRICTS REFERRED TO IN TEXT.

Dibra, and occupying the north-east corner of the present Albanian kingdom. The district is shaded in on the accompanying map, which similarly shows, in the South, the area from which the Southern Group was drawn, lying almost entirely within the prefecture of Gjinokastër (Argyrocastro). The larger towns were avoided as being perhaps more likely to be affected by racial mixture. Also Albanians of the North being partly Moslem, partly Catholic, with Catholicism as the religion of the southern Serbs; and those in the South being either Moslem or Orthodox, with Orthodox neighbours over the Greek border, some preference was given for Moslems in both districts, again with the object of avoiding as far as possible racial mixture. Among the Albanians themselves, the difference of religion corresponds to no difference of original stock, but to the subsequent accidents of conversion some centuries back. The last large-scale conversion was that to Mohammedanism, imposed by force, under the Turk. Different communities may share the same village, but more often inhabit different ones in the same district. Inter-community marriages and individual conversions are both rare; the latter are said to be discouraged almost as much by the community joined as by the community abandoned.

The individual sheets of observations record not only the man's own village (which was of course that of his father and probably of all his male progenitors for generations back), but the distance between it and his mother's village. It was interesting to discover that the men of Kukës tend to go rather farther afield for their wives than the men of Gjinokastër. Whether this can be associated, either as contributory cause or part effect, with the reputed better physique of the men in the North is a matter for speculation. The following are the distributions obtained, distance being measured by number of hours' walk:

Group	Same village	½ hr.	1 hr.	1½ hrs.	2 hrs.	2½ hrs.	3 hrs.	3½ hrs.	4 hrs.	5 hrs.	6 hrs.	7 hrs.	8 hrs.	9 hrs.	10 hrs.	11 hrs.	12 hrs.	over 12 hours	Total
Northern	7	4	3	3	9	8	9	1	6	2	8	2	5	2	2	0	4	3	78
Southern	45	8	7	6	7	1	4	1	2	2	3	0	0	0	1				87

It may be suggested that the villages are perhaps farther apart in the Kukës prefecture. Such may be the case: the most detailed map available* certainly gives that impression, but this may perhaps be accounted for by the northern villages being on the whole smaller and fewer of them recorded. Information was obtained concerning the size of the villages from which 30 of our Gegës (Northerners) came, and it gave a mean population of 303; for 47 of the Toskës (Southerners) the mean was 605†. Presumably the means for the whole of our two groups would still show the average Gegë village to be smaller than the average Toskë, and thus contribute one reason why the Gegë should need to go farther afield to seek his bride, being bound to avoid marriage with any woman whose male line of descent was known to include

* In M. Justin Godart's *L'Albanie en 1921*.

† Data obtained from the 1927 census, published in Tirana, 1928, entitled *Shqipria më* 1927. It was not possible to identify in all cases the villages entered on the measurement slips with those published in the census, owing to variations in the spelling. The writer wishes to express her thanks to His Excellency Djemil Bey Dino, Albanian Minister in London, for his kind help in the identification of the districts in question and for the loan of the book quoted.

an ancestor in his own male line. This does not seem, however, to be the whole explanation, for if we confine our analysis to those men who came from the larger villages, with over 350 inhabitants, we find that out of 19 Gegës the mothers of only two had come from the father's village; out of 37 Toskës from the larger villages, 27 had both parents from the same village. Only three Gegës came from towns numbering more than a thousand inhabitants, and two mothers out of the three came from elsewhere. Ten Toskës came from towns of this size, and the mother of only one was brought in from outside.

Without the privilege of access to the army and the assistance and facilities so courteously accorded it would have been impossible to obtain in the few weeks available the measurements and photographs which were eventually brought home, and the writer wishes to express her most grateful thanks for the great kindness she experienced in Albania. It may be imagined how different would have been the rate of progress if it had been necessary to waylay individual villagers as they came to the Scutari market, and to make to them the outrageous proposition that they should submit to the indignity of being handled, clad only in shorts, by one of the subject-sex. The reason alleged being incomprehensible and absurd, the real reason would probably be sinister, and in any case would be suspected. The results, if any, obtained in these conditions would have been relatively few and expensive.

For the much more favourable conditions in which the work was eventually carried out the writer must first of all express her indebtedness to her friend Mr Qazim Kastrati, whose help, suggestions, and initiative throughout she most gratefully records, and in whose family she was privileged to enjoy the wonderful experience of Albanian hospitality. Military photographs being forbidden, no less authority than that of the Prime Minister was needed to override this ruling. Through the kind instrumentality of the British Consul and the British head of gendarmerie she was able to state her request in person to the Premier, His Excellency Koço Kota, and to return from the capital to Scutari armed with the necessary instructions. Finally, she wishes to express her thanks to the Commandant of the regiment, and to Adjutant Ibershimi who, being detailed to provide the men, accommodation and facilities required, carried out these tasks with great courtesy and good will.

The field conditions were thus much easier than in some cases, but yet certain difficulties existed which doubtless had their effect upon the records. These were chiefly difficulties of language and of time. The regiment had to depart for Tirana three weeks after the work was begun, in order to take part in the celebration of the first anniversary of the accession of H.M. King Zog. The effective part of these three weeks was shortened by delays which sometimes occurred, soldiers being required for other duties than the duty of being measured. As my friends Mr Qazim Kastrati and Mr Teufik Kalatsi were only able to assist part of the time by recording the measurements—for which I thank them—the work had to be done in part with the assistance of two sergeants who had a smattering of English or French (too imperfect to remove all danger of mistakes) and partly

with the help of a corporal speaking only Albanian. In the latter case she either both measured and recorded, or measured and dictated the figures in Albanian. As she was not well, and the heat very great, and as the hours worked were long, fatigue was not absent; and when fatigue supervenes, the translation of figures into unfamiliar words is neither so fluent nor perhaps so accurate as it would otherwise be. It is probable also that observations made in these conditions may themselves be less accurate, in spite of every endeavour to keep them up to standard. The extent to which such factors are likely to modify the results remains unknown; but as, presumably, errors due to these causes are as likely to be in excess as in defect of the real values, the means at any rate may be very little affected.

To come now to particular measurements: the values for chest girth when the lungs are fully inflated or fully deflated can certainly not be relied upon. Complete deflation was sometimes obtained by making the soldier laugh; the tape was held with one hand, while his ribs were tickled with the other and the Albanian word for "Laugh" uttered. Whether his ribs or his sense of humour were more tickled by this procedure is not certain, but in most cases it was immediately effective and both hands were then quickly used to tighten the tape round the collapsed chest. Sometimes, however, his gravity was portentous, and the lungs far from collapsed. Deliberate inflation and deflation being quite new to him, neither for the most part was performed very successfully; and though normal chest girth was usually the best measurement of the three, there was a tendency to inflate somewhat as soon as the tape was passed round the chest. Where operator and subject have only a few words of any language in common and time is limited, it must always be very difficult to make these measurements comparable with those taken, say, by Germans on Germans who have been trained to breathing exercises from their youth. The "balling" of the upper arm muscle seemed on the other hand to be understood and performed much better, so that maximum upper arm girth is probably fairly reliable. So also is span. The closing of the teeth for measurement of total Face Height almost always offered some preliminary difficulty. The investigator is not underhung, but the soldier always responded to the demonstration of her own closed teeth by shooting out his lower jaw in front of the upper; and so convinced was he that he had to do something unusual with his lower jaw that it often took several moments of voluble explanation and demonstration by the corporal in charge and by those soldiers who had already been through the process, assisted by physical force, before his bewildered mandible could be got into the correct position. Having got it there, however, he usually kept it well clenched, and the operator's impression is that the Face Height measurement was fairly well done. The identification of the nasion in the living is notoriously open to wide differences of practice, but she felt few doubts about her practice of the instructions received on this point. It was otherwise with the identification of the terminal for measuring Knee Height, and this character was therefore abandoned half-way through. Also with that for measuring Ankle Height, though this was not abandoned. Those projective measurements which depend upon posture—height from the ground of shoulder, elbow,

wrist, top of breast bone, etc.—are well known as being very subject to observational error, error which will, however, be less with the very experienced observer. As she was not so very experienced, these must be to some extent suspect in her case, though she was not conscious of any particular misgivings while taking them. Stature seems likely to be the most reliable of these, as it is easier for the subject to assume the correct position than to maintain it for some time. Martin's method of taking Head Height was regarded by her instructors as unsatisfactory and liable to very considerable error: the results for this character must therefore doubtless be accepted with caution. Head Length and Breadth, on the other hand, would be among the most reliable measurements taken. How far the misgivings or confidence here expressed are justified, and how far any errors affected the results cannot now be tested except in so far as the comparative measurements available may tend to confirm them. Where they do not, the cause may not lie in the writer's departure from the technique she was attempting to apply, but in the nonconformity of the other observers' practice with that of her instructors. As far, however, as comparison of Gegë with Toskë is concerned, the same conditions apply equally to both. They were not measured at different times, but Gegë and Toskë interspersed, whether for measurement or photography.

The profile photographs from which Albanian type silhouettes have been obtained were taken, not in the barracks, but in the rooms of a local photographer, who was good enough to allow the use of his premises, to assist by changing the plates in the holders, and afterwards to do a good deal of the developing. His possession of a limited amount of German, acquired during the Austrian occupation in 1914—1918, gave us a medium of intercourse, up to a point, but though he could read figures he was unfortunately unable to write them. This presented a considerable difficulty in that the plates removed from the holders had each to be marked with the soldier's number, for identification. At my suggestion he represented each numeral by an appropriate number of scratches on the plate, though the noughts baffled him and were attempted in various ways—sometimes by ten strokes. This device, however, was successful as regards nine-tenths of the photographs; those in which the counting of the strokes as they were made was wrong, offered a task of some patience in identifying the men, but with the help of the measurements recorded all were identified in the end save two or three. The other difficulties attendant on the photography were those of time shortage (they were taken towards the end of the three weeks allotted) and the job of coping, often single-handed, with batches of a score of men who felt livelier out of barracks, of posing the one and keeping him motionless till photographed and at the same time preventing some of the others from interfering with the photographer's property. Sometimes there was the complication of a client for the photographer, before which all else had to give way, and one's own camera and properties to be shifted. A standard distance between camera and subject was aimed at throughout and a standard focus; these were kept as exact as possible in the rather unplacid circumstances. It is hoped that the photographs, thanks to the labour spent upon reducing them to type silhouettes, may form a definite

addition to our knowledge of the Albanian head. One omission, however, has robbed them of some of their value, and for this the writer must now publicly don the white sheet which has long been her garb in the Biometric Laboratory whenever the subject of the Albanian photographs has been raised there by the Editor of this *Journal*. She omitted to take any direct measurement on the head which would give exactly the scale of the photograph. For this reason, though the Albanian type silhouettes give the proportions of the type head, they fail to give their exact sizes, thus preventing exact comparison with other type silhouettes. The omission was due to no other circumstance than her own failure to realise the necessity of such a measurement.

It remains to record the result of certain observations on the back of the head and on the teeth. The majority of the heads examined were distinctly flat-backed, but a note was made of those that seemed "round," with a roundness that would be seen more in *N. verticalis* than in *N. lateralis*. Thirty-three out of 77 men from the North are thus recorded, as against six out of 89 from the South. Difference significant. Occipital asymmetry was also noted, and found to occur in eight out of the 77 Kukës men (three on L., five on R.), and in 15 out of 89 from Gjinokastër (three on L., 12 on R.); the difference is non-significant. As regards the teeth, edge to edge bite was observed in 16 out of 78* from Kukës, and in 14 out of 89 from Gjinokastër. Obviously-irregular dentition was noted in only two out of each group—would the same were true of our own countrymen! Only two were underhung, and only one had open bite, these three being all from Kukës.

The scientific results of the Albanian Expedition were thus twofold: on the one hand a gain in experience which the writer feels to be very much worth while to herself, and on the other hand these records, imperfect in some respects, concerning the physical characters of Gegë and Toskë. She cannot sufficiently express her gratitude to Professor Karl Pearson for the very great labour bestowed by the Biometric Laboratory on these records and for his kindness in permitting them to take a place, albeit a modest one, in *Biometrika*. She is greatly indebted to Dr Morant for the loan of his camera and telephoto lens, and to him and to Miss N. Karn for the statistical treatment of her measurements and for the reduction of the silhouettes to types; also to the draughtsmen, Miss E. Irvine and Miss M. Kirby, of the Biometric Laboratory for the large amount of drawing work involved.

(2) DISCUSSION OF MISS M. L. TILDESLEY'S MEASUREMENTS ON THE NORTHERN AND SOUTHERN ALBANIANS.

BY THE STAFF OF THE BIOMETRIC LABORATORY.

An examination of Table I shows that the group of men studied by Miss Tildesley from the North is differentiated in essential characters from the group drawn from

* Not quite all the men were available for the entire series of observations and again for photography, hence the variation in numbers.

the South. Taking the differences of absolute body sizes—North minus South—we find the deviations in terms of the probable errors of those deviations are for:

Stature: 7·7; Span: 7·8; Sitting Height: 3·4; Suprasternal Height: 6·9; Acromial Height: 7·4; Elbow Height: 7·3; Wrist Height: 6·7; Finger Height: 5·6; Foot Length*: 12·6; Foot Breadth: 4·0; Chest Breadth: 3·1; Chest Depth: 4·1; Chest Girth at rest: 5·0; Chest Girth inflated*: 7·0; Hip Breadth: 4·5; Waist Girth: 3·4; Upper Arm Girth (straight): 3·9, (flexed): 5·3; Head Circumference: 6·1.

In no case was the Southern Group greater in a bodily measurement than the Northern Group. Clearly the man of the Northern Group is in nearly all respects significantly a larger and more muscular being than the man of the Southern Group.

Turning now to the facial measurements the Minimum Frontal Breadth, Face Heights, Nasal Measurements, Orbital Measurements and Ear Diameters show no significant differences. There is no distinction in Neck Girth or Bi-zygomatic Breadth. Turning to the head measurements we find that the excess of Head Length in the Northern over the Southern Group is 14·7 times the probable error of the difference, and the defect in breadth of the former is 8·0 the probable error of the difference. There is no significant difference in the Auricular Heights.

These racial distinctions are well illustrated in the three cephalic indices. The Breadth-Length Index of the Southern Group being probably the highest known. Such an index may, of course, occur in individuals, but as a racial mean it is of the rarest occurrence. Again the Height-Length Index of the Northern Group is remarkably low. The Fronto-Mandibular Indices are almost certainly significantly different, although it is not possible in the absence of the standard deviations to assign the degree of significance of any of the indices. No other indices have obviously differences of importance.

The variability as measured by the standard deviations shows nothing like the same differentiation, the highest ratios of difference to probable error being in Hand Breadth (3·5) and Foot Breadth (4·2). This is in accordance with the general experience that racial differences are usually those of type rather than those of variability.

We may take it as proven that the Northern Albanians do differ substantially from the Southern in bodily size and, it would appear, also in head-shape. But the high value of the first cephalic index in the Southern Group together with certain vague rumours have led to statements that the Albanians distort the heads of their children. Thus Eugène Pittard states that the variability of the Albanian head measurements "peut aussi provenir des pratiques de déformations qui sont loin

* These measurements are of difficult accuracy, but there could be no bias between the two groups, as they were measured on the same spot in the same manner as the individuals occurred on the muster roll.

TABLE I. *Constants for the Characters measured.*

(Measurements in centimetres.)

Character Absolute Measurements	Albanians of the South No.	 Mean ± Prob. Error	 Stand. Deviation ± Prob. Error	Albanians of the North No.	 Mean ± Prob. Error	 Stand. Deviation ± Prob. Error
(1) Stature	85	163·72 ± ·440	6·14 ± ·318	77	166·03 ± ·520	6·08 ± ·374
(23) Sitting Height	85	87·94 ± ·275^{-}	3·73 ± ·194	77	89·19 ± ·240	3·20 ± ·174
(17) Span	85	169·32 ± ·405	6·76 ± ·350	77	175·58 ± ·630	8·20 ± ·446
(4) Suprasternal Height	84	133·58 ± ·388	5·28 ± ·275^{-}	77	137·91 ± ·493	6·42 ± ·349
(8) Acromial Height	85	133·20 ± ·400	5·47 ± ·283	77	137·86 ± ·472	6·14 ± ·334
(9) Elbow Height	85	102·80 ± ·318	4·35^{+} ± ·225^{+}	76	106·33 ± ·363	4·69 ± ·257
(10) Wrist Height	85	78·43 ± ·255^{+}	3·49 ± ·180^{5}	77	81·03 ± ·290	3·77 ± ·205^{+}
(11) Finger Height	85	60·93 ± ·235^{-}	3·21 ± ·166	77	62·84 ± ·245^{-}	3·18 ± ·173
(35) Shoulder Breadth	85	37·00 ± ·121	1·65^{+} ± ·086	75	37·48 ± ·138	1·77 ± ·098
(40) Hip Breadth	85	27·65^{+} ± ·110	1·50^{+} ± ·078	77	28·33 ± ·102	1·33 ± ·072
(36) Chest Breadth	85	25·63 ± ·096	1·31 ± ·068	77	26·05^{+} ± ·097	1·26 ± ·069
(37) Chest Depth	85	19·18 ± ·081	1·11 ± ·058	77	19·70 ± ·095^{-}	1·23 ± ·067
(52) Hand Breadth	85	8·39 ± ·035	·47 ± ·024	77	8·49 ± ·028	·36 ± ·020
(58) Foot Length	85	24·30 ± ·089	1·21 ± ·063	77	25·07 ± ·098	1·27 ± ·069
(59) Foot Breadth	85	10·05^{-} ± ·048	·66 ± ·034	77	10·29 ± ·037	·48 ± ·026
(61) Chest Girth: at rest	85	87·81 ± ·204^{5}	3·61 ± ·187	77	89·79 ± ·295^{+}	3·84 ± ·209
(61^{a}) " " inflated	85	90·06 ± ·272	3·72 ± ·193	77	93·85^{+} ± ·314	4·08 ± ·222
(61^{b}) " " deflated	85	85·67 ± ·284	3·89 ± ·201	77	86·62 ± ·262	3·40 ± ·185
(62) Waist Girth	85	72·24 ± ·304	4·15^{+} ± ·216^{-}	77	73·63 ± ·278^{5}	3·62 ± ·197
(65) Upper Arm Girth (straight)	85	24·53 ± ·107	1·46 ± ·076	77	25·14 ± ·117^{5}	1·53 ± ·083
(65^{1}) " " " (bent)	85	27·79 ± ·127	1·73 ± ·090	77	28·78 ± ·138	1·80 ± ·098
(66) Lower Arm Girth	85	25·40 ± ·099	1·35^{-} ± ·070	77	25·74 ± ·095^{+}	1·24 ± ·067
(69) Calf Girth	85	33·46 ± ·152	2·08 ± ·107	76	34·03 ± ·130	1·70 ± ·093
(63) Neck Girth	84	35·47 ± ·102^{5}	1·39 ± ·073	77	35·76 ± ·112	1·45 ± ·079
(1) Max. Head Length	85	17·70 ± ·042	·58 ± ·030	77	18·05 ± ·049	·64 ± ·035
(3) Max. Head Breadth	84	16·07 ± ·029	·57 ± ·030	77	15·05 ± ·044	·57 ± ·031
(15) Auricular Height	85	12·17 ± ·039	·54 ± ·028	77	12·10 ± ·044	·57 ± ·031
(4) Min. Frontal Breadth	84	10·93 ± ·020	·40 ± ·021	77	10·87 ± ·030	·39 ± ·021
(6) Bi-zygomatic Breadth	83	14·10 ± ·036	·48 ± ·025^{+}	76	14·07 ± ·039	·50 ± ·028
(8) Mandibular Breadth	84	10·69 ± ·037	·50^{-} ± ·026	76	10·87 ± ·041	·53 ± ·029
(45) Head Circumference	85	55·15^{-} ± ·106	1·45^{+} ± ·075	77	56·07^{5} ± ·107	1·39 ± ·076
(17) Hair Line to Chin	85	17·82 ± ·064	·88 ± ·046	76	17·98 ± ·066^{+}	·85 ± ·046
(18) Nasion to Chin	84	11·90 ± ·042	·57 ± ·029	77	12·05 ± ·046	·61 ± ·033
(21) Nasal Height	85	5·64 ± ·024	·33 ± ·017	77	5·64 ± ·033	·42 ± ·023
(13) Nasal Breadth	85	3·44 ± ·016	·22 ± ·012	77	3·41 ± ·018	·23 ± ·012
(22) Nasal Depth	84	1·71 ± ·017	·23 ± ·012	77	1·71 ± ·017	·22 ± ·012
(16) External Ocular Distance	85	8·78 ± ·020	·39 ± ·020	76	8·75^{+} ± ·027	·34 ± ·019
(9) Internal Ocular Distance	84	3·24 ± ·017	·24 ± ·012	77	3·26 ± ·018	·23 ± ·013
(12) Pupillary Distance	85	6·42 ± ·023	·32 ± ·016	77	6·41 ± ·022	·29 ± ·016
(29) Ear Length	85	6·16 ± ·028	·38 ± ·020	77	6·18 ± ·028	·37 ± ·020
(30) Ear Breadth	85	3·69 ± ·019	·26 ± ·013	76	3·71 ± ·019	·24 ± ·013

Indices, found from the Ratio of Means only.

Sitting Height Index (23)/(1)	85	53·7*	......	77	52·8*	......
Span Index (17)/(1)	85	103·4†	......	77	103·9†	......
1st Cephalic Index (B/L)	84	90·8	......	77	83·9	......
2nd Cephalic Index (H/L)	85	68·8	......	77	64·9	......
3rd Cephalic Index (H/B)	84	75·7	......	77	77·3	......
1st Nasal Index (B/H)	85	61·0	......	77	60·5	......
2nd Nasal Index (D/B)	84	49·6	......	77	50·1	......
Fronto-Mandibular Index (4)/(8)	84	102·2	......	76	100·0	......
Face Index (18)/(6)	83	84·4	......	76	85·6	......
Aural Index (30)/(29)	85	59·8	......	76	60·1	......

* English men at standard age 40·5 years: 52·7.

† English men at standard age 40·5 years: 102·8.

The numbers in brackets placed before the characters indicate the numbers in the corresponding sections of Martin's *Lehrbuch der Anthropologie*, the measurements described therein being those taken by Miss Tildesley. The English names are not intended as descriptions of the measurement. They are those provided by Miss Tildesley in the list she gave the writer of means and standard deviations; they have been occasionally contracted to allow of the table being printed on one page, and occasionally slightly expanded for the sake of lucidity.

d'être abandonnées par les populations de la Péninsule balkanique*." Pittard gives no references to deformation among the Albanians, nor is it clear what he means by the variability of the head-measurements, for as judged by the standard deviations this is not outstanding in Tildesley's data. He quotes no standard deviations and is possibly only referring to the difference in type between North and South.

Remarks on the above Table. Unfortunately when the above material was handed to the present writer for discussion, it was found that in the case of the 41 absolute measurements, seven of the chief characters were only given as far as their means and standard deviations were concerned to two decimal places, but their probable errors to three. The remaining 34 were given to four decimals. To make the table uniform, the means and standard deviations are all given only to two decimal places, but the probable errors to three that there may be at least two significant figures in all cases.

No means nor standard deviations of the indices were provided, so that all that it has been feasible to do was to take the ratio of the means of the absolute measurements. The absence of the chief index distributions is a loss not only because they would have provided further evidence of differentiation between the two groups, but because they would have allowed a wider comparison between Miss Tildesley's measurements and those of other investigators to be made. The large amount of time devoted to the enlargement of the photographic profiles, the measurement of these enlargements, and the final reduction to type contours did not permit of further work by members of the Biometric Laboratory Staff on this material.

It is extremely difficult to get definite information with regard to the treatment of infants in Albania. The present reporter applied first to Mrs M. M. Hasluck, well known for her travels in Albania. She most kindly made inquiries and wrote to him two letters which cannot be said to clear up the matter entirely but are instructive. From the first dated "Elbasan 23. xii. 30"† we make the following extracts touching this point:

Albanian babies lie beside their mothers till the latter get up. Then they are kept permanently (night and day) in cradles; the mother even lifting the cradle on to her knee to feed the child. In the cradles they lie lightly swaddled.

As to head distortion, for years I've been chasing the story of babies being strapped on boards to make their heads flat. It has met me from Zagori, a Greek district, a little north of Jannina, the capital of Epirus, to Scutari in Albania. Of all the cradles I've investigated, and they are pretty numerous by now, not one was without its little mattress and its pillow. The average head from Zagori up through Albania has obviously an unusually flat back, and so far as my present knowledge goes there is no more in the story than that. The story of the boards in

* Eugène Pittard: *Les Peuples des Balkans. Recherches anthropologiques dans la Péninsule des Balkans, spécialement dans la Dobroudja*, p. 280, Geneva, 1920.

† No two authors, no two maps agree in the spelling of Albanian place names. There appears to be as yet no standardisation. Accordingly, we have thought it best to use throughout the spelling adopted by each author cited in discussing his work, notwithstanding the apparent resulting medley. The task of standardising must be left for philologists to dispute over; we are content to show how necessary it is.

Albania may also be helped on by the curiously plank-like shape Albanian men (and many peasant women) give their faces by shaving the hair on their temples.

Miss Durham, I believe, thought the story all foolishness. So far I have found no cases in Albania of the tight bandaging of the head practised in W. Macedonia and elsewhere as recorded in my notebooks.

Some months later I received a second letter dated "Elbasan 11. iv. 31." In it Mrs Hasluck says:

I hope you will be interested to hear that I've run to earth the story that Albanian heads are flat at the back because their owners were strapped on purpose to boards when babies. Within the last weeks I have seen two new-born infants, both strapped to boards. But between each infant and its board there was always a mattress an inch thick, with a thicker pillow (of rags) under the head. Is that not enough to prevent the board from influencing the shape of the baby's head? Both mattress and pillow are as thick as any used among the various populations of Macedonia, who do not strap their infants on boards.

The women state that they strap the babies on the board to make their hold on life as strong as the board. This ignores the fact that the floor of the cradle is wooden and at least almost as thick as the board. Babies are strapped on the board till baptised, if the parents are of the Orthodox religion, and for forty days if the parents are Mohammedans, descended from Orthodox ancestors. For the moment I have no information about the Catholic Albanians, but hope to get some within a day or two. Nor have I any about Mohammedans descended from Catholic Albanians.

The Orthodox qualify their practice, however. Baptism follows birth within a week as a rule, in case the baby dies, when its soul would be lost. They state that they put the baby in a cradle as soon as baptised. In practice, however, they wait till the moon is full, to ensure the baby's living out the allotted span. I incline to think that the board is used as a worthless thing, on which the worthless, hardly human, unbaptised baby is most fittingly strapped. In the town of Elbasan, which is either pure Orthodox or Mohammedan ex-Orthodox, and has no wood, a tile is used instead of a board.

Or, it may be that the board has survived from an earlier time when cradles were not known. Albanian infants are not swaddled round and round like Macedonian and something had obviously to be done to keep their backs straight.

Some Albanians have rationalised the custom to me, saying that it has arisen because the mother's father must present the cradle, and that cannot be bought till the child has been born,—counting your chickens before they are hatched being a dangerous procedure in Albanian opinion, not merely a foolish one as with us. This explanation ignores the fact that the grandfather presents a cradle only when the first child is born, and that the (generally numerous) children who follow are laid in the same cradle. Often, too, a cradle is preserved for several generations in the same family, and the grandfather is not required to present a new one.

Head-moulding is as deliberately and as commonly practised as it was in Macedonia, and by the same means, a handkerchief tightly bound round the temples. 'Who wants a head like an apple?' they say. But they never associate the board with their attempts at head moulding."

It is probable that we may safely dismiss the strapping to a board before baptism as a cause of the brachycephaly of the Albanian Southern head. An English medical woman who had studied gynaecology in Vienna told the present writer that in the lying-in hospital in Vienna the babies were invariably strapped to boards, apparently for the convenience of handling, and that the nurse would carry about four to six of these strapped babies at a time. Yet one does not find that the Austrian Germans are as a whole more brachycephalic than Southern Germans.

The last paragraph of Mrs Hasluck's second letter appears to indicate that the Albanians are willing to head-mould, but that their attempts must be singularly ineffectual. A tight bandage round the temples should achieve two results, (a) a high skull and (b) a round horizontal section. But the Albanian skull has a very low auricular height*, and its characteristic feature is that very apple-shape which the head-moulders are seeking to avoid—the Albanian head is the roundest head in Europe! The reported distortion of the Albanian head has probably little influence on the extreme brachycephaly of the Southern Albanians. It is not unlikely to be a *post hoc* explanation of a remarkable natural character.

Accepting the standpoint that the Northern and Southern Groups studied by Miss Tildesley form in bodily and cephalic characters two distinct racial types, we may now consider how far her measurements are in accord with those of previous investigators.

We may deal with the work† of Raffaello Zampa first. On S. 209 he gives measurements of 59 Albanian men living in Italy—Calabria, Cosenza. This Albanian colony is found in the centre of the toe of Italy, but we have no clue to the part of Albania from whence the men came‡. Zampa states that their mean stature was 164 cms., which agrees with that of Miss Tildesley for her Southern Group. He gives a few undefined measurements of which the most remarkable is the Head Breadth = 148 mm. No one, as far as we are aware, has suggested a lesser breadth than 155 mm. for any Albanian district and the average for 110 cases from various localities of Haberlandt and Lebzelter is 158 mm. The mean Head Length of Zampa's Albanians is 183 mm., which agrees exactly with Haberlandt and Lebzelter's pooled value for 119 men. Zampa gives for the first Cephalic Index the value 80·6 (80·7 from his frequency distribution), a value far below anything suggested by later inquirers for any Albanian group. Thus it does not seem possible to lay any stress on Zampa's figures.

In 1897 Leopold Glück§ published a more important paper giving individual measurements of 30 male Albanians from the north of the country. Fifteen of these men came from Prizren, 11 from Djakova, 1 from Novi-Bazar and 1 from Ipek, these being their birthplaces (see map, p. 24). All these towns lie outside the present boundaries of Albania, although the first two of them are within 10—15 kilometres of the boundary of Miss Tildesley's Northern Group, and they were within the old boundary of Turkish Albania. Comparison, if any, must therefore be

* "La hauteur du crâne (diamètre auriculo-bregmatique) est également faible chez les Albanais. Toutes les populations de la Péninsule des Balkans...excepté les Serbes, ont une hauteur du crâne supérieure à celle des Albanais...c'est le plus petit crâne examiné jusqu'à présent parmi les groupes brachycéphales de la Péninsule des Balkans. Il est d'autant plus nécessaire de souligner ce fait que la taille des Albanais est, en moyenne, une des plus élevées de cette région." Eugène Pittard: *Les Peuples des Balkans*, p. 288, Geneva, 1920.

† "Vergleichende anthropologische Ethnographie von Apulien," *Zeitschrift für Ethnologie*, Bd. XVIII (1886), S. 167—198 u. S. 201—232.

‡ See for the account of this settlement Norman Douglas: *Old Calabria*, London, 1920, pp. 151—193. The migration started in the 15th century with the advance of the Turks.

§ "Zur physischen Anthropologie der Albanesen," *Wissenschaftliche Mittheilungen aus Bosnien und der Hercegovina*, Bd. V (1897), S. 365—402.

made with Miss Tildesley's Northern Group. Unfortunately Glück gives no definitions of the measurements taken.

It will be seen that Miss Tildesley's Northern Group has in all directions a bigger bodied and larger type of man. The one exception is the Head Height, but we do not know what measurement Glück was taking, and whether it was from the centre of the ear passage, or again whether it was the vertical height or not. If it were from the centre, 8 mm. was a quite reasonable difference. To Glück's "Entfernung des Ohrloches von der Nasenwurzel" we shall return later; it is probably the measurement Miss Tildesley unfortunately omitted to take.

Comparable Characters	Glück (30 men) (from his Table, S. 374—75)	Tildesley's Northern Group (77 men) (from our Table, p. 31)	*Remarks*
Stature	1684	1690·3	Of the twelve comparable absolute characters *every one* is greater in Tildesley's Northern Group, when compared with Glück's still more northerly series. The odds against this alone are 4095 to 1, if we neglect the only moderate correlations of the characters. The probability of the observed differences in Head Length and Head Breadth occurring in 12 samples from the same population is of the order ·0004.
Span	1704	1755·8	
Chest Girth	874	897·9	
Head Circumference	553·5	560·75	
Head Length	183·5	186·5	
Head Breadth	153	156·5	
Bi-zygomatic Breadth	139	140·7	
Cephalic Index	82·58	83·9	
Internal Ocular Distance	31·0	32·6	
Foot Length	259	259·7	
Ear Length	58·6	61·8	
Nasal Height	55	56·4	
Nasal Breadth	32	34·1	
Nasal Index	58·74	60·5	
Characters measured by one author only.			
Forehead Height	60	. .	Tildesley's measurements would give 59·3 were it legitimate to subtract distance from gnathion to nasion from distance from gnathion to crinion.
Forehead Breadth	[103]	108·7	There must be an error in Glück's Table, as the value 103 does not agree with his value 109 on p. 371.
Hand Length	187		
"Entfernung des Ohrloches von der Nasenwurzel"	103		Glück's measure is discussed on our pp. 48—49.
Characters non-comparable owing to definition or to personal equation.			
Face Height	184	179·8	Must be due to definition, or to personal equation in determining gnathion.
Lower Face Height	125	120·5	Difference due to personal equation in determining nasion or gnathion, probably the latter. See remark to Forehead Height.
Face Index (= (6)/(17))	75·77	78·25	Difference follows from remarks on Face Height.
Head Height	120	121·0	Possibly Glück measured from centre of auricular passage; Tildesley measured from the "tragion."
Mandibular Breadth	103	108·7	Must be due to personal equation in determining the jaw-angle.
External Ocular Distance	92·6	87·5	Glück's measurement, like Martin's, is from the "Augenwinkel," but was it really taken from orbital margin?
Hand Breadth	89	84·9	? Differences due to personal equation in flattening hand and foot.
Foot Breadth	100	102·9	

It will be seen that while Miss Tildesley's values are more in accordance with Glück's than with Zampa's, the divergence between them points to a racial change north of the present Albanian border, or else to extreme diversity in methods of measurement, which undoubtedly occurs for certain characters.

We may consider here, somewhat out of order, a paper by Eugène Pittard of 1922*, since it deals only with the Cephalic Index. He gives this index for 116 men from different districts in Albania, from the environs of Scutari in the north to those of Argirocastro (Guinokastra) in the extreme south, "ainsi que des localités appartenant à la partie centrale de l'Albanie." It is not surprising that Pittard found a value, 87·9, for the first Cephalic Index almost the weighted mean of Miss Tildesley's North and South Groups, i.e. 87·5. It is not possible, however, to pool the north, central and south Albanians in this manner, and we do not understand the basis of Pittard's remark:

Une telle homogénéité...est à souligner fortement, car on n'en trouverait guère de semblable dans la Péninsule des Balkans (p. 50).

Another paper by Pittard† deals with 26 ♂ crania‡ from two churches in the south of Albania at Argirocastro (Guinokastra), and Moscopole (? = Muskopolie = Voskopoïe, see map, p. 24), and these should accordingly belong to the Toskes, or Southern Albanians. For these 26 skulls the Length is 170·4 and the Breadth 148·2, giving a Cephalic Index of 86·7. If we assume some 7 mm. allowance for flesh and hair on the Maximum Cranial Length and 12 mm. on the Parietal Breadth, these would give a Cephalic Index on the living of about 90·3 in reasonable accord with Miss Tildesley's value of 90·8. Pittard's seven El Bassan male crania give a Cephalic Index of 84·8, already as a cranial value in excess of Miss Tildesley's 83·9 for the living head in the North Group. El Bassan is just north of the Skumbi, and it seems highly probable that Central Albania would show an index intermediate between Miss Tildesley's two groups.

A paper by Max Kässbacher§ deals with five Albanian skulls in the Anatomical Institute of Heidelberg University, one of the skulls being that of a juvenile. Very ample measurements are taken, namely 65 absolute measurements and 83 indices and angles! No means are given—indeed they would be of little value on four individuals—and the author concludes, on the basis of a graphical method due to Toldt, that the skulls belong to the so-called "Dinarischen Rassen." No locus is given for the source of the skulls, and the writer seems ignorant of Pittard's paper of 1924 dealing with much larger numbers. The question of sex does not appear to be considered. By a graphical method due to Mollison, it is asserted that the skulls

* "L'Indice céphalique chez 116 Albanais," *Revue anthropologique*, xxxIIème année (1922), pp. 48–51.

† "Contribution à l'étude anthropologique de l'Albanie. L'Indice céphalique de 58 crânes d'Albanais," *Institut International d'anthropologie*, IIème session Prague, 1924, pp. 220–226 (1926).

‡ There is one skull from Scutari and seven males from El Bassan. The 23 female skulls do not concern us here.

§ "Metrische und vergleichende Untersuchung an Albanerschädeln," *Zeitschrift für Anatomie und Entwicklungsgeschichte*, Bd. xc (1929), S. 199–221.

form "eine einheitliche Gruppe" (S. 210). The Cephalic Indices of the five skulls are given as 78·0, 81·4, 79·1, 79·7, 81·2; these values contrast strangely with Pittard's 86·3 for 34 ♂, and 87·8 for 24 ♀ skulls for the southern portion of Albania. The Heidelberg crania may have been brought from the north, in which case they would confirm the view that whatever these skulls may be the Albanians themselves are not "eine einheitliche Gruppe." The paper, notwithstanding its extensive array of figures, is of small service for our present purpose, not only on account of the smallness of the series, but also because no locus of origin is supplied.

There is still another contribution by Eugène Pittard from 1920*. In this he deals with a large variety of measurements taken on some 112 men from different parts of Albania. He says that of this number there were 27 Ghègues and 51 Toskes, defining the Albanians from the north of the river Skumbi as Ghègues and those from the south of it as Toskes. It is not clear who the remaining 34 Albanians were: they must have come from either north or south of the Skumbi, but they may be hybrids. Clearly Tildesley's North and South Groups, which come from relatively small areas in the extreme north and south, do not correspond with Pittard's Ghègues and Toskes. Thus we have:

	Tildesley's Southerners	Pittard's Toskes	Tildesley's Northerners	Pittard's Ghègues
Stature	1637 (86)	1673 (48)	1690 (77)	1683 (27)
Cephalic Index	90·8 (84)	87·0 (51)	83·9 (77)	84·7 (27)

The differences are in the same direction, but they cannot be supposed to arise from two pairs of samples of the same two populations.

Meanwhile Pittard's combined Albanian results should not show marked differences from Miss Tildesley's results in those characters for which her Northern and Southern Groups show no marked racial distinctions.

Nothing is said in the volume by Pittard as to definitions of measurements, but there is little doubt that Pittard followed Broca. Now the fact that for a considerable number of characters there is almost complete accordance between Pittard's combined Albanians and Tildesley's—i.e. for Sitting Height, Span, Maximum Head Length, Auricular Height, 2nd and 3rd Cephalic Indices (less than a unit), and Bi-zygomatic Breadth—suggests that we are dealing with the same mixed population, possibly not mixed in quite the same proportions, but the striking difference between the values of other characters compels us almost of necessity to believe that the two observers are not defining their quantities in the same manner or that the personal equation, to which measurements on the living are subject, is too great to admit of racial conclusions being safely drawn.

We take first the nose measurements. Miss Tildesley gives absolutely the same Nasal Height for both her groups and very nearly the same Nasal Breadth. But it is clear that Pittard is measuring his Nasal Height, and probably his Nasal Breadth, in quite a different manner. The result is that judged by Nasal Index we should conclude on this character alone that we are dealing with two very distinct races.

* *Les Peuples des Balkans*, Geneva (1920) (the full title is given in the footnote, p. 32): see Pittard's pp. 278—291.

TABLE II.

Character	Pittard's Combined Albanians	Tildesley's Southern Group	Tildesley's Northern Group	Tildesley's Combined
No.:	Average 112	Average 85	Average 77	162
Stature	1678	1637	1690	1662
Sitting Height	886·9	879·4	891·0	885·3
Span	1718	1693	1750	1723
Maximum Head Length (L)	181·3	177·0	186·5	181·5
Maximum Head Breadth (B)	156·6	160·7	156·5	158·7
Auricular Height (H)*	121·4	121·7	121·0	121·4
Minimum Frontal Diameter	111·1	109·3	108·7	109·0
1st Cephalic Index (B/L)	86·4	90·8	83·9	87·5
2nd Cephalic Index (H/L)	66·8	68·8	64·9	66·9
3rd Cephalic Index (H/B)	77·1	75·7	77·3	76·5
Height of Nose	51·35	56·4	56·4	56·4
Breadth of Nose	35·3	34·4	34·1	34·3
Nasal Index	68·8	61·0	60·5	60·8
Bi-zygomatic Breadth	140·7	141·0	140·7	140·9
Ear Length	63·75	61·6	61·8	61·7
Ear Breadth	36·1	36·9	37·1	37·0
Aural Index	56·6	59·8	60·1	59·9
External Ocular Distance	96·6	87·8	87·5	87·7
Internal Ocular Distance	30·7	32·4	32·0	32·5

* "Diamètre auriculo-bregmatique." Pittard does not state how the bregma is to be found on the living, but his value of H agrees closely with those of Miss Tildesley.

If the two observers are measuring from different points† then the urgency of some standardisation in definition and measurement becomes obvious. Much the same remarks apply to the aural measurements, where we reach aural indices whose differences we may be fairly sure are personal not racial. Another like case is that of the ocular distances, where Miss Tildesley appears consistent with herself, but differs widely from Pittard‡. It is quite possible that in Maximum Head Breadth and even in Stature (where a good deal of adjustment is needful) personal equation is playing its part. Minimum Frontal Breadth has so little variation that one is inclined to believe that even the two millimetres difference may be personal equation depending on the pressure of the calipers!

Another paper which deserves consideration in our present inquiry is that of Haberlandt and Lebzelter§. This paper has a considerable advantage over some

† Martin's External Ocular Distance (10) is to the external canthi and Broca's to the external margins of the orbits. This may account for the difference, but only the more emphasises the need for standardisation.

‡ Miss Tildesley, following her instructors, may be assumed to be seeking the nasal suture, which Martin considered identifiable in the living. Pittard is probably following Broca who measures from the "racine du nez" to the "point sous-nasal" (*Instructions générales*, 1879, p. 182). On pp. 139—140 Broca says that the "racine du nez" or "nasion" corresponds on the skeleton to the nasal suture.

§ "Zur physischen Anthropologie der Albanesen," *Archiv für Anthropologie*, Bd. XLV (N.F. XVII) (1918), S. 123—154.

others in that it deals with Albanian soldiers and states very definitely from what parts of Albania they were drawn. The districts are all in the northern section, i.e. north of the Skumbi, and mostly in the extreme north, covering a good deal of Miss Tildesley's Northern Group area. Two localities ought, however, to be excepted. If we may judge by Haberlandt and Lebzelter's data for 13 men only, then in the district of Kastrati there exists a very tall local race (1738·5 mm.) with extreme brachycephaly (88·2). This district is to the west and to the north-west of Miss Tildesley's area. In brachycephaly, but not in stature, it approaches Miss Tildesley's Southern Group. The second group of Haberlandt and Lebzelter's measurements which ought to be excepted when we compare their data with hers is that of 22 men from Kruja. This district is more than half-way down to the Skumbi, and in their Cephalic Index, 89·8, we are approaching that of Miss Tildesley's extreme southern area, i.e. 90·8. The mean Stature of this group is the lowest of all in Haberlandt and Lebzelter's districts, i.e. 1680·2, but it is very sensibly higher than Miss Tildesley's 1637·2 for the extreme south. Still these authors agree with her general result that stature decreases and brachycephaly increases as we pass from North to South Albania. The anomalous position of Kastrati requires further investigation*. Pooling all our authors' data, with the two exceptions named above, we have observations which can at any rate be compared with Tildesley's Northern Group. The data for Miss Tildesley's Northern Group are given in the centre of the Table below alongside Haberlandt and Lebzelter's results. On the extreme left is given Miss Tildesley's Southern Group and on the extreme right the Kastrati and Kruja groups of the former observers are combined. These two outer columns are *not* to be compared with each other. Further there is no justification for our adding an extreme northern group like Kastrati to the midland Kruja group. Our object is as follows: We have seen that while the stature and head-shape of the Albanians change widely from north to south, yet scarcely any, possibly no, change takes place in the facial characters. This fact provides us with a means of determining whether different observers professing to use the same scheme of measurements—here Martin's directions†—reach really comparable results, or whether we must

* It is the high head breadth, 161·1, of the Kastrati males which deserves special consideration. It is even greater than Miss Tildesley's 160·7 for her Albanians of the extreme south.

† Haberlandt and Lebzelter do not directly state that they followed Martin; they give *no* definitions of the characters they have measured. But they do say that they used R. Martin's "Messblätter" and in part R. Pöch's (for prisoners of war). Martin's "Messblatt" refers to the numbers in his *Handbuch*. A description of Pöch's "Beobachtungsblatt" for prisoners of war is given by him in the *Mittheilungen der anthropologischen Gesellschaft in Wien*, Bd. XLVI (1916), S. 115—128. He says (S. 123):

"Die 26 Masse dieses Messblattes sind alle in dem somatologischen Messblatte von R. Martin.... Die Numerierung sowie die Benennung der einzelnen Masse ist jedoch genau dieselbe wie im Martinschen Beobachtungsblatte, so dass sich jedermann über die bei den einzelnen Messungen gehandhabte Technik nach der Nummer, welche das Mass führt, im Martinschen Lehrbuche unterrichten kann."

It is clear from this that Haberlandt and Lebzelter are using Martin's technique, precisely as we may suppose Miss Tildesley's instructors to be doing. Personally the present writer does not wonder that they could in a number of measurements reach different results. The definitions and methods of measurement provided by Martin are in some cases so obscurely stated that one is not surprised at the amount of personal equation which flows from their use, and one wonders if Martin himself had ever applied them to long series.

infer that their personal interpretations lead to character-differences which are certainly of the order of racial differences.

Comparing in Table III the two central columns for Northern Groups we see that as far as the following characters are concerned: Stature, Head Breadth, Bi-zygomatic Breadth, Minimum Frontal Breadth, Mandibular Breadth, Nasal Breadth, and possibly Internal Ocular Distance, no distinction between the two sets of measurements of the Northern Albanians can be made. When, however, we turn to Face Height with the corresponding Face Index, to Nasal Height and Depth with the corresponding two Nasal Indices, and possibly to Head Length and the corresponding Cephalic Index, we mark divergences which in the case of cranial measurements would be said to mark racial differences. Do they do so in this case, or are they due to different interpretations of definitions, to the use of different definitions, or to personal equation in measurement? We think a partial answer can be given to this question. Take first the Face Height; Tildesley gives reasonably like Face Heights for her Northern and Southern

TABLE III.

Comparison of Tildesley's with Haberlandt and Lebzelter's results for Northern Albania.

(Measurements in mm.)

Character	Tildesley's		Haberlandt and Lebzelter's	
No.:	Southern Group (88—85)	Northern Group (76—77)	Northern Group† (83—81)	Kastrati and Kruja combined (33—35)
Stature	1637	1690	1687	1702
Head Length	177·0	186·5	184·4	179·8
Head Breadth	160·7*	156·6	160·7	160·4
Cephalic Index	90·8	83·9	85·1	89·2
Face Height	119·0	120·5	110·7	116·4
Bi-zygomatic Breadth	141·0	140·7	141·0	142·8
Face Index	(84·4)	(85·6)	83·1	81·8
Minimum Frontal Breadth	109·3	108·7	107·4	108·4
Mandibular Breadth	106·9	108·7	[107·6]‡	[107·9]‡
Fronto-Mandibular Index	(102·2)	(100·0)	99·9	100·5
Nasal Height	56·4	50·4	51·9	51·9
Nasal Breadth	34·4	34·1	33·8	34·8
Nasal Depth	17·1	17·1	[22·7]‡	[23·2]‡
Nasal B/H Index	(61·0)	(60·5)	65·2	67·2
Nasal D/B Index	(49·6)	(50·1)	[illegible]	66·6
Internal Ocular Distance	32·4	32·6	31·9	33·8

Results in curled brackets are the indices obtained from the ratio of means of absolute characters

* Omitting one breadth of 105 (!).

† Omitting the "Serben und Türken aus Podgorica" group.

‡ Absolute measurements *not* provided, only the Fronto-Mandibular and Nasal Depth/Breadth Indices recorded. The absolute values are supplied from the indices.

Groups, and Haberlandt and Lebzelter do the same for their two groups. Both observers are self-consistent but differ extremely the one from the other. There cannot be a doubt that we are not here dealing with a racial difference but with a matter of definition or interpretation of definition. Turn again to the Nasal Height, Tildesley's results are the same for the two groups and so are Haberlandt and Lebzelter's. Both agree in the statement that the Nasal Height does not vary from one group to a second. But these observers differ so substantially in the absolute height of the nose, and the resulting Nasal Index (breadth/height) that a racial differentiation would be legitimately asserted, had we not the absolute equality within their own measurements of groups which are racially distinct in other characters; the like remark applies equally to their measurements of Nasal Depth which are self-consistent within the same observers' data and inconsistent between observers.

We are inclined to think the same criticism may be applied if in a less degree to Head Length. Tildesley gives a Head Length which is greater than that of any one of the six districts or for the pooled data of Haberlandt and Lebzelter, thus she obtains a lower Cephalic Index than they do. Picking out the Austrian investigators' groups which correspond geographically most nearly to Tildesley's northern area, the Cephalic Index obtained is 85·1, as compared with Tildesley's 83·9, and the difference lies entirely in the measurement of the Head Lengths, for the Head Breadths in both cases are practically identical (156·6). Whether this difference in Head Length is due to the choice of the frontal terminus, to choice of the occipital terminus, to a difference of dealing with the hair on the back of the head, or to the use of different types of head spanners, or to conventions as to the maximum length in asymmetrical heads, we cannot say. All we can draw attention to is that in measurements on the living methods of procedure and personal interpretation may lead even in such important racial characters as the Cephalic Index or the Nasal Index to divergences as great as those which in cranial research mark definite racial differences.

With these facts before us, sad as the conclusion is, we are compelled to hold that the great and ever accumulating mass of measurements on the living are practically worthless. Observers may state that they are following the directions of such and such an authority, but this is of no avail. Obscurity always arises about verbally defined "points," which are not points. The only solution is training under instructors whose methods have been standardised one with the other. And this standardisation involves not only that of human instruments, but of the machines employed. How many observers think it needful to test their scales and calipers against standard scales before they start, and *how many remember to repeat the same on their return?* The present conception seems to be that a very large number of measurements vaguely made on relatively few individuals by wholly unstandardised observers will be of value in building up the racial history of mankind. This is the veriest delusion. Sad as the statement may seem, the work of measuring the living will have to be restarted with *highly* trained observers, standardised internationally (or better supernationally) one with another. As the astronomers started inter-

nationally to divide up the heavens for their great star-chart, so anthropologists when they have ceased to be dilletanti will divide up the world, working like the astronomers in a standardised way with standardised instruments, to reach an international racial chart. Important as the choice of measurements on the human body may be, that choice and the mode of definition is of far less value than personal and instrumental standardisation. Sadly, but without hesitation, we affirm that what has been done will have sooner or later to be scrapped, and anthropometricians must start afresh, not even from internationally accepted definitions—which do not define—but on a basis of international standardisation. Even then we scarcely believe that measurements on the living body will give as satisfactory results as measurements on the skeleton. But at the same time variations in external bodily type, if we can measure them—especially in physiognomy—are of the greatest interest and value; they are not necessarily in every case *highly* correlated with the skeletal framework beneath them. Perhaps, as in the case of astronomy, it is not to direct measurement that we shall trust in the future, but to measurement on standardised photographs. Some attempt has been made in this direction by the type silhouettes of the Biometric Laboratory. These are only on trial at present, but if they are successful the same method might be applied to other parts or aspects of the human body. The rapidity with which a hundred *standardised* photographs could be obtained in the field as compared with 30 or 40 measurements on 100 men is a great advantage, and it leaves the laborious task of measurement to be undertaken in the laboratory with proper instruments under standardised conditions. As a slight contribution to this topic the type silhouettes of the 67 Northern and 67 Southern Albanians will be discussed in the remaining section of this paper*.

(3) SILHOUETTES OF THE ALBANIANS OF THE NORTH AND SOUTH.

By the Staff of the Biometric Laboratory.

The accompanying silhouettes are based on the photographs taken by Miss Tildesley during her stay in Albania, and have been dealt with in the customary manner. This consists briefly in the most delicate process of enlarging the profile photograph by aid of a Coradi pantograph, this being done in two stages to reduce the error of enlargement. The next process is to modify this enlargement, also by aid of the pantograph, so that the distance from mesoporion to the nearest point of the nasal bridge coincides with the like distance measured with the ear-plug spanner (i.e. the spanner used to take the auricular height) on the subject of the photograph. We have now a life-size silhouette of this subject. By aid of somewhat elaborate co-ordinate systems, a very large number of measurements are taken on

* A certain number of photographs could not be used because a portion of the head was not on the plate, or because the tragion or sub-orbital point was not visible on the plate, or because the subdian sagittal plane of the subject was not parallel to the focal plane of the camera, or for other defects in photographing.

Type Silhouette of the Northern Albanian Group.

Type Silhouette of the Northern Albanian Group.

Type Silhouette of the Southern Albanian Group.

this profile outline. This process is repeated on every one of the individual photographs, which are to be pooled to obtain the composite, and then the average of each of the co-ordinates thus obtained is taken, and these are plotted afresh to give the average or type contour. Diagrams I and II give the type contours of Miss Tildesley's Northern and Southern Groups respectively, with the mean values of the

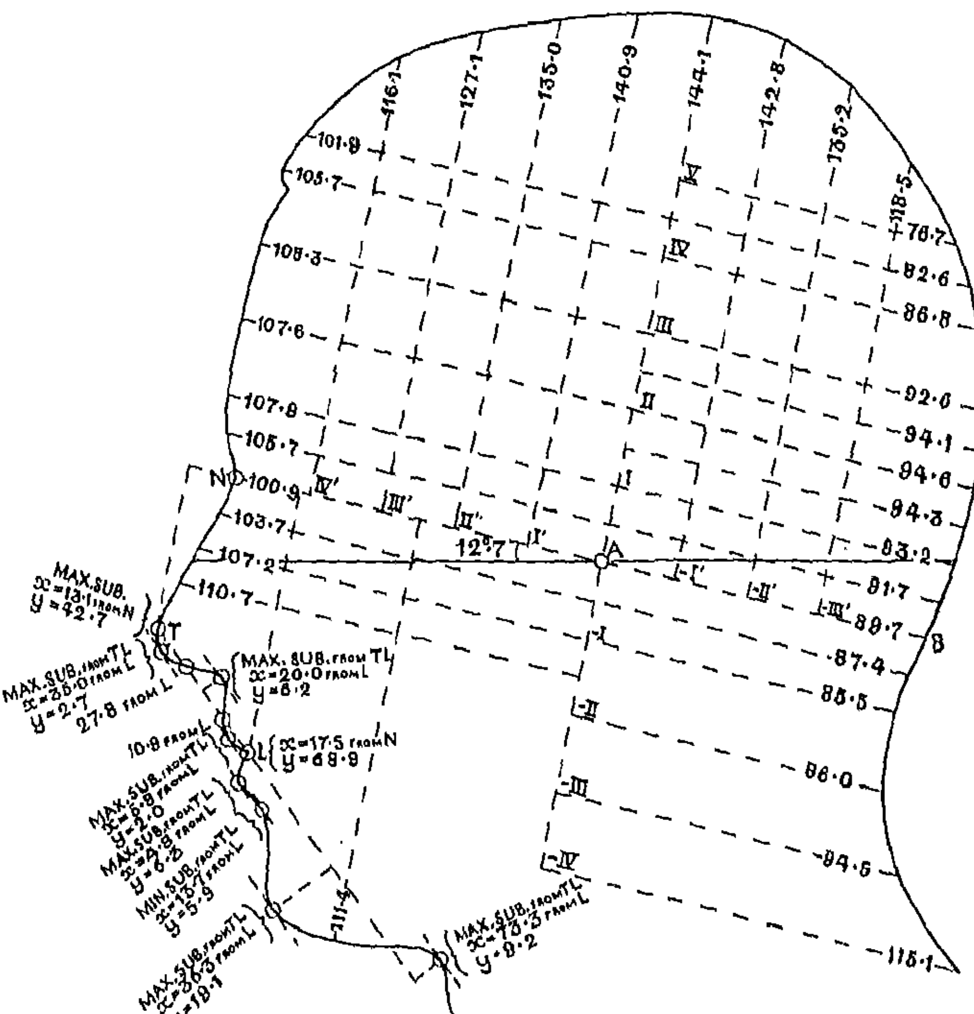


Diagram I. Type Contour of Northern Albanian Group.

co-ordinates marked upon them. They have been reduced accurately to half scale. To obtain the line for orientation a small white wafer is placed on the sub-orbital point, and this appears on the photograph. From this with the centre of the auricular passage, it is possible by a slight adjustment to obtain very approximately a line representing the Frankfurt Horizontal.

Although the suggestion that she should take standard photographs of the Albanians was made to Miss Tildesley from the Biometric Laboratory and a camera with a telephoto-lens was lent to her by one of the Staff, she failed to realise the importance of a portion of the instructions given to her. Unfortunately not one of the facial or cephalic measurements taken by Miss Tildesley is of any real service for the production of standard type silhouettes. If a photograph be taken in profile,

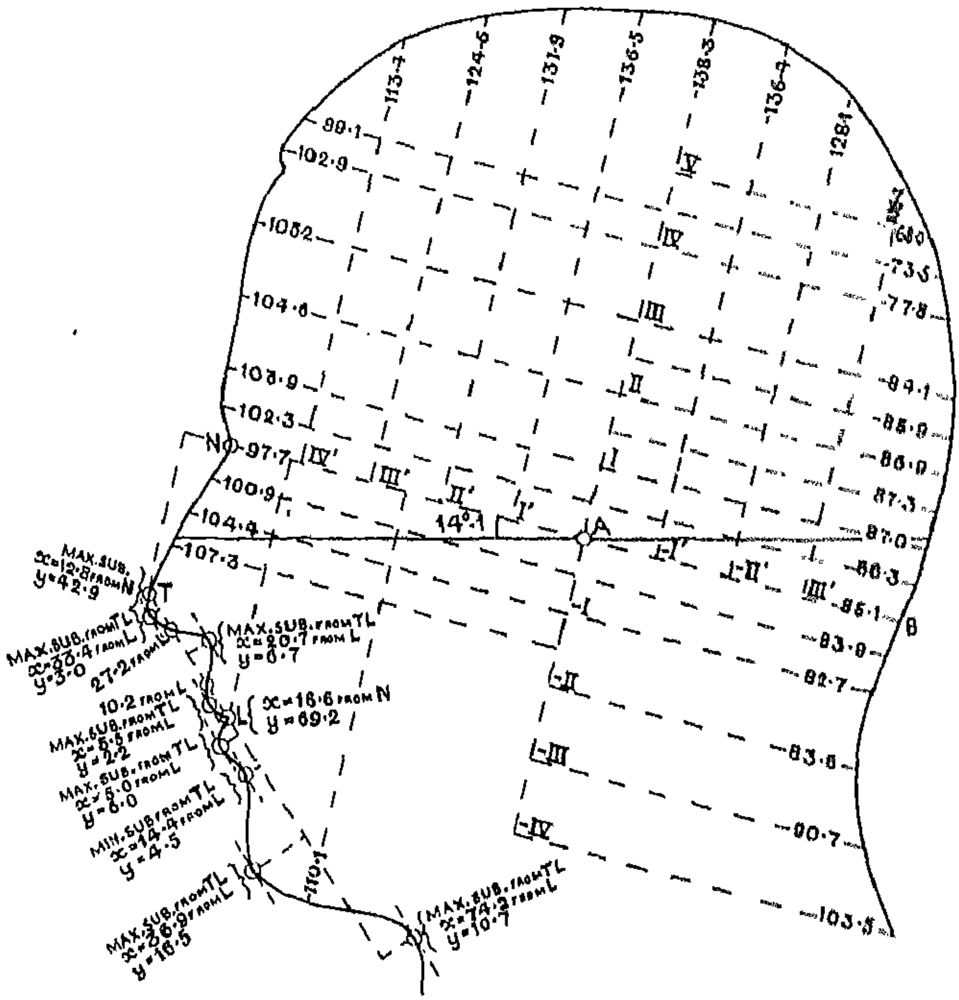


Diagram II. Type Contour of Southern Albanian Group.

no major length that could be of real service for determining absolute size must have a terminal beneath the hair. Thus maximum head length and auricular height are of little value. Facial measurements in the median plane are too small, too vaguely determined, or too arbitrary to be used. Thus in dealing with the nose height, if it be measured from the nasion (!) to the subnasal point, the length is not only too small, but both terminals are too vague, and the same remark applies to the

nasal depth. The facial heights from nasion to chin, or from hair line to chin would perhaps be satisfactory as to length, but the determination of the two terminals, however clean cut on the enlarged photograph, appears to have been hopeless on the living. That is to say, the scales of enlargement of the photograph to bring it up to life-size as judged by the several facial measurements were not only quite different for the individual, but for the average of the series. No other method was adopted by the investigator for standardising her photographs*. She did, however, mark on the subject's face the sub-orbital point and the tragion, and these are visible on the majority of the photographs. Unfortunately she did not measure, as she might have done, by aid of a projection spanner†, the distance between the sub-orbital point and the tragion projected on to the median sagittal plane‡.

The method adopted by the Biometric Laboratory is to insert an ear-plug about an inch in length and 8 mm. in diameter in the auricular passage, the plug is held in position against any resistance of the ear-lobe by a simple arrangement. The visible end is coated black, with a central white spot, which is reproduced by the photograph. On the photograph, after mechanical enlargement, the distance from this white spot to the *nearest* point of the nasal bridge, the hyperrhinion (no searching for nasion!) can be at once measured. But the actual distance from the central axis of the auricular passages to the nearest point of the nasal bridge can be at once measured by the head spanner which takes the auricular height, and is inserted in the auricular passages by aid of its ear-plugs. Thus the actual life-size of the individual of which we have the profile photograph is determined. The knowledge of the central auricular point and of the orbital point on the photograph enables us by a slight correction to obtain a good approximation to the Frankfurt Horizontal§.

Of course there are other methods of obtaining the scale of enlargement needful for the photographs. Thus:

(i) We may place a scale in the median sagittal plane of the subject's head. This is easy enough with the instrumental fittings in the laboratory, but far less easy in the field, and does not free us from the need of determining from the mesoporion‖ the standard horizontal plane.

(ii) We may place our subjects in such a position that their median sagittal planes are always at the same distance from the focal plane of the camera. This is an adjustment relatively easy in the laboratory and the photographic room, but by no means easy in the field, where the ground may be rough and the subject's seat extemporised.

* A critical examination of the photographs shows that they were not taken in the same standard manner. The chair and the camera were not in the same positions for all individuals.

† See *Biometrika*, Vol. I. p. 416.

‡ The tragion is, however, in itself a most unsatisfactory point. See *Biometrika*, Vol. xx^B, p. 393.

§ See *Biometrika*, Vol. xx^B, p. 389.

‖ *Ibid.* p. 389.

It may simplify the work to have all the photographs to standard size, but it is risky to attempt this and then fail in that standardisation, for all is then lost. It is therefore better to have a knowledge of size from a standard length on subject and on photograph. We have not been able to discover anything as good as, still less better than, the distance from the hyperrhinion to the central auricular axis.

To obtain any type contours from Miss Tildesley's photographs we had to follow a roundabout and by no means thoroughly safe road! After a long period of perplexity we determined to proceed as follows:

There were 134 suitable right profiles available, 67 of each group. It was clear as already stated that the scale of reduction was not the same for all these photographs. The sub-orbital point and a point anterior to the ear termed by Miss Tildesley the "tragion" had been marked on the subject by black dots, and these made it feasible to obtain an approximation to the Frankfurt Horizontal plane on each profile.

In the first paper on type silhouettes published in *Biometrika* in 1928*, it was pointed out in discussing the Pöch-Weninger photographs of the West African Negro, that Martin's definition of the tragion was so obscure that it was not possible to use it in practice. Miss Tildesley kindly furnished the following definition of the ear-point termed the tragion as identified by her instructors in Munich:

"In my practice the tragion was determined as the point of intersection of two tangents; the one a common tangent to the anterior margin of the tragus and *crus helicis*, the other a tangent to the upper border of the tragus passing through the nasion as seen in profile."

While this emended definition still leaves room for personal equation, and possibly is inexact, as it is difficult to draw in space tangents to curves only *seen* in profile, and difficult to see a subject in profile except on the focal plane of a camera, it is safe to assume that the "tragion" as marked by Miss Tildesley is always anterior to the opening of the auricular passage, and probably slightly above the highest point, the hyperporion on the superior margin of that opening. The point where the axis of the ear-plug inserted into the auricular passage cuts the median sagittal plane is the mesoporion†, and the auricular point A is defined to be the one where a circle of 8 mm. radius having the mesoporion as centre is met by its upper tangent from the projection of the sub-orbital point on to the median sagittal plane. The tragion as marked by Miss Tildesley probably lies close to this auricular point A, but anterior to it and slightly inferior to it. It was accepted as the auricular point A, which serves as the origin of the axes from which the co-ordinates are measured.

Now the actual origin of co-ordinates does not matter as far as the *silhouettes* are concerned, if we obtain for each subject a point which may be said to be anatomically the same. But it is of importance to have the same point on all type

* *Biometrika*, Vol. xx^B. pp. 389—400.

† Defined *Biometrika*, Vol. xx^B. p. 389.

contours, if we wish to compare their measurements one with another. Unfortunately Miss Tildesley, having deserted the hyperrhinion and mesoporion (leading to the auricular point) for the nasion and her tragion, did not provide by aid of the projection calipers the projected distance between her points. She did, indeed, record head measurements of her Albanian soldiers, but as we have seen there is no one of these which corresponds with sufficient accuracy to any length which can be measured on the photographs to make it possible to determine the individual scales of reduction.

The steps adopted were as follows: The first step was to trace from the photographs with extreme care the outline of each head, introducing the marked positions of the sub-orbital point and the tragion. These drawings were then enlarged by two stages to exactly four times their linear dimensions by aid of a Coradi precision pantograph. The outlines thus obtained, which were evidently rather smaller than those of the living head, were then divided up by the system of co-ordinates shown in Diagrams I and II, and their measurements were taken.

The origin, or point A represented by Miss Tildesley's tragion, was first joined to the hyperrhinion N, the nearest point to A on the nasal bridge, and thus we obtained the base line NAB, one of the co-ordinate axes, the second being the perpendicular to it through A. From these axes the co-ordinates of all points on the outline were determined, except those of the mouth and chin. This method of measurements, and those used in the case of the facial outline, were very similar to those adopted in constructing the English male type silhouette, though slight modifications which can be seen by comparing the figures were introduced in order to make the best use of the new material*.

It was next necessary to reduce the measurements of the enlarged profiles to a uniform scale as well as this could be done under the circumstances. After comparing several caliper measurements made on the living head with the corresponding lengths on the profiles to which they may roughly be supposed to correspond, the following method was found to be the least unsatisfactory. Neither of the terminals of the facial height as measured by Miss Tildesley can be located at all accurately on the profiles; but it is not unreasonable to suppose that their facial height will bear an approximately constant ratio to the line joining the hyperrhinion to the progenion on the actual life-sized silhouette. Now we define the progenion as the point on the outline of the chin farthest removed from the line joining the protion to the stomion. The protion is the most anterior point of the nose obtained by drawing a tangent to it perpendicular to the NAB axis and the stómion is the meet of the lips. We require only to draw a tangent to the chin parallel to the protion-stomion join†. All this can be done on the outline. By means of this ratio of measured nasion-gnathion length on the living to the measured hyperrhinion-progenion lengths on the enlarged photographic profiles, it was possible to reduce all the enlarged photographic profiles to a common scale,

* Cf. *Biometrika*, Vol. xx[B]. pp. 390—396.
† *Biometrika*, Vol. xx[B]. p. 396.

which would not, however, be necessarily the true actual size. These ratios ranged from 1·250 to 1·553 for the Northern Group and 1·274 – 1·567 for the Southern; the nasion-gnathic length being the greater in every case. All the measurements of each enlarged profile were then multiplied by its particular ratio and the means of these adjusted measurements were found for each co-ordinate. Thus type profiles ϵ_1 and ϵ_2 were obtained with these means for the Northern and Southern Groups. But these type contours will not yet be of life-size. A further adjustment was needed for the contours were clearly too large. Now the fifty English male students whose silhouettes were used in the construction of the English type had a mean caliper glabellar occipital length of 194·4 mm. The maximum length from the glabella to the back of the head on the type silhouette was 211·8. These two lengths will not necessarily coincide in direction, but their difference was treated as the same for Albanian and English types. This difference was 17·4 mm. which would in the first place measure thickness of the hair, in the second pressure of the calipers, and lastly be possibly a result of non-coincidence of direction, the hair not being equally thick over the back of the head. Thus we obtained a rough process for finding the absolute size of the silhouettes. The Albanians being soldiers had shorter hair (see photographs) than the peasants as a rule would have and would be likely to correspond in this respect more closely with the English undergraduates. Measuring the maximum length from the glabella on the 67 Northern Albanians of the photographs the mean gave 186·6 mm. If we add 17·4 mm. we have for maximum length from glabella on silhouette 204·0 mm. For the 67 southern Albanians the mean length on the living head was 176·6 mm. or the corresponding silhouette length should be 194·0 mm.

On the enlarged type contours, ϵ_1 and ϵ_2, obtained as described above, the maximum Head Length for the Northerners was 217·7 and this has to be reduced to give 204·0*, the reducing factor is therefore ·937. In the same manner for the Southerners the enlarged type contour gave a maximum Head Length of 207·2 and this needed to be reduced to 194·0 or the reducing factor was ·936. The near equality of these two reduction ratios is satisfactory as the Northerners and Southerners were photographed in random order, and it indicates that displacements of camera, chair and subject in chair were fairly randomly distributed between both groups. These reducing factors were applied to all the measurements on the enlarged type contours, and final presumed life-size contours obtained for the two groups. These are reproduced to half-scale in Diagrams I and II, the corresponding silhouettes being given on Plates I and II to half life-size. The method by which they have been obtained from the photographs is frankly admitted to be a crude one, but it appeared the best that could be applied to the defective data available.

Having obtained in the above manner the Northern and Southern Albanian male silhouettes we puzzled once more to find any quite independent method of checking our result. Looking up the literature we discovered Leopold Glück's

* Actually Diagram I and Plate I have been drawn slightly in excess of the true values; they require reducing in the ratio 102 to 103.

statement (see our p. 35) that the "Entfernung des Ohrloches von der Nasenwurzel" for the measurements on his 30 Northern Albanians was in the mean 103 mm. Now "the distance of the ear-passage from the root of the nose" is a very vague expression. Glück probably took the expression from Virchow's list in Neumayer's *Anleitung zu wissenschaftlichen Beobachtungen auf Reisen*, 1875 (see also *Zeitschrift für Ethnologie*, Bd. XVII, 1885, S. 99—102), where the measurement is named in precisely the same way, but this is not helpful, as Virchow does not define it, nor refer to any instrument for taking the measurement*.

If we suppose the "root of the nose" to be the most posterior point of the nasal bridge we have still to define the word "posterior." We shall not reach precisely the same point, if we define posterior in regard to the Frankfurt Plane, or as the point nearest to the "Ohrloch." Finally if we agree to suppose Glück's measurement to correspond very closely to our hyperrhinion to mesoporion distance, we have to remember that this is *not* Tildesley's nasion to tragion length. We must first reduce Glück's 103 to our distance from auricular point to hyperrhinion and this will be equal to $\sqrt{103^2-8^2} = 102{\cdot}7$ nearly. Now the auricular point is not Miss Tildesley's tragion, which latter may easily be 1·5 to 2·5 mm. anterior to the auricular point. Hence we should expect on the basis of Glück's measurement that the distance of A to N on silhouettes of the Northern Albanians might lie between 100·2 and 101·2, with a mid-value of 100·7. On our silhouette it is 100·9 for the Northern Group. Considering the small number of Glück's cases, the area, wider than Miss Tildesley's northern area from which he drew them†, and further, the fact that the distance from mesoporion to hyperrhinion varies from 98·3 to 101·2 as we pass from south to north of Albania, the above accordance is, perhaps, all we could look for. It, of course, depends on our interpretation of the term "Entfernung des Ohrloches von der Nasenwurzel" being correct. Assuming that is so, our silhouettes are hardly likely to be at most more than 1 % to 2 % in error as to their absolute size. Had Miss Tildesley simply recorded the measurement we suspect to be that taken by Glück‡, and used the ear-plug when photographing, all the laborious computing work just described would have been saved, and the absolute sizes of the silhouettes would have been ascertained without any of the doubts arising from hypotheses such as we have been compelled to make.

* Schmidt (*Anthropologische Methoden*, 1888, S. 107) identifies the "Nasenwurzel" with "der tiefsten Stelle der Eingesattelung zwischen Stirn und Nase," but if a point on a curve is to be "deepest" it must be with regard to some chord of the curve and he does not define this chord. In the next sentence he speaks about the nasal suture and apparently identifies the nasion with this "deepest point of the nasal bridge." Martin (*Lehrbuch der Anthropologie*, 1928, Bd. I, S. 147) very properly says that "von manchen Autoren wird fälschlich die am tiefsten eingesattelte Stelle der Nase als 'Nasenwurzel' bezeichnet." He identifies the "Nasenwurzel" with the Nasion, but he does not say how the "tiefsten eingesattelte Stelle der Nase," if required is to be found. We define it as the point on the median sagittal section of the nose nearest to the mesoporionic axis. This is easy to find in the case of the type silhouettes.

† His men came from Novibazar, Ipek, Djakova and Prizren (see map, p. 24), part of the former Turkish Albania, now outside the northern boundary of present Albania.

‡ It is quite simply taken with Pearson's head spanner, and the mesoporion quite simply found on the photograph by aid of Pearson's ear-plug.

If the two silhouettes, i.e. that for the Northern and that for the Southern Albanians, be superposed so that hyperrhinion and stomion agree, it will be found that the whole of the faces are in almost complete accordance. This facial likeness we had already seen must arise from Miss Tildesley's facial measurements. On the other hand the tops and backs of their heads show marked divergence; the Northern head projects beyond the Southern in the neighbourhood of the ophryon. From the crinion the hair line* of the former lies increasingly outside that of the latter, and at the back of the head between obelion and hystation the difference amounts to 8 to 9 mm. This excess, if it diminishes somewhat, is maintained right down to the lophion. If the auricular points be superposed, and the two lines from these points to the hyperrhinions be brought into contact we find that the Southern type lies entirely inside the Northern right away round from gulion to lophion, the two faces having practically parallel contours from progenion to the ophryonic region.

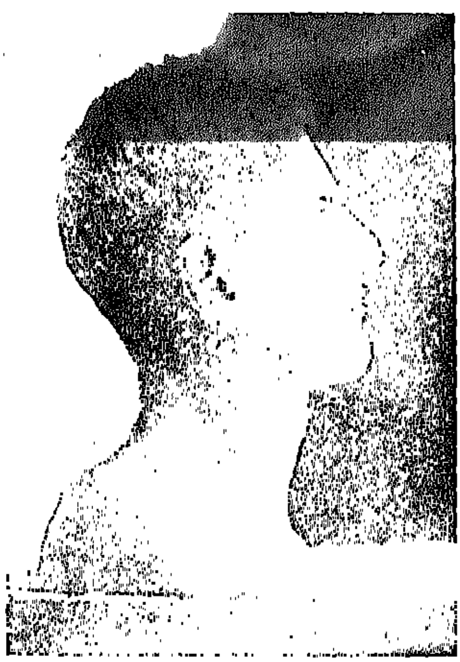
To emphasise the extreme smallness of the Albanian skull, whether north or south, we may take the silhouette type contour of the Englishman from the pocket of *Biometrika*, Vol. XX^{B}, and superpose it on that of the Northern Albanian, so that the nose contours, which fit well, are in agreement, the Englishman's hyperrhinion being slightly above that of the Albanian. The Englishman has a slightly longer upper lip, but his chin and forehead retreat, his crinion being slightly below and inward of the Albanians. Beyond the crinion the English type is bigger than the Albanian, rising about 4 mm. above it at the apex, and it is 5 to 6 mm. horizontally outside it between obelion and hystation. This excess is maintained, if decreasingly, right down to the lophion. If as previously we make the auricular points and the hyperrhinion-auricular point lines to coincide, the northern Albanian head lies almost inside the English, except for a slight projection of the former in the region of the crinion, and a somewhat more significant projection between plakion and obelion. The southern Albanian type lies wholly inside the English male type whether we make the auricular points and the auriculo-hyperrhinion lines, or the auricular points and Frankfurt Horizontals to coincide. Further evidence if required of the smallness of the Albanian, especially the southern, heads may be found by superposing the type silhouette contours of the English female head or even the West African Negro head.

The comparative value of the silhouettes of racial types can only be settled when many others have been constructed. It will be interesting to compare those of the Albanians with silhouettes of other Balkan peoples. Some of these are in preparation, and they will, we have reason to hope, be free of the omission which detracts so much from the value of the present pair.

* This term "hair line" is used purposely because the measured *auricular* height of the Southern type is actually 0·7 mm. in excess of that of the Northern type. (See Table I.) The vertical height from A on the Northerners' silhouette is 8·8 mm. greater than that of the corresponding height on the Southerners'. An examination of Plates III—V suggests that the Northerners' hair may well account for this difference; the average thickness of hair at the back of the head of 50 young Englishmen was of the order of 17 mm.

(i) Moslem.

(ii) Moslem.



(iii) Moslem.

Albanians of the North (1st series).

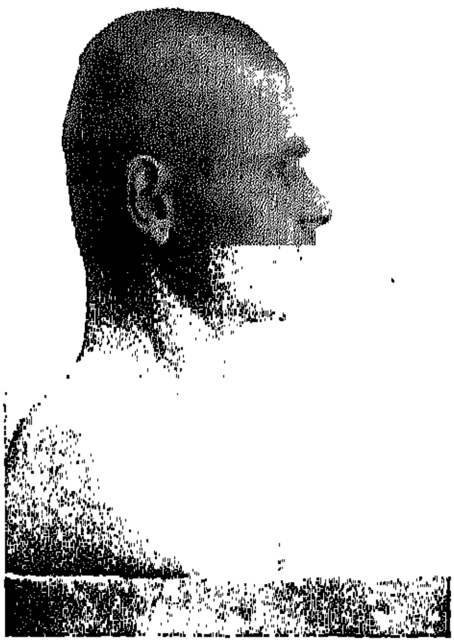
(i) Moslem.

(ii) Moslem.

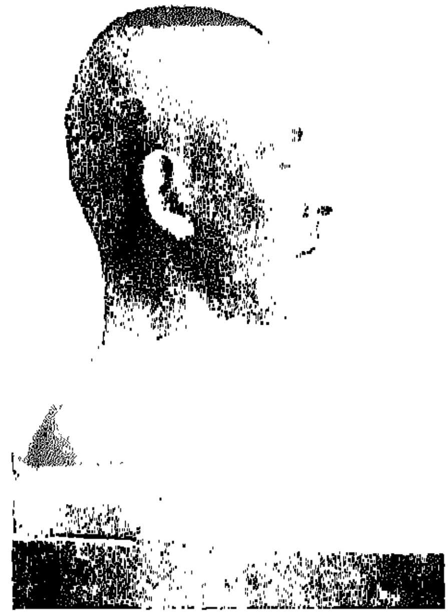
(iii) Moslem.

(iv) Catholic.

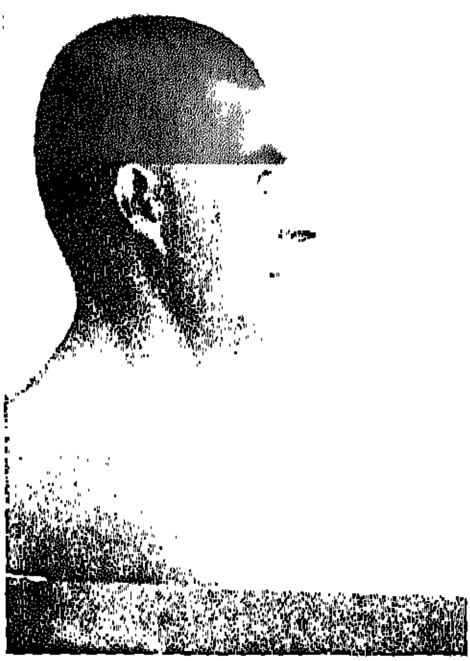
Albanians of the North (2nd series).



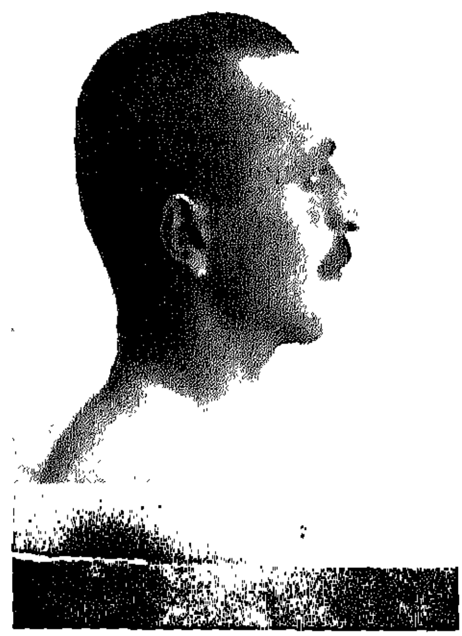
(i) Orthodox Greek.



(ii) Orthodox Greek.



(iii) Moslem.



(iv) Moslem.

Albanians of the South

Two points, which, if noticed before, are at any rate re-emphasised by Miss Tildesley's work, are the following:

(i) There are at least two differentiated groups (or we might say races) in Albania, those of the extreme North and of the extreme South.

(ii) Both races have from the European standpoint small, and in the case of the Southern Group extremely small heads. We might add that it is possible for two differentiated groups to have faces closely alike, or we must accept the fact that a strong facial resemblance by no means connotes racial identity. Thus physiognomic characters do not necessarily provide the best method of discriminating races.

DESCRIPTION OF PLATES

I. Type Silhouette of the Northern Albanian Group.

II. Type Silhouette of the Southern Albanian Group.

III. Examples of the Photographs of the Albanians of the North (1st Series).

IV. Examples of the Photographs of the Albanians of the North (2nd Series).

V. Examples of the Photographs of the Albanians of the South.

A COMPARISON OF THE SEMI-INVARIANTS OF THE DISTRIBUTIONS OF MOMENT AND SEMI-INVARIANT ESTIMATES IN SAMPLES FROM AN INFINITE POPULATION.

By JOHN WISHART, M.A., D.Sc., Clare College, Cambridge.

THE appearance of yet another paper on the sampling problem* directs attention to the success which has attended of recent years the efforts of workers in this field. The general problem considered by one group of workers is the following. Let there be given a population, supposed infinite in extent, but subject to this having any law of distribution with finite moments. It may be a population of one or many variables. The population may be regarded as completely specified by a knowledge of all its characteristic parameters, which may be moment coefficients or semi-invariants, or expressible in terms of these. For a sample of size n drawn at random from this population we may calculate in some manner certain functions which are to be regarded as estimates of the population moment coefficients, or semi-invariants. The simultaneous distribution in repeated samples of the various estimates will depend upon that of the parent population, and the problem I wish to take up deals with the determination of the moment coefficients, or semi-invariants of this simultaneous distribution. Prior to 1928 certain individual results only had been worked out; in that year two independent papers of great importance appeared. R. A. Fisher† showed that if we define as estimates of the population semi-invariants (κ_r) certain functions (k_r) of the sample observations by means of the simple property that the mean value of k_r for an infinite number of samples is to be κ_r‡, then the semi-invariants of the simultaneous distribution of the k's are peculiarly simple in form, compared to analogous expressions derived in other ways. These semi-invariants can be derived algebraically, or more simply by following out certain straightforward combinatorial rules. Fisher's paper marked a great advance in showing the possibility of beginning a systematic tabulation of the required formulae, a thing that had not before been possible. In his paper all formulae up to the tenth degree were given, together with a number of special interest of the twelfth degree. In addition the paper showed how the methods could readily be applied to multivariate populations, and a number of the more general formulae were given.

The other paper was by Craig§, who dealt with the simultaneous distribution

* N. St Georgescu: *Biometrika*, XXIV. 1932, pp. 65—107.

† R. A. Fisher: *Proc. Lond. Math. Soc.* (2), 30, 1929, pp. 199—238.

‡ [The reader must bear in mind that the mean value of k_r is not the value more likely than any other to occur, i.e. it is not the modal value. ED.]

§ C. C. Craig: *Metron*, VII. 1928—9, pp. 3—74.

of the sample moment coefficients (m_r), as ordinarily defined, but chose to express this distribution by means of its semi-invariants. His results do not, therefore, have the same peculiar simplicity of expression that Fisher's have. Craig was able, by algebraic methods, to deduce quite a number of formulae involving moments not higher than the fourth. Now in the paper already mentioned St Georgescu derives precisely the same functions as Craig, i.e. the semi-invariants of the simultaneous distribution of sample moments. The interest of his paper is in the presentation of a different method, for he describes a combinatorial procedure rather like that of Fisher, although it is not as simple in its rules, just as the final results of Craig and St Georgescu are not so simple and compact as those of Fisher. Quite obviously, then, those formulae which are given both by Craig and St Georgescu are identical, although the identity is not immediately evident, since St Georgescu has expressed his results in terms of N, one less than the size of the sample, and has used a different notation. He has, however, a number of formulae which do not appear in Craig's paper, for, confining himself to moments not higher than the fourth, St Georgescu gives all, or nearly all*, the formulae up to weight 11, together with certain only of the formulae of weight 12, and two high order results for normal populations only. Of these, the formulae of weight 11 are new in the sense that there is nothing to correspond with them in Fisher or Craig, although in the case of the former methods have been devised† for deriving in a fairly simple way new patterns from those of lower order, and thus it would not be difficult to add to the results already published.

Now $m_r = \sum_{t=1}^{n} (x_t - \bar{x})^r/n$, and the problem, as first taken up by "Student," Tschouprow and Church, dealt with the distribution in random samples of the estimates m_r, expressing this distribution by means of its moment coefficients, which were worked out in terms of the moment coefficients of the parent population. As results in this form are still required by some workers, though not by all, it becomes important to see how the Fisher results may as required be transformed. There are three stages in the process: the first consists in finding the semi-invariants of the distribution of the m_r estimates, expressed in terms of the population semi-invariants, from those of the k_r estimates (i.e. deducing the Craig-St Georgescu results from those of Fisher‡); the second consists in turning these semi-invariants of the required distribution into moments; then, since the results are still expressed in terms of the population semi-invariants, the last stage is to turn these latter into moments. The last two stages are a matter of routine algebraic transformation, using the known relations between moments and semi-invariants, but the methods of the first stage are less obvious, and it is one of the purposes of the present paper to describe this transformation, which of course is reciprocal. Later a new result is worked out, and it is also shown how many of the terms in certain Fisher formulae of high order may be deduced from corresponding terms in formulae of lower degree.

* He has not given $S(3\ 2^4)$.

† R. A. Fisher and J. Wishart: *Proc. Lond. Math. Soc.* (2), 33, 1931, pp. 195—208.

‡ [Is not this to admit that the Craig-St Georgescu results relieve the worker, who is dealing with moment coefficients, from at least one stage of his labours? ED.]

Formulae which involve sample estimates no higher than the third degree can be readily transformed from the one notation into the other. Since

$$k_2 = nm_2/(n-1) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1),$$

$$k_3 = n^2 m_3/\{(n-1)(n-2)\} \dots\dots\dots\dots\dots\dots\dots\dots\dots (2),$$

the only change evidently consists in introducing a constant multiplier. Thus let us begin with Fisher's formula for $\kappa(2^2 3)$, the (2 1) product semi-invariant, or moment about the mean (since up to the third degree semi-invariant and moment are identical), of the simultaneous distribution of k_2 and k_3 in repeated samples. The notation is easy to follow (and has been adopted by St Georgescu), for the figures in large type relate to the estimates involved, while the exponents give the nature of the semi-invariant evaluated. The formula is Fisher's no. (8),

$$\kappa(2^2 3) = \kappa_7/n^2 + 16\kappa_5\kappa_2/\{n(n-1)\} + 12(2n-3)\kappa_4\kappa_3/\{n(n-1)^2\} + 48\kappa_3\kappa_2^2/(n-1)^2.$$

In the notation of St Georgescu, we now obtain $S(2^2 3)$, i.e. the (2 1) semi-invariant of the simultaneous distribution of m_2 and m_3, by multiplying the above result throughout (see (1) and (2)) by

$$\left(\frac{n-1}{n}\right)^2 \cdot \frac{(n-1)(n-2)}{n^2}.$$

We then have

$$S(2^2 3) = (n-1)^3(n-2)\kappa_7/n^6 + 16(n-1)^2(n-2)\kappa_5\kappa_2/n^5$$
$$+ 12(n-1)(n-2)(2n-3)\kappa_4\kappa_3/n^5 + 48(n-1)(n-2)\kappa_3\kappa_2^2/n^4.$$

This agrees with Craig's result (*loc. cit.* p. 55, formula for $S_{21}(\nu_2, \nu_3)$), and also with St Georgescu's if we make the substitution $n = N+1$.

It is to be hoped that we shall in time settle down to a uniform and satisfactory notation for the semi-invariants. What Fisher writes κ_r is written λ_r by Craig and s_r by St Georgescu. In both these cases the influence of Thiele is apparent, but to both there are objections. If we are to extend the practice of having corresponding Latin and Greek letters for sample estimate and population parameter respectively, a practice that has much to commend it, then that rules out s straight away as not being a Greek letter. In any case s is already appropriated for standard deviation. There is not much to choose between λ_r and κ_r, but on the ground that the Latin l for sample estimate is less satisfactory than k, liable as it is to be confused with the numeral *one*, we would advocate the use of k and κ throughout.

When a fourth or higher order estimate comes in, the transformation is less simple. Fisher, in his original paper (*loc. cit.* para. 10), gave a general demonstration of the method to be followed, but it will not be out of place here to give the details, illustrating by means of an example or two. Let us take the case of $\kappa(4^2)$, the variance of k_4, compared with that of m_4, denoted by $S(4^2)$. The latter formula occupies three and a half lines of print in St Georgescu's paper, and contains a number of quite complex terms. The former reads

$$\kappa(4^2) = \kappa_8/n + 16\kappa_6\kappa_2/(n-1) + 48\kappa_5\kappa_3/(n-1) + 34\kappa_4^2/(n-1) + 72n\kappa_4\kappa_2^2/\{(n-1)(n-2)\}$$
$$+ 144n\kappa_3^2\kappa_2/\{(n-1)(n-2)\} + 24n(n+1)\kappa_2^4/\{(n-1)(n-2)(n-3)\}.$$

The number of terms is the same, but the simplicity of each is very marked. In fact the occurrence of common factors in the second and third, and fifth and sixth, terms would enable the formula to be abbreviated by taking these terms together, while the writing of N for $n-1$ would also somewhat shorten the formula. It is of more immediate interest, however, to see how the Craig-St Georgescu result can be derived from the above. By definition we have

$$k_4 = \frac{n^2(n+1)\,m_4}{(n-1)(n-2)(n-3)} - \frac{3n^2 m_2{}^2}{(n-2)(n-3)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus we may write $m_4 = pk_4 + qk_2{}^2$, utilising the value of k_2 already given in (1), where p and q stand for the two factors involving the size of the sample

$$p = (n-1)(n-2)(n-3)/\{n^2(n+1)\}, \quad q = 3(n-1)^3/\{n^2(n+1)\}.$$

We are to consider now the simultaneous distribution of k_4 and k_2, and to find the distribution of m_4, a certain known function of these quantities. The moment generator of the distribution is given by the operator

$$\exp\{\tau(p\partial/\partial t_1 + q\partial^2/\partial t_2{}^2)\},$$

while the operand is

$$1 + \mu(4)\,t_1 + \mu(2)\,t_2 + \mu(4^2)\frac{t_1{}^2}{2!} + \mu(4\,2)\frac{t_1 t_2}{1!\,1!} + \mu(2^2)\frac{t_2{}^2}{2!} + \ldots.$$

The operator is expanded and the differentiations carried out, after which t_1 and t_2 are put equal to zero. We have the following series in τ:

$$1 + \{p\mu(4) + q\mu(2^2)\}\,\tau + \{p^2\mu(4^2) + 2pq\mu(4\,2^2) + q^2\mu(2^4)\}\,\tau^2/2! + \ldots\ldots(4),$$

the binomial character of the coefficient of $\tau^r/r!$ being evident. In general the notation $\mu(a^b)$ denotes the moment coefficient corresponding to the semi-invariant $\kappa(a^b)$. The series in τ is the moment generator of the distribution of m_4. To obtain the semi-invariant generator we expand the logarithm of (4) in powers of $\tau^r/r!$. We get

$$\begin{aligned}\{p\mu(4) + q\mu(2^2)\}\,\tau + [p^2\{\mu(4^2) - \mu^2(4)\} + 2pq\,\{\mu(4\,2^2) - \mu(4)\,\mu(2^2)\}\\ + q^2\{\mu(2^4) - \mu^2(2^2)\}]\,\tau^2/2! + \ldots\ \ldots(5).\end{aligned}$$

The term in $\tau^2/2!$ in this is the κ_2, or variance, of the distribution of m_4, i.e. the required result. But it is expressed in terms of the moment coefficients of the simultaneous distribution of k_4 and k_2, whereas it is the semi-invariants which are known from Fisher's work. The relations connecting moments and semi-invariants are, however, well known. Those we shall require are

$$\begin{aligned}\mu_1 &= \kappa_1,\\ \mu_2 &= \kappa_2 + \kappa_1{}^2,\\ \mu_{12} &= \kappa_{12} + 2\kappa_{11}\kappa_{01} + \kappa_{02}\kappa_{10} + \kappa_{10}\kappa_{01}{}^2,\\ \mu_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2{}^2 + 6\kappa_2\kappa_1{}^2 + \kappa_1{}^4.\end{aligned}$$

The term in $\tau^2/2!$ in (5) may then be written

$$\begin{aligned}S(4^2) = p^2\kappa(4^2) + 2pq\,\{\kappa(4\,2^2) + 2\kappa(4\,2)\,\kappa_2\}\\ + q^2\{\kappa(2^4) + 4\kappa(2^3)\,\kappa_2 + 2\kappa^2(2^2) + 4\kappa(2^2)\,\kappa_2{}^2\} \quad \ldots\ldots(6).\end{aligned}$$

This is the relation sought, into which the known values of p and q may be inserted. It is seen to involve, in addition to $\kappa(4^2)$, a number of other results, all tabulated in Fisher's paper. Reciprocally, of course, it is also possible to express $\kappa(4^2)$ in terms of a series of the Craig-St Georgescu results.

The terms of $S(4^2)$ may now be worked out one by one. Take for example the term in κ_2^4, the only term that survives when the parent population is normal. In this case the entire middle term of (6) (that in pq) vanishes, since $\kappa(a2^b)$ is zero for normal populations when $a > 2$. We make use of the known results

$$\kappa(4^2) = 24n(n+1)\kappa_2^4/\{(n-1)(n-2)(n-3)\} \quad \ldots\ldots\ldots\ldots\ldots(7),$$

$$\kappa(2^r) = 2^{r-1}(r-1)!\,\kappa_2^r/(n-1)^{r-1}.$$

Substitution in the above formula leads without difficulty to

$$S(4^2) = 24(n-1)(4n^2-9n+6)\kappa_2^4/n^4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

as given by Craig, St Georgescu, and others.

Actually, in this case, it would probably be nearly as simple to obtain the result by direct algebraic methods, but the example is only an illustration of what is possible. The series (5) has in fact been extended by the writer to the terms in $\tau^4/4!$ and used to work out the normal term of $S(4^4)$ in terms of $\kappa(4^4)$ and other κ results, thus checking the result given by Craig and later by St Georgescu. With such a high order result it is obvious that direct algebraic methods would be exceedingly laborious.

For this, the normal, term of $\kappa(4^2)$ or $S(4^2)$, there is not a great deal to choose between the formulae (7) and (8) on the ground of simplicity. It is perhaps instructive to choose another term in the formula to show a more striking difference. Suppose we are required to find the term in κ_4^2. For $\kappa(4^2)$ this is simply $34/(n-1)$, being derived from the two patterns

$$\begin{array}{cc|c} 3 & 1 & 4 \\ 1 & 3 & 4 \\ \hline 4 & 4 & \end{array} \quad \text{and} \quad \begin{array}{cc|c} 2 & 2 & 4 \\ 2 & 2 & 4 \\ \hline 4 & 4 & \end{array}$$

which can be set up in 16 and 18 ways respectively, and with coefficient $1/(n-1)$ in each case. To find from (6) the term in κ_4^2 of $S(4^2)$ we require in addition the following results:

$$\kappa(4\,2^2) = \ldots\, 4(7n-10)\kappa_4^2/\{n(n-1)^2\} \ldots,$$

$$\kappa(2^4) = \ldots\, 8(4n^2-9n+6)\kappa_4^2/\{n^2(n-1)^3\}; \ldots,$$

$$\kappa(2^2) = \kappa_4/n \ldots.$$

These are taken from formulae nos. (12), (14) and (1) of Fisher's paper, already cited.

After substituting in (6) and reducing as far as practicable, we have, for the required term in κ_4^2,

$$2(n-1)(17n^4-111n^3+309n^2-405n+207)/n^6.$$

So much, then, for the direct semi-invariants of a single moment or semi-invariant estimate. When we come to consider the simultaneous distribution of two or more sample estimates, the procedure is a little modified. To serve as an illustration of method let us find $\kappa(2\,4)$, and from it $S(2\,4)$. The first of these is the first order product semi-invariant (or moment) of the joint distribution of k_2 and k_4 (in fact its κ_{11}). The second is the corresponding parameter of the joint distribution of m_4 and m_2. For $\kappa(2\,4)$ the required patterns are

$$\begin{array}{cc|c} 2 & 4 & 6 \\ 2 & 4 & \end{array} \qquad \begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 1 & 2 \\ \hline 2 & 4 & \end{array} \qquad \begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 3 \\ \hline 2 & 4 & \end{array}$$

$$\kappa(2\,4) = \kappa_6/n + 8\kappa_4\kappa_2/(n-1) + 6\kappa_3^2/(n-1).$$

To transform this into $S(2\,4)$ we shall also require $\kappa(2^2)$ and $\kappa(2^3)$,

$$\begin{array}{cc|c} 2 & 2 & 4 \\ 2 & 2 & \end{array} \qquad \begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 2 \\ \hline 2 & 2 & \end{array}$$

$$\kappa(2^2) = \kappa_4/n + 2\kappa_2^2/(n-1),$$

$$\begin{array}{ccc|c} 2 & 2 & 2 & 6 \\ 2 & 2 & 2 & \end{array} \qquad \begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ \cdot & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & \end{array} \qquad \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \end{array} \qquad \begin{array}{ccc|c} 1 & 1 & \cdot & 2 \\ 1 & \cdot & 1 & 2 \\ \cdot & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & \end{array}$$

$$\kappa(2^3) = \kappa_6/n^2 + 12\kappa_4\kappa_2/\{n(n-1)\} + 4(n-2)\kappa_3^2/\{n(n-1)^2\} + 8\kappa_2^3/(n-1)^2.$$

We now consider the simultaneous transformations

$$m_4 = pk_4 + qk_2^2,$$
$$m_2 = rk_2,$$

where
$$p = (n-1)(n-2)(n-3)/\{n^3(n+1)\},$$
$$q = 3(n-1)^3/\{n^3(n+1)\},$$
$$r = (n-1)/n.$$

The μ-generator of the simultaneous distribution is given by

$$\exp\left\{\tau_1\left(p\frac{\partial}{\partial t_1} + q\frac{\partial^2}{\partial t_2^2}\right) + \tau_2 r\frac{\partial}{\partial t_2}\right\}$$

and the operand is, as before,

$$1 + \mu(4)\,t_1 + \mu(2)\,t_2 + \mu(4^2)\frac{t_1^2}{2!} + \mu(4\,2)\frac{t_1}{1!}\frac{t_2}{1!} + \mu(2^2)\frac{t_2^2}{2!} + \dots.$$

Performing the differentiations and then putting $t_1 = t_2 = 0$, we have, for the μ-generator,

$$1 + \tau_1\{p\mu(4) + q\mu(2^2)\} + \tau_2 r\mu(2) + \tau_1\tau_2\{pr\mu(4\,2) + qr\mu(2^3)\} + \dots.$$

The κ-generator is the logarithm of this, and to find the $(1\,1)$ product moment of m_4 and m_2 we require to expand the logarithm and find the term in $\tau_1\tau_2$. Quite obviously this is

$$pr\mu(4\,2) + qr\mu(2^3) - \{p\mu(4) + q\mu(2^2)\}\,r\mu(2)$$
$$= pr\,\{\mu(4\,2) - \mu(4)\,\mu(2)\} + qr\,\{\mu(2^3) - \mu(2^2)\,\mu(2)\}.$$

On converting μ's to κ's this becomes

$$S(2\,4) = pr\kappa(4\,2) + qr\,\{\kappa(2^3) + 2\kappa_2\kappa(2^2)\}.$$

On substitution of the values of p, q and r and of the expressions given above for the semi-invariants involved, we get

$$S(2\,4) = (n-1)^2(n^2-3n+3)\,\kappa_6/n^5 + 2\,(n-1)(7n^2-18n+15)\,\kappa_4\kappa_2/n^4 \\ + 6\,(n-1)(n-2)^2\,\kappa_3^2/n^4 + 12\,(n-1)^2\,\kappa_2^3/n^3,$$

as already given by Craig and St Georgescu.

Enough has been done to indicate the procedure to be followed in the case of more complex formulae. In all cases the simplest final results are obtained when we deal with the semi-invariants of the k-estimates, which fact renders them particularly suitable when a storehouse of information is required for reference purposes. In addition, anyone who has mastered the technique may work out *ab initio* many of the Fisher results in a few minutes, a procedure that will often be quicker than looking up the required formula. In such cases the transformation which has been described will be useful when the results are required in other forms.

Two results of high order, namely $S(4^4)$ and $S(3^6)$, have been given by St Georgescu for the special case of the parent population being normal. The first agrees with the corresponding result given earlier by Craig, which, as we have said, checks with the Fisher result for $\kappa(4^4)$ *. But it ought to be pointed out that the St Georgescu result for $S(3^6)$ is in error. $\kappa(3^6)$ was first worked out in full for the normal case some two years ago by Dr Fisher and the present writer*. The result is

$$\kappa(3^6) = 466560n^3(22n^2-111n+142)\,\kappa_2^9/\{(n-1)^5(n-2)^5\} \quad \text{......(9)}.$$

Now since $m_3 = (n-1)(n-2)\,k_3/n^2$, we may obtain $S(3^6)$ by multiplying the above result by $\{(n-1)(n-2)/n^2\}^6$, and we have

$$S(3^6) = 466560\,(n-1)(n-2)(22n^2-111n+142)\,\kappa_2^9/n^9 \quad \text{......(10)}.$$

If we write n for $N+1$ in the St Georgescu result, we get

$$3265920\,(n-1)(n-2)(4n^2-21n+28)\,\kappa_2^9/n^9.$$

There is no doubt remaining as to the correctness of (9), and therefore (10). It was determined on more than one occasion, and has recently been reworked, by the combinatorial method, and carefully checked. A further check is provided by the relationship between $\mu(3^6)$ and $\mu(3^6\,2^{-9})$, which latter is the sixth moment of the distribution of the ratio $g_1 = k_3/k_2^{3/2}$, the first measure of departure from normality, differing by only a constant from $\sqrt{\beta_1}$. Fisher's recurrence relation for g_1† enables one, after some fairly heavy algebra, to obtain the second, fourth and sixth moments of g_1 in succession, and so to check $\kappa(3^6)$. This also has been done more than once, with the same result each time‡.

* See J. Wishart: *Biometrika*, XXII. 1930, p. 237.

† R. A. Fisher: *Proc. Roy. Soc.* A, 130, 1930, pp. 16—28.

‡ J. Pepper, in *Biometrika*, XXIV. 1932, p. 60, has calculated $\mu(3^6\,2^{-12})$, the result being checked by Dr Fisher by his combinatorial method.

To fill a gap in St Georgescu's table of results, I have worked out $\kappa(3\,2^4)$, from which $S(3\,2^4)$ may be directly derived by multiplying by $(n-1)^5(n-2)/n^6$. The former was obtained by the combinatorial method, and carefully checked by the rather lengthy process of direct algebra. The result is

$$\begin{aligned}\kappa(3\,2^4) = {} & \kappa_{11}/n^4 + 48\kappa_9\kappa_2/\{n^3(n-1)\} + 8(16n-29)\kappa_8\kappa_3/\{n^3(n-1)^2\} \\ & + 8(38n^2-99n+75)\kappa_7\kappa_4/\{n^3(n-1)^3\} \\ & + 16(26n^3-98n^2+127n-58)\kappa_6\kappa_5/\{n^3(n-1)^4\} \\ & + 720\kappa_7\kappa_2^2/\{n^2(n-1)^2\} + 96(31n-53)\kappa_6\kappa_3\kappa_2/\{n^2(n-1)^3\} \\ & + 576(9n^2-23n+16)\kappa_5\kappa_4\kappa_2/\{n^2(n-1)^4\} \\ & + 288(9n^2-32n+26)\kappa_5\kappa_3^2/\{n^2(n-1)^4\} \\ & + 96(41n^2-129n+111)\kappa_4^2\kappa_3/\{n^2(n-1)^4\} \\ & + 3840\kappa_5\kappa_2^3/\{n(n-1)^3\} + 8640(2n-3)\kappa_4\kappa_3\kappa_2^2/\{n(n-1)^4\} \\ & + 1152(5n-12)\kappa_3^3\kappa_2/\{n(n-1)^4\} + 5760\kappa_3\kappa_2^4/\{(n-1)^4\} \quad\ldots\ldots(11).\end{aligned}$$

One point of some interest that arises out of the derivation of such high order formulae is that under certain circumstances part of the result at any rate can be derived by the application of simple formulae from corresponding terms of results of lower degree that have been already evaluated. This point has not been studied at all systematically, and in fact a wide field of study awaits the research worker who cares to take the problem up. The formulae arise from consideration of the way in which patterns of a certain size may be expanded by the addition of new columns and rows, and of the number of ways in which such a change can be effected. One or two results of this character have already been given, but only for the case of the parent population being normal. Thus we have the result given in a previous paper* that

$$\kappa(p^q\,2^r) = \frac{2^r(r+\frac{1}{2}pq-1)!}{(\frac{1}{2}pq-1)!\,(n-1)^r}\kappa_2^r\kappa(p^q) \quad\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

which gives the normal term of $\kappa(p^q\,2^r)$, which will only exist for pq even, in terms of that of $\kappa(p^q)$. This relation should in fact have been extended to $\kappa(3^a4^b5^c\ldots 2^r)$, where $3a+4b+5c+\ldots=pq$, as has been indicated by St Georgescu† with the parallel formula for his form. A particular case is the well-known one, putting $p=2$, $q=1$,

$$\kappa(2^{r+1}) = 2^r r!\,\kappa_2^{r+1}/(n-1)^r \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

giving immediately the normal term of the $(r+1)$th semi-invariant of the distribution of k_2.

An extension of this work to the case of non-normal populations can be made under certain circumstances. It is obvious that if any term of a formula which involves a power of κ_2 can be derived from the corresponding term of a formula of lower degree, then this will enormously reduce the number of terms to be evaluated by combinatorial methods. In particular it will seldom be needful to evaluate patterns of large size in which most of the cells are unoccupied, a type to which it is usually somewhat difficult to assign numerical coefficients. For example 8 out of the 14 terms in $\kappa(3\,2^4)$ contain a κ_2, leaving only 6, 4 of which

* J. Wishart: *Biometrika*, XXII. 1930, p. 234. † *Loc. cit.* p. 97.

are very easy, to be directly determined. Again $\kappa(2^7)$ contains 34 terms, 21 of which contain at least one κ_2. A formula which embraces all such terms from formulae of the type $\kappa(p\,2^r)$ is the following:

The term in $\kappa_3{}^a\kappa_4{}^b\kappa_5{}^c \dots \kappa_2{}^{r-s+1}$ in $\kappa(p\,2^r)$ is obtained from the term in $\kappa_3{}^a\kappa_4{}^b\kappa_5{}^c \dots$ in $\kappa(p\,2^{s-1})$ by multiplying the coefficient of the latter by

$$\frac{2^{r-s+1}\,r!\,(r+p-1)!}{(s-1)!\,(s+p-2)!\,(r-s+1)!\,(n-1)^{r-s+1}} \quad \dots\dots\dots\dots\dots (14).$$

We have, of course, that $3a+4b+5c+\dots=2(s-1)+p$, and p may take any integral value from 2 upwards. The formula is subject to the one exception that owing to symmetry it has to be slightly modified to give the normal term in $\kappa(2\,2^r)$, but for this we have formula (13) above. The other terms in $\kappa(2^{r+1})$ are given correctly by (14).

As an illustration let it be desired to find the term in $\kappa_5\kappa_3{}^2\kappa_2{}^2$ in $\kappa(3\,2^6)$. Here $a=2$, $b=0$, $c=1$, $p=3$ and $r=6$. It follows that $s=5$, and the required term will be obtained from that in $\kappa_5\kappa_3{}^2$ of $\kappa(3\,2^4)$ by multiplying the coefficient of the latter by

$$\frac{2^2\,6!\,8!}{4!\,6!\,2!\,(n-1)^2},$$

or by $3360/(n-1)^2$. As $\kappa(3\,2^4)$ is given by our formula (11) we see that the required term is

$$967680\,(9n^2-32n+26)\,\kappa_5\kappa_3{}^2\kappa_2{}^2/\{n^2(n-1)^6\}.$$

A further application of the rule gives us, for the last three terms in $\kappa(2^r)$,

$$\kappa(2^r)=\dots+2^{r-2}\frac{r(r-1)(r-1)!}{2!\,n(n-1)^{r-2}}\kappa_4\kappa_2{}^{r-2}$$
$$+2^{r-3}\frac{r(r-1)(r-2)(r-1)!\,(n-2)}{3!\,n(n-1)^{r-2}}\kappa_3{}^2\kappa_2{}^{r-3}+2^{r-1}\frac{(r-1)!}{(n-1)^{r-1}}\kappa_2{}^r.$$

Suppose, then, we have a population which is removed to some extent, but not greatly, from normality, so that κ_2, κ_3 and κ_4 only exist, the higher semi-invariants being zero or negligible. If also κ_3 and κ_4 are small, so that their squares and higher powers can be neglected in comparison with their first powers (or second in the case of κ_3), then under these conditions we may write

$$\kappa(2^r) \doteqdot \frac{2^{r-1}(r-1)!}{(n-1)^{r-1}}\kappa_2{}^r\left[1+\frac{1}{2}\cdot\frac{n-1}{n}\cdot\frac{r(r-1)}{2!}\gamma_2+\frac{1}{2}\cdot\frac{n-2}{n}\cdot\frac{r(r-1)(r-2)}{3!}\gamma_1{}^2\right]$$

approximately, where $\gamma_1=\kappa_3\kappa_2{}^{-\frac{3}{2}}$ and $\gamma_2=\kappa_4\kappa_2{}^{-2}$. γ_1 and γ_2 are the $\sqrt{\beta_1}$ and β_2-3 of a more familiar notation. Such a formula may serve to indicate in what way the distribution of k_2, the estimate of variance, changes for a slight departure from normality of the parent population. The case considered may, however, be of limited interest, but it does not seem as if much progress will be made in determining the distribution of the estimated variance in samples from non-normal populations except under certain simplifying assumptions as to the nature of the moment or semi-invariant law. With our present knowledge it would be possible to write down quite a large number of the terms in $\kappa(2^r)$, but it is not clear that any useful purpose would be served by doing so.

AN EMPIRICAL AGE SCALE.

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General.

In many tropical countries the opening up to modern methods is very recent; annoying gaps occur in essential information in unexpected places. When carrying out work on the subject of vital statistics, knowledge of the different age groups of the population is fundamental and yet more often than not it is impossible to obtain even approximate figures.

A study was recently being made among one of the large race units in Southern Nigeria, the Yoruba nation, on the subject of malarial endemicity; in this work the enlargement of the spleen is of importance when read in the light of the child's age. As with many another native people, the idea of birth registration or that of counting numbers of persons is contrary to the general ideas of good luck, and consequently it is very rare to find a person who knows his or her age. Until an age scale could be devised, the work referred to was brought to a standstill.

Anthropometrical research has been carried out on a large scale in many parts of the world with different peoples. Among these the American Negro has been extensively weighed and measured, and as birth registration is fairly widespread in the United States the ages are also known. In times past, the Yoruba nation has contributed in a considerable degree to the present coloured population of North America.

The Method Adopted.

As many Yoruba children as could be obtained in the schools of Abeokuta, a representative town in the middle of the national area, were measured for weight and height, and the one was plotted against the other. A second graph was then drawn from the weights and heights of the American coloured children at the age units. The curves were not unlike in general appearance.

As a check, a similar set of curves was drawn for English, Australian and American white children. It will be seen from Fig. 1 that there is a considerable amount of variation in the five curves which cannot be accounted for by the different methods of measurement such as the amount of clothes worn. This variation is most striking in the younger ages of English children. Also it is obvious that the curves for the two African peoples are as significantly different the one from the other as are either one of them from any of the white curves.

If the American coloured curve had fitted that of the Yorubas, one would have been fairly justified in inserting the age points on the latter as on the former, but

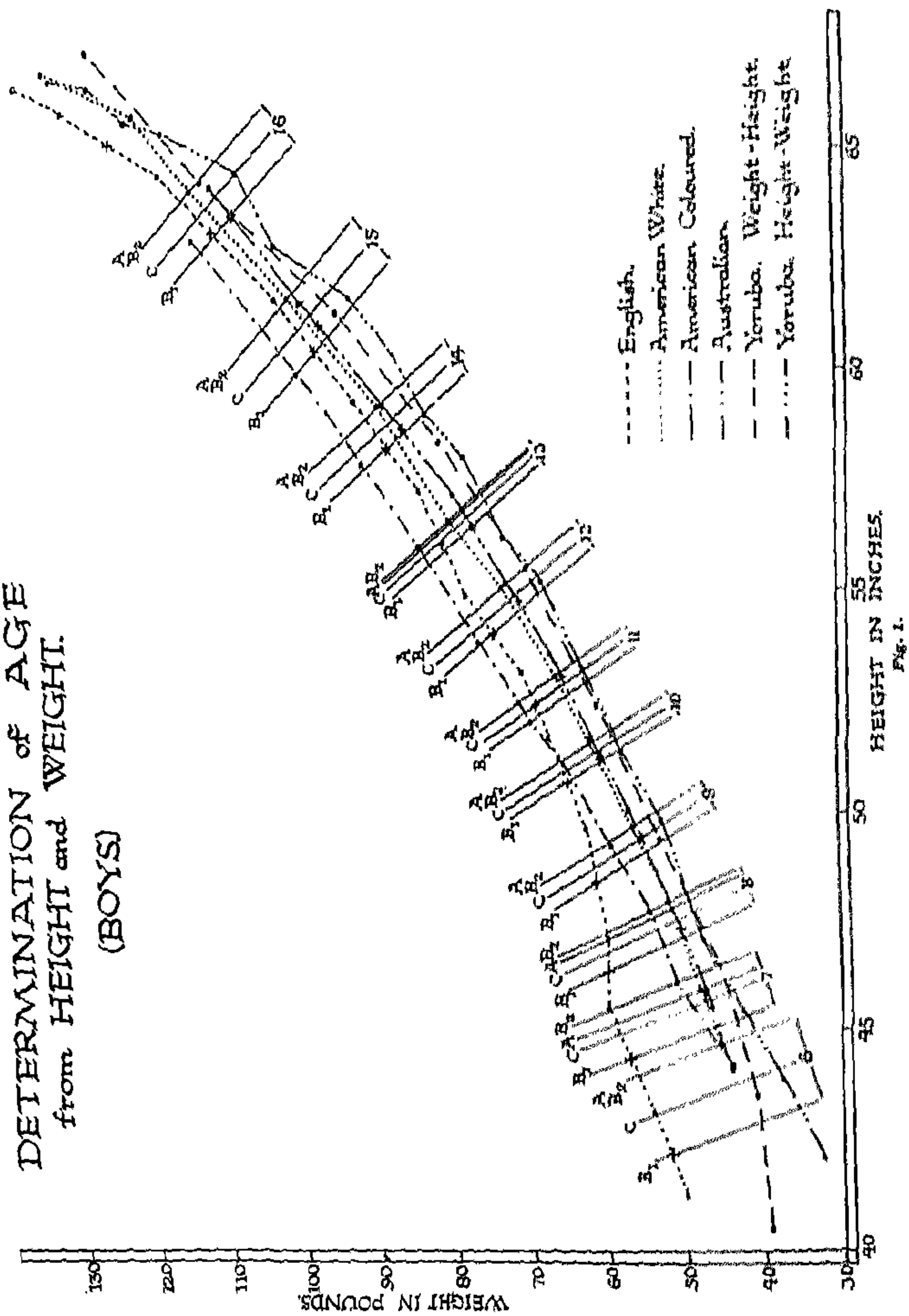

DETERMINATION of AGE
from HEIGHT and WEIGHT.
(BOYS)
English.
American White.
American Coloured.
Australian.
Yoruba. Weight-Height.
Yoruba. Height-Weight.
WEIGHT IN POUNDS.
HEIGHT IN INCHES.
30
40
50
60
70
80
90
100
110
120
130
40
45
50
55
60
65
6
7
8
9
10
11
12
13
14
15
16

Fig. 1.

this procedure is impracticable with the two curves as different in detail as they are. On inspecting the five curves, one thing emerges; age points for all of them appear to be in straight bands lying radially across the curves. The difficulty is where to locate in these bands the average age on the Yoruba curve. There is no information available to act as a criterion to a method for choosing a particular set of points.

As the inquiry had a practical rather than a theoretical end in view—that of assigning an age to a child without too large a possible error—the following method was formulated. It probably has no greater virtue than any other which might be devised, but to those carrying on the work it appeared the simplest and most satisfactory although altogether empirical.

On this understanding the American white curve, being the most regular of the four curves of known age, was taken as the index. At all age points on it, the angle between the two contiguous straight lines was bisected by lines A; parallel lines B_1 and B_2 were then drawn through the corresponding age points farthest apart on the curves of known age. In each group the distance between B_1 and B_2 was bisected and an indicator line C drawn through the point of bisection, parallel with A, B_1 and B_2, to cut the Yoruba curve at a point which became the empirical age point.

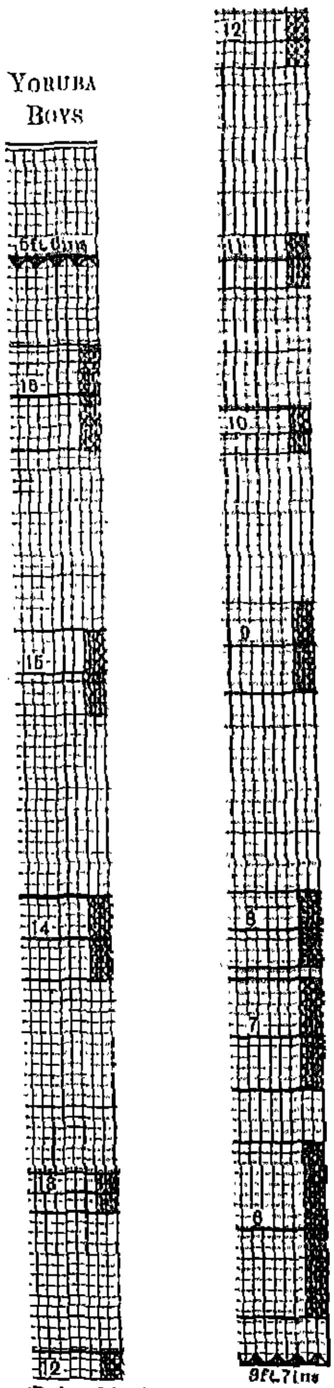
The Yoruba Curve.

This could be drawn from two sets of data, either those obtained from working out the average heights of weight groups, or from those of the average weights of height groups. Both sets were worked out and the two graphs drawn. There is a slight difference between the two due to the larger number of the weight groups as compared with those of height*.

Although this difference becomes more conspicuous at the upper extreme, it is outside the school age (six to sixteen years) and is of no importance. The variation at the young end can be accounted for by considering the group table. It will be seen that there are obviously persons of lower weight than 30+ lbs. in the height groups 39+ ins. and 42+ ins. which have been left out. This may be due to a parent judging whether a child is old enough to go to school from his height. One does not know, this is merely a suggestion. The points on one curve are relatively too heavy for the heights in these two groups.

To differentiate between the two curves they have been named differently. That plotted from the average heights to weight groups has been named the "Height-Weight" curve, whereas that from the average weights to height groups has been called the "Weight-Height" curve. As the probable errors of the mean weights of the height groups is much larger than the probable errors of the mean heights of the weight groups, the Height-Weight curve was chosen as that from which to work. Incidentally it will be noticed that the probable error of the mean weights become so large at the upper extremes as to show the result in one case to be non-significant.

[* The two regression curves should not coincide, and the reason for their non-coincidence lies in the nature of regression curves, and cannot be attributed to the cause suggested by the author. ED.]



(Reduced in the ratio of 92 to 53)

Fig. 2. Scale on the pole running from 3 ft. 7 in. to 5 ft. 6 in. to determine from height of a boy his supposed age.

Discussion of the Sources of Information.

The Australian Curve. The data are taken from Part II of the Reports of the Principal Medical Officer of the Commonwealth Department of Education for 1918 and 1919, from the work of H. Sutton, "Rate and Growth of Australian Children," and of F. A. Mecham, "Physical Condition of Children attending Public Schools in New South Wales." The figures have been extracted in six monthly age groups, and the age groups 6+, 7+, etc., used. An adjustment must therefore be made for the age points. An age group contains all the children of ages from x years to x years plus six months. The mid-point is therefore x years plus three months. The reading on the curve is really at the point x plus $\frac{1}{4}x$. The age point has thus to be set back one-quarter of the distance towards the previous one. It is definitely stated by the authors that all children were weighed without clothes.

The American Coloured Curve. The information is provided by A. MacDonald in his *Experimental Study of Children*, 1899, in which the exact age is taken as the age point. There is no statement as to the clothes worn.

The American White Curve. The same source as that of the American Coloured Curve provided the data used.

The English Curve. Information was obtained from the Report of the Anthropometric Committee of the British Association for the Advancement of Science in their *Proceedings* for 1879 in which Roberts's *Manual of Anthropometry* is quoted. The age units points are age last birthday. By the same reasoning as that used above with the Australian data, it is necessary to set back the age points by half the distance to the immediately preceding ones. There is no statement as to clothing, if any, which was worn, or whether measurements were taken barefoot or not.

The Yoruba Curve. Children were measured barefoot and were dressed in light cotton drill shorts and a jumper of the same material.

The Scale.

This consists of a strip of stiff paper two feet long. See Fig. 2, where it has been needful

to divide the scale for publication into two parts at age 12 years. With one end marked "3 ft. 7 in." it is graduated in ages according to the corresponding heights read off from the Yoruba Height-Weight curve. The shaded areas on either side of the age heights represent the distance B_1 and B_2 on either side of C. Any reading coming inside a shaded area may therefore be read as the exact age in question; readings between the shaded areas are interpolated in quarter years.

The strip is glued to a flat pole and then varnished with the 3 ft. 7 in. point at that distance from one end; this end serves as the foot of the scale.

TABLE I.

Yoruba Boys.

Weight in pounds	Height in inches													Total
	36+	39+	42+	45+	48+	51+	54+	57+	60+	63+	66+	69+	72+	
30+	1	7	10	2	—	—	—	—	—	—	—	—	—	20
35+	1	7	17	13	—	—	—	—	—	—	—	—	—	38
40+	—	4	27	39	5	2	—	—	—	—	—	—	—	77
45+	—	3	10	47	30	1	1	—	—	—	—	—	—	92
50+	—	—	2	22	58	21	2	3	1	—	—	—	—	109
55+	—	—	—	3	42	74	14	—	—	—	—	—	—	133
60+	—	—	—	2	6	53	36	2	—	—	—	—	—	99
65+	—	—	—	—	3	26	47	21	2	—	—	—	—	99
70+	—	—	—	—	—	10	65	36	1	1	—	—	—	113
75+	—	—	—	—	—	1	18	27	10	—	—	—	—	56
80+	—	—	—	—	—	2	6	47	26	1	—	—	—	82
85+	—	—	—	—	—	—	1	23	32	4	1	1	—	62
90+	—	—	—	—	—	—	—	7	22	8	1	—	—	38
95+	—	—	—	—	—	—	1	4	29	21	1	—	—	56
100+	—	—	—	—	—	—	1	1	24	30	1	—	—	57
105+	—	—	—	—	—	—	—	—	11	40	12	1	—	64
110+	—	—	—	—	—	—	2	3	6	19	12	—	—	42
115+	—	—	—	—	—	—	—	—	4	15	14	1	—	34
120+	—	—	—	—	—	—	—	—	3	12	17	—	—	32
125+	—	—	—	—	—	—	—	—	2	11	16	7	—	36
130+	—	—	—	—	—	—	—	1	1	3	9	2	1	17
135+	—	—	—	—	—	—	—	—	—	2	6	3	3	14
140+	—	—	—	—	—	—	—	—	—	1	4	1	—	6
145+	—	—	—	—	—	—	—	—	—	5	7	1	—	13
150+	—	—	—	—	—	—	—	—	—	1	4	—	—	5
155+	—	—	—	—	—	—	—	—	—	1	3	2	—	6
160+	—	—	—	—	—	—	—	—	—	—	2	—	—	2
165+	—	—	—	—	—	—	—	—	—	1	—	1	—	2
170+	—	—	—	—	—	—	—	—	—	—	1	—	—	1
175+	—	—	—	—	—	—	—	—	—	—	2	—	1	3
180+	—	—	—	—	—	—	—	—	—	—	1	1	1	3
Total	2	21	66	128	144	190	194	175	174	176	114	21	6	1411

The Girls.

It was seen when the corresponding graph for the girls was drawn that this method of obtaining an age scale was inapplicable because of the irregular variations which occur about and after the age of puberty.

Conclusions.

A method of reading a boy's age from a height scale has been devised.

Although applicable when children are enumerated in groups or in large numbers, the result when applied to the individual will not be inaccurate to such a degree as to render it of no value.

The scale worked out is for boys of the Yoruba nation. It is not known yet whether it is applicable to members of other African peoples.

NOTE. Any official information contained in this paper is used by the kind permission of the Director of the Medical and Sanitary Service of Nigeria. The thanks of the writer are due to Major P. Granville Edge for his valuable co-operation and criticism during the carrying out of the investigation.

TABLE II.

Yoruba Boys' Measurements.

Weight in pounds	Mean Height in inches	P.E. of Mean Height	Number in Group
30+	42·15	±0·31	20
35+	43·88	±0·24	38
40+	45·48	±0·18	77
45+	47·12	±0·17	92
50+	49·77	±0·19	103
55+	51·41	±0·14	133
60+	53·35	±0·15	99
65+	55·29	±0·17	99
70+	56·32	±0·13	113
75+	58·15	±0·17	66
80+	59·10	±0·10	82
85+	60·63	±0·07	91
90+	61·81	±0·22	38
95+	62·41	±0·06	50
100+	62·97	±0·17	57
105+	64·04	±0·19	64
110+	64·01	±0·33	42
115+	65·56	±0·25	34
120+	65·91	±0·09	32
125+	66·83	±0·28	36
130+	66·07	±0·33	17
		Total	1365

Height in inches	Mean Weight in pounds	P.E. of Mean Weight	Number in Group
39+	38·21	±0·70	21
42+	39·79	±0·41	68
45+	46·06	±0·32	128
48+	52·88	±0·25	144
51+	60·06	±0·24	180
54+	69·50	±0·38	194
57+	80·72	±0·50	175
60+	94·16	±0·62	174
63+	110·74	±0·72	176
66+	127·89	±1·09	114
69+	130·31	±2·62	21
		Total	1403

TABLE III.

American White and Coloured Boys' Measurements.

Age	White				Coloured			
	Height in inches	Number in Group	Weight in pounds	Number in Group	Height in inches	Number in Group	Weight in pounds	Number in Group
6	44·60	102	42·24	102	44·17	73	43·44	69
7	45·93	395	47·66	399	46·08	246	50·10	220
8	47·81	603	51·42	605	47·74	280	53·99	276
9	49·76	640	56·18	638	49·26	294	59·04	287
10	51·66	698	61·57	696	51·14	333	65·17	228
11	53·19	660	66·14	668	52·10	268	69·44	270
12	55·15	746	72·70	751	53·94	283	75·07	278
13	56·98	677	79·34	683	56·08	318	83·50	317
14	59·32	586	88·75	582	57·98	280	90·00	275
15	61·85	403	100·91	398	60·09	218	99·42	215
16	64·31	283	113·83	263	63·13	124	113·45	121
17	65·97	119	121·18	119				
18	67·06	38	133·90	38				
Totals		5930	—	5942	Totals	2717	—	2562

TABLE IV.

English and Australian Boys' Measurements.

Age	English				Australian			
	Height in inches	Number in Group	Weight in pounds	Number in Group	Height in inches	Number in Group	Weight in pounds	Number in Group
5	41·16	175	40·99	176				
6	43·18	327	54·10	327	44·07	3906	43·37	3906
7	45·58	784	58·89	631	45·88	6070	47·04	6070
8	47·15	1052	59·50	1038	48·01	7171	51·91	7171
9	49·70	1241	62·29	1202	50·08	6174	56·99	6174
10	51·79	1193	66·87	1200	51·74	6902	61·49	6902
11	53·21	1230	71·20	1129	53·63	6370	66·55	6370
12	54·98	868	77·00	803	55·23	6009	72·20	6009
13	57·36	1464	83·43	1527	57·27	5758	78·80	5758
14	59·43	2424	91·91	2571	59·34	3378	88·16	3378
15	62·02	1207	103·60	1451	61·76	1144	92·22	1144
16	64·66	1704	118·52	1724	64·38	420	111·43	420
17	66·15	2055	131·28	2106				
18	66·88	1675	137·57	1669				
Totals		17489	—	17674	Totals	53302	—	53302

TABLE V.

Yoruba Boys: Age from Height-Weight Curve.

Age	Height in inches	P.E.
5	—	—
6	44·27	±0·83
7	46·10	±0·52
8	47·10	±0·35
9	49·76	±0·40
10	51·73	±0·22
11	53·35	±0·25
12	55·41	±0·25
13	57·17	±0·20
14	59·59	±0·38
15	62·09	±0·40
16	64·68	±0·50

N.B. *The following data were sent at a later date to the Editor of Biometrika* by Dr Anderson. They allow a comparison between the Height-Weight curves of two Nigerian races to be made.

TABLE VI

Height-Weight Curve for Ibo Boys.

Weight in pounds	Height in inches	P.E.	Number in Group
30+	41·00	±0.24	17
35+	43·07	±0·12	61
40+	45·41	±0·14	88
45+	47·28	∓0·13	118
50+	49·50	±0·08	130
55+	51·08	±0·10	169
60+	53·07	±0·10	145
65+	54·81	±0·11	109
70+	55·95	±0·10	122
75+	57·00	±0·12	96
80+	58·47	±0·10	87
85+	59·73	±0·14	91
90+	60·06	±0·14	73
95+	61·59	±0·17	88
100+	62·64	±0·13	81
105+	62·94	±0·16	69
110+	64·80	±0·16	79
115+	65·10	±0·14	76
120+	65·58	±0·14	73
125+	66·09	±0·16	62
130+	66·60	±0·20	40
135+	66·66	±0·24	25
140+	67·68	±0·33	16
145+	68·01	±0·57	10
		Total	1925

These measurements indicate a very considerable difference between the Height-Weight curves of the Ibo and Yoruba Boys. They agree moderately up to age 11, but then the Ibo rises above the Yoruba curve, and after age 16 above all others but the English.

[NOTE. If the three correlation tables, Age and Height, Age and Weight, Weight and Height, were known to give straight regression lines, then the probable Age for a given Weight and Height would be determined at once for a given race by the regression of Age on the two variates Weight and Height, and what is more the scatter round this mean or probable age would be given by the usual formula. Now what are the difficulties of applying this customary method in the present case?

(i) Growth curves are not as a rule linear; they are certainly not so if we take the Height or Weight from birth to prime*. But if we take only the portion of these growth curves from age 5 to 14, the curves are approximately straight†.

(ii) Tables of Height and Weight when Age is neglected have as far as the annotator is aware not been formed, or are not usually formed. Correlation tables of Height and Weight are usually given for each age group; there is no serious difficulty in adding such tables together, however, and then working out their constants. But will these Height and Weight tables give approximately straight regression lines? The Yoruba and Ibo data at least suggest that nothing really fitter than straight lines could be found for the range from 5 to 14.

(iii) The bivariate regression formula can only be applied to races where the age at birth is known. We do not know this in the case of the African children. To this the answer must be that it is an essential feature of Dr Anderson's method that he applies results from other races and expects no great difference to exist in the distributions of Height and Weight with Age between widely different races.

The special advantage of Dr Anderson's method is that it may empirically allow for some curvature of the regression lines, but what it gains in this way will we think be more than compensated by the fact that on the hypothesis of linearity the bivariate regression formula is sound theoretically and not empirical, and further provides a reasonable measure of the scatter of age round the mean or probable value.

It will be of interest accordingly to determine the regression equation of Age on Weight and Height for two fairly diverse races:

(i) A white race, say English, Scottish or American. The first two will require only one further correlation table to be made for each. Until one has examined MacDonald's data it is not possible to say whether it contains for American children the requisite material for the Height-Weight correlation table.

(ii) A coloured race, say the American negro children. Again one cannot say until MacDonald's work is examined whether the requisite material is available.

* Cf. *Annals of Eugenics*, Vol. II. pp. 100—102.

† Cf. reference in last footnote, and, better, *Biometrika*, Vol. X. pp. 292—293. Note especially the degree of linearity in the Glasgow children. The B. A. returns are based on very heterogeneous material.

Having obtained the two bivariate formulae we can then predict from *both* formulae the age of various white and coloured children of selected heights and weights and test how far the results are independent of which *formula* is used, *i.e.* of race. If the difference be not very great, we shall then feel justified in applying either formula to African children.

When this work has been completed—and it will take some time to make from the data the additional tables requisite*—we shall be in a position to test Dr Anderson's method, and replace it, if needful, by a more theoretically correct process.

In this matter it is well to recall to mind that formulae to reconstruct stature from the measurement of the long bones (i) appear to vary from race to race to a sensible extent, (ii) that the long bones are more highly correlated with stature than Weight and Height are with Age, and (iii) that the probable error in stature of a single individual (determined from a single set of bones, not that of a race from numerous skeletons) is very considerable. One would not be surprised *a priori* to find a standard error of 2 to 3 years in determining a probable Age from Height and Weight, and such a range would possibly render the determination of small value for Dr Anderson's purposes. However, that is only a surmise to be verified or not after the standard error has been found. ED.]

* The writer has not so far been able to see a copy of A. MacDonald's *Experimental Study of Children*, 1899.

THE PROBABILITY INTEGRAL OF THE CORRELATION COEFFICIENT IN SAMPLES FROM A NORMAL BI-VARIATE POPULATION.

By F. GARWOOD, B.A.

1. SAMPLES of size n are drawn from a normally distributed bi-variate population in which the coefficient of correlation is ρ. Dr R. A. Fisher has shown* that the probability that the coefficient of correlation r of the sample will lie between r and $r + dr$ is

$$y_n(r)\,dr,$$

where
$$y_n(r) = \frac{(1-\rho^2)^{\frac{n-1}{2}}}{\lfloor n-3 \, \pi}(1-r^2)^{\frac{n-4}{2}} \frac{d^{n-2}}{d(r\rho)^{n-2}}\left[\frac{\cos^{-1} - \rho r}{\sqrt{1-\rho^2 r^2}}\right].$$

The problem is to evaluate $\int_{-1}^{r} y_n(r)\,dr$, which is the chance of drawing a sample with a correlation coefficient less than r. We shall obtain the integral for the first few values of n, a symbolic method for the case of n odd also being shown.

2. $n = 3$. We shall use the notation

$$U = \frac{\cos^{-1}(-r\rho)}{\sqrt{1-r^2\rho^2}}, \quad D^p U = \frac{d^p U}{d(r\rho)^p} \text{ and } y_n \text{ for } y_n(r).$$

Thus
$$\frac{\pi}{1-\rho^2}\int_{-1}^{r} y_3\,dr = \int_{-1}^{r} \frac{1}{\sqrt{1-r^2}} DU\,dr.$$

Put
$$r = \sin\theta,\ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2};\ \therefore \frac{\pi}{1-\rho^2}\int_{-1}^{r} y_3\,dr = \int_{-1}^{r} DU\,d\theta.$$

Now we find later (6) that if $s > p$, then

$$\int_{-\pi/2}^{\theta} \sin^{2p}\theta \left(\frac{d^{2s-1}U}{dx^{2s-1}}\right)_{x=\rho\sin\theta} d\theta = \left(\theta + \frac{\pi}{2}\right)\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\frac{1}{1-\rho^2} - \cos\theta\,\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\frac{\rho^{2s-2p-1}}{1-\rho^2}(U)_{\rho\sin\theta},$$

where
$$\Delta^p y = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\ldots (p \text{ times})\ldots y.$$

Thus taking $p = 0$, $s = 1$, we have

$$\frac{\pi}{1-\rho^2}\int_{-1}^{r} y_3\,dr = \left(\theta + \frac{\pi}{2}\right)\frac{1}{1-\rho^2} - \cos\theta\,\frac{\rho U}{1-\rho^2},$$

and since $\cos\left(\theta + \frac{\pi}{2}\right) = -r$, we have

$$\int_{-1}^{r} y_3\,dr = \frac{\cos^{-1} - r}{\pi} - \frac{\sqrt{1-r^2}}{\pi}\,\rho\,\frac{\cos^{-1} - \rho r}{\sqrt{1-\rho^2 r^2}}.$$

* *Biometrika*, Vol. x. p. 507.

3. $n=5$. Using integration by parts, we find that

$$\frac{2!\,\pi}{(1-\rho^2)^2}\int_{-1}^{r} y_5\,dr = \int_{-1}^{r}(1-r^2)^{\frac{1}{2}}\,D^3U\,dr = \frac{(1-r^2)^{\frac{1}{2}}}{\rho}\,D^2U + \frac{1}{\rho}\int_{-1}^{r}\frac{r}{\sqrt{1-r^2}}\,D^2U\,dr.$$

We thus have to find $\int_{-\pi/2}^{\theta}\sin\theta\,D^2U\,d\theta$, and using the formula obtained below in Section 7:

$$\int_{-\pi/2}^{\theta}\sin^{2p+1}\theta\left(\frac{d^{2s}U}{dx^{2s}}\right)_{x=\rho\sin\theta}d\theta = \left(\theta+\frac{\pi}{2}\right)\frac{\partial^{2p+1}}{\partial\rho^{2p+1}}\Delta^{s-p-1}\frac{1}{1-\rho^2}$$
$$-\cos\theta\,\frac{\partial^{2p+1}}{\partial\rho^{2p+1}}\Delta^{s-p-1}\frac{\rho^{2s-2p-1}}{1-\rho^2}\,U \quad (2s>2p+1),$$

we see that its value is

$$\left(\theta+\frac{\pi}{2}\right)\frac{\partial}{\partial\rho}\frac{1}{1-\rho^2}-\cos\theta\frac{\partial}{\partial\rho}\frac{\rho U}{1-\rho^2} = \left(\theta+\frac{\pi}{2}\right)\frac{2\rho}{(1-\rho^2)^2}-\cos\theta\left[\frac{1+\rho^2}{(1-\rho^2)^2}\,U+\frac{\rho r}{1-\rho^2}\,DU\right].$$

Hence

$$\int_{-1}^{r} y_5\,dr = \frac{(1-\rho^2)^2\sqrt{1-r^2}}{2\pi\rho}\,D^2U - \frac{\sqrt{1-r^2}}{2\pi}\left[r(1-\rho^2)\,DU+\frac{1}{\rho}(1+\rho^2)\,U\right]+\frac{\cos^{-1}-r}{\pi}.$$

It is convenient to express $\int_{-1}^{r} y_5\,dr$ in terms of the corresponding ordinates of the frequency curves for smaller samples: we use the substitutions

$$DU = \frac{\pi}{1-\rho^2}\sqrt{1-r^2}\,y_3,$$

$$D^2U = \frac{\pi}{(1-\rho^2)^{\frac{3}{2}}}\,y_4,$$

$$D^3U = \frac{2\pi}{(1-\rho^2)^2}\,\frac{1}{\sqrt{1-r^2}}\,y_5, \text{ etc.}$$

Thus we obtain

$$\int_{-1}^{r} y_5\,dr = \frac{\sqrt{1-\rho^2}\sqrt{1-r^2}}{2\rho}\,y_4-\frac{r}{2}(1-r^2)\,y_3-\frac{\sqrt{1-r^2}(1+\rho^2)}{2\pi\rho}\,U+\frac{\cos^{-1}-r}{\pi}.$$

4. $n=7$. Integrating by parts twice, we have

$$\frac{4!\,\pi}{(1-\rho^2)^3}\int_{-1}^{r} y_7\,dr = \int_{-1}^{r}(1-r^2)^{\frac{3}{2}}\,D^5U\,dr = \frac{(1-r^2)^{\frac{3}{2}}}{\rho}\,D^4U+\frac{3}{\rho}\int_{-1}^{r} r(1-r^2)^{\frac{1}{2}}\,D^4U\,dr$$
$$= \frac{(1-r^2)^{\frac{3}{2}}}{\rho}\,D^4U+\frac{3r(1-r^2)^{\frac{1}{2}}}{\rho^2}\,D^3U-\frac{3}{\rho^2}\int_{-1}^{r}\frac{D^3U}{\sqrt{1-r^2}}(1-r^2-r^2)\,dr.$$

We thus have to find $\int_{-\pi/2}^{\theta}(1-2\sin^2\theta)\,D^3U\,d\theta$, and using the formula in Section 6 below with $s=2$, $p=0$ and 1, we find its value to be

$$\left(\theta+\frac{\pi}{2}\right)\Delta\,\frac{1}{1-\rho^2}-\cos\theta\,\Delta\,\frac{\rho^3U}{1-\rho^2} \quad (\sin\theta=r)$$
$$-2\left(\theta+\frac{\pi}{2}\right)\frac{\partial^3}{\partial\rho^3}\frac{1}{1-\rho^2}+2\cos\theta\,\frac{\partial^3}{\partial\rho^3}\frac{\rho U}{1-\rho^2}$$

$$= \left(\theta + \frac{\pi}{2}\right)\left[\frac{4(1+\rho^2)}{(1-\rho^2)^3} - \frac{4(1+3\rho^2)}{(1-\rho^2)^3}\right] - \cos\theta\,\Delta\left(\frac{\rho U}{1-\rho^2} - \rho U\right) + 2\cos\theta\,\frac{\partial^2}{\partial\rho^2}\frac{\rho U}{1-\rho^2}$$

$$= -\left(\theta + \frac{\pi}{2}\right)\frac{8\rho^2}{(1-\rho^2)^3} + \frac{\rho U\cos\theta}{(1-\rho^2)^3}(3+6\rho^2-\rho^4) + \frac{rDU\cos\theta}{(1-\rho^2)^2}(4-3\rho^2+3\rho^4) + r^2D^2U\cos\theta\,\frac{\rho(2-\rho^2)}{1-\rho^2}.$$

Substituting as before, we obtain

$$\int_{-1}^{r} y_7\,dr = \frac{\sqrt{1-r^2}\sqrt{1-\rho^2}}{4\rho}y_6 + \frac{(1-\rho^2)r}{4\rho^2}y_5 - \frac{r^2\sqrt{1-r^2}\sqrt{1-\rho^2}(2-\rho^2)}{8\rho}y_4 - \frac{r(1-r^2)(4-3\rho^2+3\rho^4)}{8\rho^3}y_3 - \frac{3+6\rho^2-\rho^4}{8\pi\rho}U\sqrt{1-r^2} + \frac{\cos^{-1}-r}{\pi}.$$

5. *Any odd integer* $n = 2s+3$.

We shall have

$$\frac{\overline{n-3}\,!\,\pi}{(1-\rho^2)^{\frac{n-1}{2}}}\int_{-1}^{r} y_n\,dr = \int_{-1}^{r}(1-r^2)^{\frac{2s-1}{2}}D^{2s+1}U\,dr = \int_{-\pi/2}^{\theta}(1-\sin^2\theta)^s D^{2s+1}U\,d\theta \qquad (\sin\theta = r),$$

and by the formulae obtained below this is equal to

$$-\cos\theta\left[\left\{\Delta^s\rho^{2s} - \binom{s}{1}\frac{\partial^2}{\partial\rho^2}\Delta^{s-1}\rho^{2s-2} + \binom{s}{2}\frac{\partial^4}{\partial\rho^4}\Delta^{s-2}\rho^{2s-4} - \ldots + (-)^s\frac{\partial^{2s}}{\partial\rho^{2s}}\right\}\frac{\rho U}{1-\rho^2}\right] + \left(\theta+\frac{\pi}{2}\right)\left[\left\{\Delta^s - \binom{s}{1}\frac{\partial^2}{\partial\rho^2}\Delta^{s-1} + \binom{s}{2}\frac{\partial^4}{\partial\rho^4}\Delta^{s-2} - \ldots + (-)^s\frac{\partial^{2s}}{\partial\rho^{2s}}\right\}\frac{1}{1-\rho^2}\right].$$

We can immediately obtain the coefficient of $\left(\theta + \frac{\pi}{2}\right)$ by putting $\theta = \frac{\pi}{2}$, when $\int_{-1}^{r} y_n\,dr = 1$ and $\cos\theta = 0$. Thus we have, finally,

$$\int_{-1}^{r} y_n\,dr = \frac{\cos^{-1}-r}{\pi} - \sqrt{1-r^2}\,\frac{(1-\rho^2)^{s+1}}{2s\,!\,\pi}\left[\left\{\Delta^s\rho^{2s} - \binom{s}{1}\frac{\partial^2}{\partial\rho^2}\Delta^{s-1}\rho^{2s-2} + \binom{s}{2}\frac{\partial^4}{\partial\rho^4}\Delta^{s-2}\rho^{2s-4} - \ldots + (-)^s\frac{\partial^{2s}}{\partial\rho^{2s}}\right\}\frac{\rho U}{1-\rho^2}\right],$$

where $$n = 2s+3.$$

Owing to the complexity of the operator Δ, the above methods of calculating $\int_{-1}^{r} y_n\,dr$ for $n = 3, 5, 7$ have been used in preference to the above formula, which has been included for theoretical interest.

We now proceed to obtain the formulae mentioned above.

6. *Evaluation of the integral* $I(\theta) = \int_{-\pi/2}^{\theta}\sin^{2p}\theta\left(\frac{d^{2s-1}U}{dx^{2s-1}}\right)_{x=\rho\sin\theta}d\theta \quad (s > p)$.

This is achieved by expanding $\left(\frac{d^{2s-1}U}{dx^{2s-1}}\right)_{x=\rho\sin\theta}$, i.e. $D^{2s-1}U$, in an infinite series,

multiplying each term by $\sin^{2p}\theta$, integrating, and then summing in a particular way. The infinite series* is

$$\frac{d^{2s-1}}{dx^{2s-1}}\left[\frac{\cos^{-1}-x}{\sqrt{1-x^2}}\right]=\frac{\pi}{2}1^2.3^2\ldots(2s-1)^2\left[x+(2s+1)^2\frac{x^3}{3!}+(2s+1)^2(2s+3)^2\frac{x^5}{5!}+\ldots\right]$$
$$+2^2.4^2\ldots(2s-2)^2\left[1+(2s)^2\frac{x^2}{2!}+(2s)^2(2s+2)^2\frac{x^4}{4!}+\ldots\right],$$

so that we shall be concerned with the integrals

$$\int_{-\pi/2}^{\theta}\sin^{2m}\theta\,d\theta \text{ and } \int_{-\pi/2}^{\theta}\sin^{2m+1}\theta\,d\theta,$$

and these we proceed to evaluate. Calling the first u_m and integrating by parts, we have

$$u_m=\int_{-\pi/2}^{\theta}\sin^{2m-1}\theta\sin\theta\,d\theta=-\cos\theta\sin^{2m-1}\theta+(2m-1)\int_{-\pi/2}^{\theta}\sin^{2m-2}\theta\cos^2\theta\,d\theta$$
$$=-\cos\theta\sin^{2m-1}\theta+(2m-1)u_{m-1}-(2m-1)u_m.$$
$$\therefore\ u_m=-\cos\theta\frac{\sin^{2m-1}\theta}{2m}+\frac{2m-1}{2m}u_{m-1}.$$
$$\therefore\ u_m=-\cos\theta\left[\frac{1}{2m}\sin^{2m-1}\theta+\frac{2m-1}{2m(2m-2)}\sin^{2m-3}\theta\right.$$
$$\left.+\frac{(2m-1)(2m-3)}{2m(2m-2)(2m-4)}\sin^{2m-5}\theta+\ldots+\frac{(2m-1)(2m-3)\ldots3}{2m(2m-2)(2m-4)\ldots2}\sin\theta\right]$$
$$+\frac{(2m-1)(2m-3)\ldots3.1}{2m(2m-2)\ldots4.2}\left(\theta+\frac{\pi}{2}\right)\qquad\text{since } u_0=\left(\theta+\frac{\pi}{2}\right).$$

Similarly we obtain the second integral v_m:

$$v_m=-\cos\theta\left[\frac{1}{2m+1}\sin^{2m}\theta+\frac{2m}{(2m+1)(2m-1)}\sin^{2m-2}\theta\right.$$
$$\left.+\frac{2m(2m-2)}{(2m+1)(2m-1)(2m-3)}\sin^{2m-4}\theta+\ldots+\frac{2m(2m-2)\ldots2}{(2m+1)(2m-1)\ldots3}\right].$$

The power series in ρ is uniformly convergent for all values of θ, so that the series may be summed after integration. We then have

$$I(\theta)=\int_{-\pi/2}^{\theta}\sin^{2p}\theta\left[\frac{\pi}{2}1^2.3^2\ldots(2s-1)^2\left(\rho\sin\theta+\frac{(2s+1)^2}{3!}\rho^3\sin^3\theta+\ldots\right)\right.$$
$$\left.+2^2.4^2\ldots(2s-2)^2\left(1+\frac{(2s)^2}{2!}\rho^2\sin^2\theta+\ldots\right)\right]d\theta$$
$$=\frac{\pi}{2}1^2.3^2\ldots(2s-1)^2\left[\rho v_p+\frac{(2s+1)^2}{3!}\rho^3v_{p+1}+\frac{(2s+1)^2(2s+3)^2}{5!}\rho^5v_{p+2}+\ldots\right]$$
$$+2^2.4^2\ldots(2s-2)^2\left[u_p+\frac{(2s)^2}{2!}\rho^2u_{p+1}+\frac{(2s)^2(2s+2)^2}{4!}\rho^4u_{p+2}+\ldots\right]$$
$$=-\frac{\pi}{2}\cos\theta.1^2.3^2\ldots(2s-1)^2\left[\rho\left\{\frac{\sin^{2p}\theta}{2p+1}+\frac{2p}{(2p+1)(2p-1)}\sin^{2p-2}\theta+\ldots\right.\right.$$
$$\left.+\frac{2p(2p-2)\ldots2}{(2p+1)(2p-1)\ldots3}\right\}$$

* *Biometrika*, Vol. XI. p. 330.

$$+(2s+1)^2\frac{\rho^3}{3!}\left\{\frac{\sin^{2p+2}\theta}{2p+3}+\frac{2p+2}{(2p+3)(2p+1)}\sin^{2p}\theta+\ldots+\frac{(2p+2)(2p)\ldots 2}{(2p+3)(2p+1)\ldots 3}\right\}+\ldots\Bigg]$$

$$-\cos\theta\, 2^2.4^2\ldots(2s-2)^2\Bigg[\left\{\frac{\sin^{2p-1}\theta}{2p}+\frac{2p-1}{2p(2p-2)}\sin^{2p-3}\theta+\ldots\right.$$

$$\left.+\frac{(2p-1)(2p-3)\ldots 3}{2p(2p-2)\ldots 2}\sin\theta\right\}$$

$$+(2s)^2\frac{\rho^2}{2!}\left\{\frac{\sin^{2p+1}\theta}{2p+2}+\frac{2p+1}{(2p+2)2p}\sin^{2p-1}\theta+\ldots+\frac{(2p+1)(2p-1)\ldots 3}{(2p+2)(2p)\ldots 2}\sin\theta\right\}+\ldots\Bigg]$$

$$+\left(\theta+\frac{\pi}{2}\right)2^2.4^2\ldots(2s-2)^2\left[\frac{(2p-1)(2p-3)\ldots 3}{2p(2p-2)\ldots 2}+(2s)^2\frac{\rho^2}{2!}\frac{(2p+1)(2p-1)\ldots 3}{(2p+2)(2p)\ldots 2}+\ldots\right].$$

The series inside the two sets of brackets [ ] both converge when $\theta=\frac{\pi}{2}$, and for this value of θ each series can be expressed as a double series, the terms being those inside each set of brackets { }. The corresponding double series for any value of θ will have its terms less in absolute value than those of the series for $\theta=\frac{\pi}{2}$, *and so converges absolutely and may be summed in any of the standard ways.* We sum by diagonals, viz. we sum the last terms in each of the brackets { }, then the next to last, and so on. On the cancellation of common factors we have for the value of the integral:

$$\begin{aligned}
I(\theta)=-\frac{\pi}{2}\cos\theta\Bigg[&\{a_1\rho+a_3\rho^3+a_5\rho^5+\ldots\}\\
&+1^2\frac{\sin^2\theta}{2!}\{a_1\rho+a_3\rho^3+a_5\rho^5+\ldots\}\\
&+1^2.3^2\frac{\sin^4\theta}{4!}\{a_1\rho+a_3\rho^3+a_5\rho^5+\ldots\}\\
&+\ldots\\
&+1^2.3^2\ldots(2p-1)^2\frac{\sin^{2p}\theta}{2p!}\{a_1\rho+a_3\rho^3+a_5\rho^5+\ldots\}\\
&+1^2.3^2\ldots(2p+1)^2\frac{\sin^{2p+2}\theta}{2p+2!}\{a_3\rho^3+a_5\rho^5+a_7\rho^7+\ldots\}\\
&+1^2.3^2\ldots(2p+3)^2\frac{\sin^{2p+4}\theta}{2p+4!}\{a_5\rho^5+a_7\rho^7+a_9\rho^9+\ldots\}\\
&+\ldots\Bigg]\\
-\cos\theta\Bigg[&\frac{\sin\theta}{1!}\{a_0+a_2\rho^2+a_4\rho^4+\ldots\}\\
&+2^2\frac{\sin^3\theta}{3!}\{a_0+a_2\rho^2+a_4\rho^4+\ldots\}\\
&+2^2.4^2\frac{\sin^5\theta}{5!}\{a_0+a_2\rho^2+a_4\rho^4+\ldots\}\\
&+\ldots
\end{aligned}$$

$$+\,2^2.4^2\ldots(2p-2)^2\frac{\sin^{2p-1}\theta}{2p-1\,!}\{a_0+a_2\rho^2+a_4\rho^4+\ldots\}$$

$$+\,2^2.4^2\ldots(2p)^2\frac{\sin^{2p+1}\theta}{2p+1\,!}\{a_2\rho^2+a_4\rho^4+a_6\rho^6+\ldots\}$$

$$+\,2^2.4^2\ldots(2p+2)^2\frac{\sin^{2p+3}\theta}{2p+3\,!}\{a_4\rho^4+a_6\rho^6+a_8\rho^8+\ldots\}$$

$$+\ldots\Big]$$

$$+\left(\theta+\frac{\pi}{2}\right)[a_0+a_2\rho^2+a_4\rho^4+\ldots],$$

where

$$a_r=(r+1)(r+2)(r+3)\ldots(r+2p)(r+2p+2)^2(r+2p+4)^2\ldots(r+2s-2)^2.$$

Now Δ is used to denote the operator $\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}$, so that

$$\Delta^p\rho^n=n^2.(n-2)^2.(n-4)^2\ldots(n-2p+2)^2\rho^{n-2p}\qquad(n\geqslant 2p).$$

Thus it is clear that

$$a_r\rho^r=\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\rho^{r+2s-2},$$

and so

$$a_1\rho+a_3\rho^3+a_5\rho^5+\ldots=\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}[\rho^{2s-1}+\rho^{2s+1}+\rho^{2s+3}+\ldots];$$

inside the brackets [ ] we may add any number of the terms previous to ρ^{2s-1}, since they will all be annihilated by the operator.

Thus
$$a_1\rho+a_3\rho^3+a_5\rho^5+\ldots=\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\left[\frac{\rho^{2m-1}}{1-\rho^2}\right]\qquad(0<m\leqslant s),$$

similarly
$$a_0+a_2\rho^2+a_4\rho^4+\ldots=\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\left[\frac{\rho^{2m-2}}{1-\rho^2}\right]\qquad(0<m\leqslant s).$$

Substituting these values in $I(\theta)$, choosing in each case a suitable value of m, we have

$$I(\theta)=-\cos\theta\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\left\{\frac{\pi}{2}\frac{\rho^{2s-2p-1}}{1-\rho^2}+\frac{\pi}{2}.1^2.\frac{\sin^2\theta}{2\,!}\frac{\rho^{2s-2p+1}}{1-\rho^2}+\ldots\right.$$

$$+\frac{\pi}{2}.1^2.3^2\ldots(2p-1)^2\frac{\sin^{2p}\theta}{2p\,!}\frac{\rho^{2s-1}}{1-\rho^2}$$

$$\left.+\frac{\pi}{2}.1^2.3^2\ldots(2p+1)^2\frac{\sin^{2p+2}\theta}{2p+2\,!}\frac{\rho^{2s+1}}{1-\rho^2}+\ldots\right\}$$

$$-\cos\theta\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\left\{\frac{\sin\theta}{1\,!}\frac{\rho^{2s-2p}}{1-\rho^2}+2^2.\frac{\sin^3\theta}{3\,!}\frac{\rho^{2s-2p+2}}{1-\rho^2}+\ldots\right.$$

$$\left.+2^2.4^2\ldots(2p-2)^2.\frac{\sin^{2p-1}\theta}{2p-1\,!}\frac{\rho^{2s-2}}{1-\rho^2}+\ldots\right\}$$

$$+\left(\theta+\frac{\pi}{2}\right)\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\frac{1}{1-\rho^2}.$$

The sum of the two series inside the brackets { } is

$$\frac{\rho^{2s-2p-1}}{1-\rho^2}\left[\frac{\pi}{2}\left\{1+1^2\frac{\rho^2\sin^2\theta}{2!}+1^2.3^2\frac{\rho^4\sin^4\theta}{4!}+\ldots\right\}\right.$$
$$\left.+\left\{\frac{\rho\sin\theta}{1!}+2^2\frac{\rho^3\sin^3\theta}{3!}+2^2.4^2\frac{\rho^5\sin^5\theta}{5!}+\ldots\right\}\right],$$

so that we have, finally,

$$\int_{-\pi/2}^{\theta}\sin^{2p}\theta\, D^{2s-1}U\,d\theta=\left(\theta+\frac{\pi}{2}\right)\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\frac{1}{1-\rho^2}$$
$$-\cos\theta\frac{\partial^{2p}}{\partial\rho^{2p}}\Delta^{s-p-1}\left\{\frac{\rho^{2s-2p-1}}{1-\rho^2}\frac{\cos^{-1}-\rho\sin\theta}{\sqrt{1-\rho^2\sin^2\theta}}\right\}.$$

7. *Evaluation of the integral*

$$\int_{-\pi/2}^{\theta}\sin^{2p+1}\theta\left(\frac{d^{2s}U}{dx^{2s}}\right)_{\rho\sin\theta}d\theta\qquad(2s>2p-1).$$

By an exactly similar method we can prove that the value of this integral is

$$\left(\theta+\frac{\pi}{2}\right)\frac{\partial^{2p+1}}{\partial\rho^{2p+1}}\Delta^{s-p-1}\frac{1}{1-\rho^2}-\cos\theta\frac{\partial^{2p+1}}{\partial\rho^{2p+1}}\Delta^{s-p-1}\left\{\frac{\rho^{2s-2p-1}}{1-\rho^2}\frac{\cos^{-1}-\rho\sin\theta}{\sqrt{1-\rho^2\sin^2\theta}}\right\}.$$

8. *Even values of n.*

The term $(1-r^2)^{\frac{n-4}{2}}$ is now rational, so that the substitution $r=\sin\theta$ is not used. The integration proceeds either by expansion of this term and repeated integration by parts, or by the method given below, which has the advantage of giving the integral in terms of the previous ordinates.

For completeness we include the easily solved case of $n=4$. We have

$$\int_{-1}^{r}y_4\,dr=\frac{(1-\rho^2)^{\frac{1}{2}}}{\pi}\int_{-1}^{r}D^2U\,dr=\frac{(1-\rho^2)^{\frac{1}{2}}}{\pi\rho}\left[DU\right]_{-\rho}^{r\rho}.$$

Now we know that

$$\frac{dU}{dx}=\frac{1}{1-x^2}[1+xU],\text{ and }DU=\frac{\sqrt{1-r^2}\,\pi}{1-\rho^2}y_3,$$
$$\therefore\int_{-1}^{r}y_4\,dr=\frac{\sqrt{1-\rho^2}\sqrt{1-r^2}}{\rho}y_3-\frac{\sqrt{1-\rho^2}}{\pi\rho}+\frac{\cos^{-1}\rho}{\pi}.$$

9. $n=6$.

Assume that $\displaystyle\int_{-1}^{r}y_6\,dr=A+BU+F_1DU+\ldots+F_pD^pU,$

where A, B, $F_1\ldots F_p$ are functions of r. Differentiating with respect to r, we see that

$$A'+B'U+(F_1'+\rho B)DU+(F_2'+\rho F_1)D^2U+\ldots+(F_p'+\rho F_{p-1})D^pU+\rho F_pD^{p+1}U$$
$$=y_6=\frac{(1-\rho^2)^{\frac{1}{2}}}{3!\,\pi}(1-r^2)D^4U.$$

$\therefore\ p+1=4$, i.e. $p=3$, and $A'=0$, $B'=0$, $F_1'+\rho B=0$, $F_2'+\rho F_1=0$,

$$\rho F_3'+F_2=0,\ \rho F_3=\frac{(1-\rho^2)^{\frac{1}{2}}(1-r^2)}{6\pi}.$$

$$\therefore\ F_3 = \frac{(1-\rho^2)^{\frac32}(1-r^2)}{6\pi\rho} \text{ and } F_3 D^3U = \frac{(1-\rho^2)(1-r^2)^{\frac32}}{2\pi} D^3U \frac{(1-\rho^2)^{\frac32}(1-r^2)^{\frac32}}{3\rho}$$

$$= \frac{(1-\rho^2)^{\frac12}(1-r^2)^{\frac32}}{3\rho} y_5$$

$$F_2 = \frac{(1-\rho^2)^{\frac32} r}{3\pi\rho},\ F_2 D^2U = \frac{(1-\rho^2)^{\frac32} D^2U}{\pi} \cdot \frac{(1-\rho^2) r}{3\rho^2} = \frac{(1-\rho^2) r}{3\rho^2} y_4,$$

$$F_1 = -\frac{(1-\rho^2)^{\frac52}}{3\pi\rho^3},\ F_1 DU = \frac{(1-\rho^2)\, DU}{\pi\sqrt{1-r^2}}\left(-\frac{(1-\rho^2)^{\frac32}(1-r^2)^{\frac12}}{3\rho^3}\right) = -\frac{(1-\rho^2)^{\frac32}(1-r^2)^{\frac12}}{3\rho^3} y_3$$

$B=0$ and $A=$ const., and to find A let $r \to -1$; we have

$$DU = \frac{1}{1-r^2\rho^2} + \frac{r\rho\cos^{-1}(-\rho r)}{(1-r^2\rho^2)^{\frac32}} \to \frac{1}{1-\rho^2} - \frac{\rho\cos^{-1}\rho}{(1-\rho^2)^{\frac32}},$$

and $$D^2U = \frac{1}{1-r^2\rho^2}[U + 3r\rho\, DU] \to \frac{1}{(1-\rho^2)^{\frac52}}[(1+2\rho^2)\cos^{-1}\rho - 3\rho\sqrt{1-\rho^2}]$$

$$\therefore\ A = \frac{(1-\rho^2)^{\frac12}(1-4\rho^2)}{3\pi\rho^2} + \frac{\cos^{-1}\rho}{\pi}.$$

Thus

$$\int_{-1}^{r} y_6\, dr = \frac{\sqrt{1-\rho^2}(1-4\rho^2)}{3\pi\rho^2} + \frac{\cos^{-1}\rho}{\pi} - \frac{(1-\rho^2)^{\frac32}\sqrt{1-r^2}}{3\rho^3} y_3 + \frac{(1-\rho^2) r}{3\rho^2} y_4$$
$$+ \frac{\sqrt{1-\rho^2}\sqrt{1-r^2}}{3\rho} y_5.$$

10. $n=8$.

Using the same method, we obtain

$$\int_{-1}^{r} y_8\, dr = -\sqrt{1-\rho^2}\frac{(3-11\rho^2+23\rho^4)}{15\pi\rho^5} + \frac{\cos^{-1}\rho}{\pi} + \frac{(1-\rho^2)^{\frac52}\sqrt{1-r^2}}{5\rho^5} y_3 - \frac{r(1-\rho^2)^2}{5\rho^4} y_4$$
$$+ \frac{(3r^2-1)(1-\rho^2)^{\frac32}(1-r^2)^{-\frac12}}{15\rho^3} y_5 + \frac{(1-\rho^2) r}{5\rho^2} y_6 + \frac{\sqrt{1-\rho^2}\sqrt{1-r^2}}{5\rho} y_7.$$

11. Proceeding in the like manner we can obtain the probability integral for larger samples, but the formulae then become too complicated for practical use. Miss F. N. David, who is computing the Probability Integral Table for the distribution of r in small samples, is using the above formulae for the smallest samples, but proceeds in a different manner for the larger sample sizes.

A FURTHER NOTE ON THE RELATION BETWEEN THE MEDIAN AND THE QUARTILES IN SMALL SAMPLES FROM A NORMAL POPULATION.

By TOKISHIGE HOJO.

1. *Introductory.*

In a previous paper* I have considered certain properties of the median and quartiles in small samples from a normal population, and have investigated methods of linking these up with the limiting values in large samples. The latter problem has since been further investigated by K. Pearson†. If the population mean and standard deviation are unknown, the most satisfactory estimates of these, when sampling from a normal population, are obtained from the sample mean and standard deviation. But a variety of alternative estimates may be calculated from the median and quartiles, whose relative value may be judged by a comparison of their standard errors. In the earlier paper the following comparisons were made:

(1) Standard error of mean with that of median. (Tables II^{a-d}, *loc. cit.*)

(2) Standard error of an estimate of $\tilde{\sigma}$ obtained from the sample standard deviation with that of another estimate obtained from the interquartile distance. (Table VIII, *loc. cit.*)

In the present paper I shall add to these previous results:

(3) Standard error of the mid-point between the two quartiles, i.e. of $\frac{1}{2}(q_1+q_2)$.

(4) Standard error of an estimate of $\tilde{\sigma}$ obtained from the distance between a quartile and the median, i.e. from q_1-m or $m-q_2$.

The first result follows from calculations previously carried out; for the second it is necessary to determine the coefficient of correlation $r_{q,m}$ between quartiles and median.

Certain experimental sampling results will also be referred to.

2. *The Standard Error of the Mid-quartile Point*, $\frac{1}{2}(q_1+q_2)$.

In the limiting case of large samples, the standard error of the point mid-way between the two quartiles has been given by K. Pearson as

$$\sigma_{\frac{1}{2}(q_1+q_2)} = 1{\cdot}1126\tilde{\sigma}/\sqrt{n} \quad \text{.............................(1).}$$

It is easily seen that
$$\sigma_{\frac{1}{2}(q_1+q_2)} = \sigma_q\sqrt{\tfrac{1}{2}(1+r_{q_1q_2})} \quad \text{..........................(2),}$$

from which the ratio of $\sigma_{\frac{1}{2}(q_1+q_2)}$ to the standard error of the mean, $\tilde{\sigma}/\sqrt{n}$, may be readily calculated from the results given in Tables IV^b and VI^b of my earlier paper. Some comparative results are shown in Table I below, the figures in the last column being taken from my previous Tables II^c and II^d.

* *Biometrika*, Vol. XXIII. pp. 315—360.

† *Biometrika*, Vol. XXIII. pp. 361—397.

TABLE I.

n	$\sigma_q \div \frac{\bar{\sigma}}{\sqrt{n}}$	$r_{q_1 q_2}$	$\sigma_{\frac{1}{2}(q_1+q_2)} \div \frac{\bar{\sigma}}{\sqrt{n}}$	$\sigma_m \div \frac{\bar{\sigma}}{\sqrt{n}}$
4	1·1590	·4888	1·000	1·092
7	1·3406	·3116	1·086	1·214
10	1·3229	·3717	1·096	1·177
12	1·2740	·3895	1·062	1·190
22*	1·3419	·3463	1·101	1·217
∞	1·3626	·3333	1·113	1·253

Owing to the four different forms of quartile definition, corresponding to the cases $n = 4p$, $4p+1$, $4p+2$ and $4p+3$, the tabled standard errors do not change uniformly as n increases, but it is seen that in all cases the standard error of $\frac{1}{2}(q_1 + q_2)$ is less than that of m. In other words a more reliable estimate of the population mean is obtained from the mid-quartile point than from the median, though both estimates have a larger standard error than the sample mean†.

3. *The Correlation between the Quartile and Median Points.*

The chance that the qth individual in a ranked sample of n lies in the interval $x \pm \frac{1}{2}dx$ is given by

$$df = \frac{n!}{(q-1)!\,(n-q)!}\alpha_x^{q-1}(1-\alpha_x)^{n-q}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx \quad \text{..............(3)},$$

where

$$\alpha_x = \int_{-\infty}^{x}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx \quad \text{..............................(4)}.$$

This is the fundamental relation used in the following analysis.

Since the mean value of m, the sample median, is at the origin, the numerator of the correlation coefficient, or

$$P_{m,q} = r_{m,q} \times \sigma_m \times \sigma_q \quad \text{..............................(5)},$$

will be the mean product of $\{m \times q\}$, and it is necessary to consider separately the four cases previously defined.

(*a*) *Case* $n = 4m + 3$‡.

Here both median and quartile correspond to values of observations, and we are concerned with the mean value of a product $\{x_1 x_2\}$; in fact

$$P_{mq} = \frac{n!}{(2m+1)!\,|m|^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+1}(\alpha_{x_1}-\alpha_{x_2})^m(1-\alpha_{x_1})^m x_1 x_2\, d\alpha_{x_1}\, d\alpha_{x_2}$$

* The figures 1·3419 and ·3463 are due to K. Pearson's calculations, *loc. cit.* pp. 890 and 390; the figure 1·217 is obtained from my equation (*b*), p. 328, *loc. cit.*

† For the limiting values of other forms of estimate of the mean from ranked individuals, see K. Pearson, *Biometrika*, Vol. XIII. pp. 113—125.

‡ This m is an integer, and must be distinguished from the m used as a subscript which denotes the median.

$$= \frac{n!}{(2m+1)!\,m!^2}\int_{-\infty}^{+\infty}(1-\alpha_{x_1})^m x_1\,d\alpha_{x_1}\sum_{r=0}^{m}(-1)^r\,{}_mC_r\,\alpha_{x_1}^{m-r}$$

$$\times\left\{\left[-\alpha_{x_2}^{2m+r+1}\frac{e^{-\frac{x_2^2}{2}}}{\sqrt{2\pi}}\right]_{-\infty}^{x_1}+\int_{-\infty}^{x_1}\frac{2m+r+1}{2\pi}\,\alpha_{x_2}^{2m+r}\,e^{-x_2^2}\,dx_2\right\}$$

$$= \frac{n!}{2\pi\,(2m+1)!\,m!^2}\sum_{s=0}^{m}(-1)^s\,{}_mC_s\sum_{r=0}^{m}(-1)^r\,{}_mC_r\,(2m+r+1)$$

$$\times\int_{-\infty}^{+\infty}\alpha_{x_1}^{m+s-r}\,x_1\,d\alpha_{x_1}\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+r}\,e^{-x_2^2}\,dx_2$$

$$= \frac{n!}{2\pi\,(2m+1)!\,m!^2}\sum_{s=0}^{m}(-1)^s\,{}_mC_s\sum_{r=0}^{m}(-1)^r\,{}_mC_r\,(2m+r+1)$$

$$\times\left\{\left[-\alpha_{x_1}^{m+s-r}\frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+r}\,e^{-x_2^2}\,dx_2\right]_{-\infty}^{+\infty}\right.$$

$$\left.+(m+s-r)\int_{-\infty}^{+\infty}\alpha_{x_1}^{m+s-r-1}\,\frac{e^{-x_1^2}}{2\pi}\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+r}\,e^{-x_2^2}\,dx_2+\int_{-\infty}^{+\infty}\alpha_{x_1}^{3m+s}\,\frac{e^{-\frac{1}{2}x_1^2}}{\sqrt{2\pi}}\,dx_1\right\}$$

$$= \frac{n!}{2\pi\,(2m+1)!\,m!^2}\sum_{s=0}^{m}(-1)^s\,{}_mC_s\left\{T_{3m+s}\sum_{r=0}^{m}(-1)^r\,{}_mC_r\,(2m+r+1)\right.$$

$$\left.+\sum_{r=0}^{m}(-1)^r\,{}_mC_r\,(2m+r+1)\,(m+s-r)\,{}_{m+s-r-1}I_{2m+r}\right\}\quad\text{for } m \geqq 1 \;\ldots(6),$$

where
$$T_r=\int_{-\infty}^{+\infty}\alpha_x^r\,\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx \;\ldots\ldots(7),$$

and
$${}_sI_r=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\alpha_{x_1}^s\,e^{-x_1^2}\,dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^r\,e^{-x_2^2}\,dx_2 \;\ldots\ldots(8).$$

(*b*) *Case* $n = 4m + 2$.

Here P_{mq} is the mean value of the product $\{\frac{1}{2}(x_3+x_2)\times x_1\}$, since the median (as defined) lies midway between the two central observations.

$$P_{mq}=\frac{n!}{(2m)!\,m!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\alpha_{x_3}^{2m}\,(\alpha_{x_1}-\alpha_{x_2})^{m-1}\,(1-\alpha_{x_1})^m$$

$$\times\left(\frac{x_3+x_2}{2}\right)x_1\,d\alpha_{x_3}\,d\alpha_{x_2}\,d\alpha_{x_1}$$

$$=\frac{n!}{2\,(2m)!\,m!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}(\alpha_{x_1}-\alpha_{x_2})^{m-1}\,(1-\alpha_{x_1})^m\,x_1\,d\alpha_{x_1}\,d\alpha_{x_2}$$

$$\times\left\{\left[-\alpha_{x_3}^{2m}\frac{e^{-\frac{x_3^2}{2}}}{\sqrt{2\pi}}\right]_{-\infty}^{x_2}+2m\int_{-\infty}^{x_2}\alpha_{x_3}^{2m-1}\,\frac{e^{-x_3^2}}{2\pi}\,dx_3+\frac{\alpha_{x_2}^{2m+1}}{2m+1}\,x_2\right\}$$

$$=\frac{n!}{2\,(2m)!\,m!\,(m-1)!}\sum_{s=0}^{m}(-1)^s\,{}_mC_s\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\int_{-\infty}^{+\infty}\alpha_{x_1}^{m+s-r-1}\,x_1\,d\alpha_{x_1}$$

$$\times\int_{-\infty}^{x_1}\left\{-\alpha_{x_2}^{2m+r}\,\frac{e^{-x_2^2}}{2\pi}\,dx_2\right.$$

$$\left.+2m\,\alpha_{x_2}^r\,d\alpha_{x_2}\int_{-\infty}^{x_2}\alpha_{x_3}^{2m-1}\,\frac{e^{-x_3^2}}{2\pi}\,dx_3+\frac{\alpha_{x_2}^{2m+r+1}}{2m+1}\,x_2\,d\alpha_{x_2}\right\}\;\ldots\ldots(9),$$

in which

$$\int_{-\infty}^{+\infty} \alpha_{x_1}^{m+s-r-1} x_1\, d\alpha_{x_1} \int_{-\infty}^{x_1} \alpha_{x_2}^{2m+r} e^{-x_2^2} dx_2 = \left[-\alpha_{x_1}^{m+s-r-1} \frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{x_1} \alpha_{x_2}^{2m+r} e^{-x_2^2} dx_2\right]_{-\infty}^{+\infty}$$

$$+ \frac{m+s-r-1}{2\pi} \int_{-\infty}^{+\infty} \alpha_{x_1}^{m+s-r-2} e^{-x_1^2} dx_1 \int_{-\infty}^{x_1} \alpha_{x_2}^{2m+r} e^{-x_2^2} dx_2$$

$$+ \int_{-\infty}^{+\infty} \alpha_{x_1}^{3m+s-1} \frac{e^{-\frac{3}{2}x_1^2}}{\sqrt{2\pi}} dx_1$$

$$= (m+s-r-1)\, {}_{m+s-r-2}I_{2m+r} + T_{3m+s-1} \quad \ldots\ldots\ldots\ldots(10).$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \alpha_{x_1}^{m+s-r-1}\alpha_{x_2}^{r} \alpha_{x_3}^{2m-1} x_1 e^{-\frac{x_1^2+x_2^2+2x_3^2}{2}} dx_3\, dx_2\, dx_1$$

$$= \int_{-\infty}^{+\infty}\int_0^\infty \alpha_{x_1+x_3}^{r} \alpha_{x_1}^{2m-1} e^{-\frac{(x_1+x_3)^2+2x_1^2}{2}} dx_3\, dx_1 \left\{\left[-\alpha_{x_1+x_2+x_3}^{m+s-r-1} e^{-\frac{(x_1+x_2+x_3)^2}{2}}\right]_0^\infty\right.$$

$$\left. + \frac{m+s-r-1}{\sqrt{2\pi}}\int_0^\infty \alpha_{x_1+x_2+x_3}^{m+s-r-2} e^{-(x_1+x_2+x_3)^2} dx_2\right\}$$

$$= \int_{-\infty}^{+\infty}\alpha_{x_1}^{2m-1} e^{-x_1^2} dx_1 \left(\sqrt{2\pi}\, I_{m+s-1} - \int_{-\infty}^{x_1}\alpha_x^{m+s-1} e^{-x^2}dx\right)$$

$$+ (m+s-r-1)\int_{-\infty}^{+\infty}\int_0^\infty \alpha_{x_1+x_3}^{r}\alpha_{x_1}^{2m-1} e^{-\frac{(x_1+x_3)^2+2x_1^2}{2}} dx_3\, dx_1$$

$$\times \left(I_{m+s-r-2} - \int_{-\infty}^{x_1+x_3}\alpha_x^{m+s-r-2}\frac{e^{-x^2}}{\sqrt{2\pi}}dx\right)$$

$$= 2\pi\left(I_{2m-1}I_{m+s-1} - {}_{2m-1}I_{m+s-1}\right) + (m+s-r-1)\int_{-\infty}^{+\infty}\alpha_{x_1}^{2m-1}e^{-x_1^2}$$

$$\times\left\{I_{m+s-r-2}\frac{\sqrt{2\pi}}{r+1}\left(1-\alpha_{x_1}^{r+1}\right)\right.$$

$$\left. - \left[\frac{\sqrt{2\pi}}{r+1}\alpha_{x_1}^{r+1}\int_{-\infty}^{x_1+x_3}\alpha_x^{m+s-r-2}\frac{e^{-x^2}}{\sqrt{2\pi}}dx\right]_0^\infty + \int_0^\infty\frac{\sqrt{2\pi}}{r+1}\alpha_{x_1+x_3}^{m+s-1}\frac{e^{-(x_1+x_3)^2}}{\sqrt{2\pi}}dx_3\right\}$$

$$= 2\pi\left\{{}_{m+s-1}I_{2m-1} + \frac{m+s-r-1}{r+1}\left[I_{m+s-r-2}\left(I_{2m-1}-I_{2m+r}\right)\right.\right.$$

$$\left.\left. - I_{m+s-r-2}I_{2m-1} + {}_{2m+r}I_{m+s-r-2} + I_{m+s-1}I_{2m-1} - {}_{2m-1}I_{m+s-1}\right]\right\}$$

$$= 2\pi\left\{{}_{m+s-1}I_{2m-1} + \frac{m+s-r-1}{r+1}\left({}_{m+s-1}I_{2m-1} - {}_{m+s-r-2}I_{2m+r}\right)\right\} \quad \ldots\ldots\ldots(11)$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\alpha_{x_1}^{m+s-r-1}\alpha_{x_2}^{2m+r+1}x_1x_2\, e^{-\frac{x_1^2+x_2^2}{2}}dx_2\, dx_1 = \int_{-\infty}^{+\infty}\alpha_{x_1}^{m+s-r-1}x_1e^{-\frac{x_1^2}{2}}dx_1$$

$$\times\left\{\left[-\alpha_{x_2}^{2m+r+1}e^{-\frac{x_2^2}{2}}\right]_{-\infty}^{x_1} + \frac{2m+r+1}{\sqrt{2\pi}}\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+r}e^{-x_2^2}dx_2\right\}$$

$$= \left[\alpha_{x_1}^{3m+s}\frac{e^{-x_1^2}}{2}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{3m+s}{2\sqrt{2\pi}}e^{-\frac{3}{2}x_1^2}\alpha_{x_1}^{3m+s-1}dx_1 + (2m+r+1)$$

$$\times\left\{\left[-\alpha_{x_1}^{m+s-r-1}e^{-\frac{x_1^2}{2}}\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+r}\frac{e^{-x_2^2}}{\sqrt{2\pi}}dx_2\right]_{-\infty}^{+\infty}\right.$$

$$+\int_{-\infty}^{+\infty}\frac{m+s-r-1}{\sqrt{2\pi}}\alpha_{x_1}^{m+s-r-2}e^{-x_1^2}dx_1\int_{-\infty}^{x_1}\alpha_{x_2}^{2m+r}\frac{e^{-x_2^2}}{\sqrt{2\pi}}dx_2$$

$$+\int_{-\infty}^{+\infty}\alpha_{x_1}^{3m+s-1}\frac{e^{-\frac{3}{2}x_1^2}}{\sqrt{2\pi}}dx_1\Big\}$$

$$=-\frac{3m+s}{2}T_{3m+s-1}+(2m+r+1)\{(m+s-r-1)\,{}_{m+s-r-2}I_{2m+r}+T_{3m+s-1}\} \quad \text{......(12)}.$$

On substituting (10), (11) and (12) into (9) we obtain, on collecting terms,

$$P_{mq}=\frac{n!}{(2m)!\,m!\,(m-1)!\,4\pi}\sum_{s=0}^{m}(-1)^s\,{}_mC_s\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\Big\{\frac{2m(m+s)}{r+1}\,{}_{m+s-1}I_{2m-1}$$
$$-\frac{(2m-r)(2m+r+1)(m+s-r-1)}{(2m+1)(r+1)}\,{}_{m+s-r-2}I_{2m+r}-\frac{3m+s-2r}{4m+2}T_{3m+s-1}\Big\}$$
$$\text{for } m\geqq 1 \quad \text{......(13)}.$$

(c) *Case* $n=4m+1$.

Here the median corresponds to an observation value, but the quartile point lies midway between two observations. P_{mq} is therefore the mean value of the product $\{x_3\times\frac{1}{2}(x_2+x_1)\}$, or

$$P_{mq}=\frac{n!}{(2m)!\,\{(m-1)!\}^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\alpha_{x_3}^{2m}(\alpha_{x_2}-\alpha_{x_3})^{m-1}(1-\alpha_{x_1})^{m-1}$$
$$\times x_3\tfrac{1}{2}(x_2+x_1)\,d\alpha_{x_3}d\alpha_{x_2}d\alpha_{x_1} \quad \text{......(14)}.$$

Following a similar process of development to that used in the previous cases, it may be shown finally that

$$P_{mq}=\frac{n!}{(2m)!\,\{(m-1)!\}^2\,4\pi}\sum_{s=0}^{m-1}(-1)^s\,{}_{m-1}C_s\sum_{r=0}^{m-1}(-1)^r\,{}_{m-1}C_r\frac{2m+r}{(m-r)(s+1)}$$
$$\times\{(s+m-r)(s-m+r+1)\,{}_{m+s-r-1}I_{2m+r-1}$$
$$+(m-r)(m-r-1)\,{}_{m-r-2}I_{2m+r-1}-s(s+1)\,{}_{s-1}I_{3m-1}$$
$$+(m+r)(T_{3m-2}-T_{3m+s-1})\}\quad \text{for } m\geqq 2 \quad \text{......(15)}.$$

(d) *Case* $n=4m$.

Here both median and quartile fall midway between observations, and P_{mq} is the mean value of the product $\{\frac{1}{2}(x_4+x_3)\times\frac{1}{2}(x_2+x_1)\}$, or

$$P_{mq}=\frac{n!}{(2m-1)!\,(m-2)!\,(m-1)!}\int_{-\infty}^{+\infty}\int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3}\alpha_{x_4}^{2m-1}(\alpha_{x_2}-\alpha_{x_3})^{m-2}(1-\alpha_{x_1})^{m-1}$$
$$\times\tfrac{1}{2}(x_4+x_3)\tfrac{1}{2}(x_2+x_1)\,d\alpha_{x_4}d\alpha_{x_3}d\alpha_{x_2}d\alpha_{x_1} \quad \text{......(16)},$$

and it can be shown after a rather lengthy process of reduction that we obtain, as a final result,

$$P_{mq}=\frac{n!}{(2m-1)!\,(m-2)!\,(m-1)!\,16m\pi}\sum_{s=0}^{m-1}(-1)^s\,{}_{m-1}C_s\sum_{r=0}^{m-2}(-1)^r\,{}_{m-2}C_r$$
$$\times\Big\{\frac{(m+s-r-1)(s-m+r+2)(2m+r)(r-2m+1)}{(r+1)(s+1)(m-r-1)}\,{}_{m+s-r-2}I_{2m+r-1}$$

$$+\frac{(m-r-2)(2m+r)(r-2m+1)}{(s+1)(r+1)}\,{}_{m-r-3}I_{2m+r-1}+\frac{(3m-1)s}{m-r-1}\,{}_{s-1}I_{3m-2}$$
$$+\frac{2m(2m-1)(m-1)}{(r+1)(s+1)}\,{}_{m-2}I_{2m-2}$$
$$+\frac{2m(2m-1)(m+s)}{r+1}\left(\frac{1}{m}-\frac{1}{s+1}\right){}_{m+s-1}I_{2m-2}+\frac{2r-3m+2}{2(s+1)}T_{3m-3}$$
$$\left.+\frac{3m-s-2r-3}{2(s+1)}T_{3m+s-2}\right\}\text{ for } m \geq 2 \text{ ....(17).}$$

From the preceding equations the values of P_{mq} may be calculated for $n=1, 2, \ldots 12$, using the numerical values of the integral forms T_r, I_r and ${}_pI_r$ previously computed*. The values of σ_m and σ_q required to find r_{mq} have already been calculated. The results are collected in Table II.

TABLE II.

Size of sample, n	P_{mq}	σ_m	σ_q	$r_{mq}=P_{mq}/\sigma_m\sigma_q$			
1	1·00000	1·00000	1·00000	1·0000			
2	·50000	·70711	·82565		·8564		
3	·27566	·66983	·74798			·5502	
4	·25000	·54608	·57952				·7900
5	·17629	·53557	·54948	·6058			
6	·16432	·46340	·52875		·6706		
7	·13074	·45874	·50669			·5625	
8	·12354	·41011	·43777				·6881
9	·10354	·40755	·42436	·5990			
10	·09842	·37214	·41835		·6322		
11	·08590	·37044	·40708			·5696	
12	·08259	·34346	·36777				·6539

4. *Limiting values for r_{mq}.*

We may use again the method given by K. Pearson †; four cases will arise as before.

(a) *Case* $n=4m+3$.

Here the median and quartile points correspond to observations, and the result has been given in the paper just mentioned, i.e.

$$r_{mq}=\sqrt{\tfrac{1}{3}}=\text{·}5774 \text{ ....(18).}$$

(b) *Case* $n=4m+2$.

Here the median and quartile are of form $\frac{1}{2}(x_3+x_2)$ and x_1 respectively, and

$$P_{mq}=\frac{1}{2}\{\text{Mean}(x_2x_1)+\text{Mean}(x_3x_1)\}$$
$$=\frac{1}{2}\left\{\frac{\bar{\sigma}^2}{nz_hz_k}\frac{s_2}{n}\left(1-\frac{s_1}{n}\right)+\frac{\bar{\sigma}^2}{nz_hz_k}\frac{s_3}{n}\left(1-\frac{s_1}{n}\right)\right\},$$

* See Table I. pp. 325–326, *loc. cit.*

† *Biometrika*, Vol. XIII. pp. 115—117

where $$\frac{s_3}{n}=\frac{1}{2}+\frac{1}{2n},\quad \frac{s_2}{n}=\frac{1}{2}-\frac{1}{2n},\quad \frac{s_1}{n}=\frac{3}{4},$$

and z_h and z_k are the tabled ordinates of the normal probability function, corresponding to the proportional areas

$$\frac{1}{2}(1+\alpha_h)=\frac{1}{2}\left(1+\frac{1}{n}\right),\quad \frac{1}{2}(1+\alpha_k)=\frac{3}{4}.$$

It follows that $$P_{mq}=\frac{\tilde{\sigma}^2}{8n}\frac{1}{z_h z_k}\quad\text{.................(19).}$$

Further, from equations (1·2) and (15) of my previous paper, we know that

$$\sigma_m=\frac{\tilde{\sigma}}{\sqrt{n}}\frac{1}{2z_h}\sqrt{1-\frac{1}{n}}\quad\text{.................(20),}$$

$$\sigma_q=\frac{\tilde{\sigma}}{\sqrt{n}}\frac{1}{2z_k}\sqrt{\frac{3}{4}}\quad\text{.................(21).}$$

Values of r_{mq} may be calculated from (19), (20) and (21). For example

$$\text{if } n=50\quad \begin{cases}\alpha_h=\cdot 02,\quad z_h=\cdot 3988169,\quad z_k=\cdot 3177766,\\ P_{mq}=\cdot 986313\,\dfrac{\tilde{\sigma}^2}{n},\quad \sigma_m=1\cdot 24111\,\dfrac{\tilde{\sigma}}{\sqrt{n}},\quad \sigma_q=1\cdot 36263\,\dfrac{\tilde{\sigma}}{\sqrt{n}},\\ r_{mq}=\cdot 5832,\end{cases}$$

while if $$n=102,\quad r_{mq}=\cdot 5802.$$

(c) *Case* $n=4m+1$.

The median and quartile are now of form x_3 and $\frac{1}{2}(x_2+x_1)$ respectively, and it is found that

$$P_{mq}=\frac{\tilde{\sigma}^2}{16n}\frac{1}{z_h}\left\{\frac{1}{z_{k_1}}\left(1-\frac{2}{n}\right)+\frac{1}{z_{k_2}}\left(1+\frac{2}{n}\right)\right\}\quad\text{.................(22),}$$

where z_{k_1} and z_{k_2} are the ordinates corresponding to

$$\frac{1}{2}(1+\alpha_{k_1})=\frac{3}{4}+\frac{1}{2n},\quad \frac{1}{2}(1+\alpha_{k_2})=\frac{3}{4}-\frac{1}{2n}.$$

Further, $$\sigma_m=1\cdot 25331\,\frac{\tilde{\sigma}}{\sqrt{n}}\quad\text{.................(23).}$$

$$\sigma_q{}^*=\frac{\tilde{\sigma}}{\sqrt{n}}\frac{\sqrt{3}}{8}\left\{\frac{1}{z_{k_1}{}^2}\left(1-\frac{4}{3n}-\frac{4}{3n^2}\right)+\frac{1}{z_{k_2}{}^2}\left(1+\frac{4}{3n}-\frac{4}{3n^2}\right)+\frac{2}{z_{k_1}z_{k_2}}\left(1-\frac{8}{3n}+\frac{4}{3n^2}\right)\right\}^{\frac{1}{2}}$$
$$\text{......(24).}$$

If $$n=49,\quad r_{mq}=\cdot 5850,$$
$$n=101,\quad r_{mq}=\cdot 5811.$$

[* The limiting value of σ_q given in Equation (18·1), *Biometrika*, Vol. XXIII, p. 333—Professor Hojo's previous paper—appears to be in error. I do not follow the values given for s_1, s_2, H_1 and H_2. On p. 346 the correct values are given for the s's, but the values of the H's, differing from those on p. 333, appear to be in error. The limiting values for $n=\infty$ would not, however, be affected. ED.]

(*d*) *Case* $n=4m$.

The median and quartile are of form $\frac{1}{2}(x_4+x_3)$ and $\frac{1}{2}(x_2+x_1)$ respectively and again as in (22) it is found that

$$P_{mq}=\frac{\tilde{\sigma}^2}{16nz_h}\left\{\frac{1}{z_{k_1}}\left(1-\frac{2}{n}\right)+\frac{1}{z_{k_2}}\left(1+\frac{2}{n}\right)\right\} \quad \text{.................(25)},$$

where z_h, z_{k_1} and z_{k_2} are as before, while σ_m and σ_q are found from (20) and (24).

If
$$n=52,\quad r_{mq}=\cdot 5903.$$
$$n=100,\quad r_{mq}=\cdot 5841.$$
$$n=500,\quad r_{mq}=\cdot 5787.$$

We see that as n increases, the values of r_{mq} in cases (*b*), (*c*) and (*d*) tend to the limiting value of case (*a*), viz. $r_{mq}=\cdot 5774$.

5. *Empirical formulae to bridge the Gap between limiting and small Sample Values.*

In the previous paper the following formulae were given to provide values for σ_m between the computed series ($n\leqslant 12$) and the limiting value:

$$\left.\begin{aligned}\sigma_m\Big/\frac{\tilde{\sigma}}{\sqrt{n}}&=1\cdot2533-\cdot2653/n-\cdot0699/n^2+\cdot0822/n^3 && \text{for } n \text{ odd}\\ &=1\cdot2533-\cdot8261/n+\cdot7826/n^2-\cdot3478/n^3+\cdot1304/n^4 && \text{for } n \text{ even}\end{aligned}\right\}\text{...(26)}$$

These "best-fitting" asymptotic curves were obtained using the method of moments. Similar equations have been calculated for both σ_q and r_{mq}, but as here there were only three computed points for each of the four cases, the curves actually pass through these points The equations are as follows:

$$\left.\begin{aligned}\sigma_q\Big/\frac{\tilde{\sigma}}{\sqrt{n}}&=1\cdot3626-1\cdot2115/n+1\cdot8693/n^2-1\cdot1251/n^3, && \text{for } n=4m\\ &=1\cdot3626-1\cdot0046/n+1\cdot9330/n^2-1\cdot2910/n^3, && \text{for } n=4m+1\\ &=1\cdot3626-\cdot3800/n-\cdot2110/n^2+\cdot3819/n^3, && \text{for } n=4m+2\\ &=1\cdot3626-\cdot1055/n-\cdot3805/n^2+\cdot2789/n^3, && \text{for } n=4m+3\end{aligned}\right\}\text{...(27)}.$$

$$\left.\begin{aligned}r_{mq}&=\cdot5774+1\cdot0150/n-1\cdot4064/n^2+2\cdot9952/n^3, && \text{for } n=4m\\ &=\cdot5774+\cdot2842/n-\cdot9215/n^2+1\cdot0600/n^3, && \text{for } n=4m+1\\ &=\cdot5774+\cdot5250/n+\cdot2778/n^2-\cdot4230/n^3, && \text{for } n=4m+2\\ &=\cdot5774-\cdot0269/n-\cdot8210/n^2+1\cdot9722/n^3, && \text{for } n=4m+3\end{aligned}\right\}\text{...(28)}.$$

A combination of results is given in Table III, consisting of (*a*) correct computed values, (*b*) limiting values obtained by the method of section (4) above, (*c*) results calculated from equations (26), (27) and (28).

The column dealing with σ_{q-m} is referred to in the following section.

TABLE III.

(*Cases:* A, $n = 4m + 1$; B, $n = 4m + 2$; C, $n = 4m + 3$; D, $n = 4m$.)

Case	Size of sample, n	$\sigma_m \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$	$\sigma_q \Big/ \frac{\tilde{\sigma}}{\sqrt{n}}$	r_{mq}	$\sigma_{q-m} \Big/ \frac{\tilde{\sigma}}{\sqrt{2n}}$
A	1	1·0000	1·0000	1·0000	0·0000
B	2	1·0000	1·1676	·8564	0·8525
C	3	1·1602	1·2955	·5502	1·6556
D	4	1·0922	1·1590	·7900	1·0355
A	5	1·1976	1·2287	·6058	1·5237
B	6	1·1351	1·2952	·6706	1·4100
C	7	1·2137	1·3406	·5625	1·6970
D	8	1·1600	1·2382	·6881	1·3431
A	9	1·2226	1·2731	·5990	1·5816
B	10	1·1768	1·3229	·6322	1·5275
C	11	1·2286	1·3501	·5696	1·6986
D	12	1·1898	1·2740	·6539	1·4536
A	13	1·2325	1·2962	·5942	1·6129
B	14	1·1982	1·3345	·6161	1·5788
C	15	1·2353	1·3540	·5725	1·6995
D	16	1·2047	1·2939	·6360	1·5118
A	17	1·2375	1·3100	·5911	1·6316
B	18	1·2098	1·3409	·6073	1·6070
C	19	1·2392	1·3561	·5739	1·7005
D	20	1·2139	1·3066	·6250	1·5480
A	25	1·2426	1·3255	·5873	1·6531
B	30	1·2266	1·3497	·5951	1·6467
C	35	1·2457	1·3593	·5760	1·7022
D	40	1·2331	1·3335	·6019	1·6244
A	45	1·2474	1·3412	·5832	1·6754
	(49)	(1·2533)	(1·3449)	(·5850)	(1·6777)
B	50	1·2371	1·3549	·5880	1·6703
	(50)	(1·2411)	(1·3626)	(·5832)	(1·6879)
D	(52)	(1·2416)	(1·3459)	(·5903)	(1·6614)
C	55	1·2485	1·3606	·5766	1·7035
D	60	1·2398	1·3430	·5939	1·6511
C	99	1·2506	1·3615	·5770	1·7046
	(99)	(1·2533)	(1·3626)	(·5774)	(1·7062)
D	100	1·2451	1·3507	·5874	1·6727
	(100)	(1·2471)	(1·3538)	(·5841)	(1·6827)
A	101	1·2507	1·3529	·5801	1·6920
	(101)	(1·2533)	(1·3539)	(·5811)	(1·6921)
B	102	1·2453	1·3589	·5825	1·6887
	(102)	(1·2473)	(1·3626)	(·5802)	(1·6972)
D	500	1·2527	1·3608	·5787	1·7018
	(500)	(1·2521)	(1·3609)	(·5787)	(1·7015)
	∞	1·2533	1·3626	·5774	1·7062

The numbers up to $n = 12$ are exact, those in brackets are the limiting values, obtained by the methods of Section (4), and the remainder are obtained from the Equations of Section (5).

6. *The Standard Error of the Distance between the Median and Quartile Points.*

The mean value of this distance in repeated samples is clearly the mean value of the distance of the quartile point from the population mean, or $\bar{q}\tilde{\sigma}$. A table

of $\bar{q}$ was given in my previous paper (*loc. cit.* p. 341), and this was extended by K. Pearson (*loc. cit.* p. 372).

The standard error of $(q-m)$ may be calculated from the following relation

$$\sigma^2_{q-m} = \sigma_q^2 + \sigma_m^2 - 2r_{mq}\sigma_q\sigma_m \quad \text{.......................(29)},$$

which has been used to give the values of the ratio of σ_{q-m} to the approximate standard error of a standard deviation, $\tilde{\sigma}/\sqrt{2n}$, shown in the last column of Table III.

On p. 358 of my other paper two estimates of the population standard deviation, $\tilde{\sigma}$, were compared; E_1 based on the sample standard deviation, and E_2 based on the interquartile distance. Both estimates were so adjusted that their mean values in repeated samples would be $\tilde{\sigma}$. It is possible to obtain a third estimate from the distance, $q-m$, between the median point and a quartile point, namely

$$E_3 = (q-m)/\bar{q} \quad \text{.................................(30)}$$

We shall have

$$\text{Mean } E_3 = \tilde{\sigma} \quad \text{.................................(31)},$$

$$\sigma_{E_3} = \frac{\sigma_{q-m}}{\bar{q}} = \frac{1}{\bar{q}} \times \frac{\sigma_{q-m}}{\dfrac{\tilde{\sigma}}{\sqrt{2n}}} \times \frac{\tilde{\sigma}}{\sqrt{2n}} = \theta_3 \times \frac{\tilde{\sigma}}{\sqrt{2n}} \quad \text{..............(32)}.$$

Values of θ_3 and of the corresponding multipliers for E_1 and E_2 are shown in Table IV.

TABLE IV.

Comparison of Standard Errors of Estimates.

n	θ_1	θ_2	θ_3
2	1·511	1·511	1·511
3	1·280	1·280	1·956
4	1·194	1·250	1·561
5	1·148	1·185	1·838
6	1·120	1·591	2·197
7	1·100	1·469	2·241
8	1·086	1·429	2·027
9	1·076	1·360	2·103
10	1·068	1·598	2·328
11	1·061	1·526	2·331
12	1·056	1·498	2·187
∞	1·000	1·649	2·530

It is clear that E_3 would be an estimate of $\tilde{\sigma}$ of very little value, and one which is distinctly worse than E_2. In addition to its large standard error, it is far from normally distributed (see Table V below).

7. *Comparison of interpolated Values with those found by K. Pearson's Method.*

Certain of the values of $r_{m,q}$ and σ_q obtained from the empirical equation of Section (5) were recomputed by K. Pearson's method*, in order to obtain some measure of their accuracy. The appropriate formula for the mean value of the product $\{x_1 x_2\}$ of two ranked individuals is his (xxi)^{ter}. Taking the case $n = 4p + 3$, for which both median and quartile points correspond to observations, we obtain using his notation, in which m refers to the median,

$$\text{Mean } \{x_m x_q\}/\sigma^2 = b_0 {}_q\bar{v}_n - b_1 \frac{n-m+1}{n+1} {}_q\bar{v}_{n+1} + b_2 \frac{(n-m+1)(n-m+2)}{(n+1)(n+2)} {}_q\bar{v}_{n+2}$$
$$- b_3 \frac{(n-m+1)(n-m+2)(n-m+3)}{(n+1)(n+2)(n+3)} {}_q\bar{v}_{n+3} + \dots \quad \dots\dots(33),$$

where he has given numerical values for the b's up to b_{13} (*loc. cit.* Equations (xi) and (xii)).

Case (i). Samples of 15; $n = 15$, $q = 4$, $m = 8$.

After calculations similar to those carried out by K. Pearson (*loc. cit.* pp. 388—389) I find that

$$\text{Mean } \{x_m x_q\}/\tilde{\sigma}^2 = \cdot 063,271,899.$$

Further, σ_q and σ_m may be obtained from his formula (xvii) and Table VI respectively, as follows,

$$\sigma_q/\tilde{\sigma} = \cdot 347,0143, \quad \sigma_m/\tilde{\sigma} = \cdot 318,692.$$

The former leads to a value of 1·34398 for the ratio of σ_q to $\tilde{\sigma}/\sqrt{n}$, which may be compared with the value of 1·3540 obtained from my formula (27) above.

Since $\bar{x}_m = 0$, it follows that

$$r_{mq} = \text{Mean } \{x_m x_q\}/(\sigma_m \sigma_q) = \cdot 57213,$$

which may be compared with the value of ·5725 obtained from formula (28), and given in my Table III.

Case (ii). Samples of 35; $n = 35$, $q = 9$, $m = 18$.

Here I find

$$\text{Mean } \{x_m x_q\}/\tilde{\sigma}^2 = \cdot 027,753,212,$$

and from K. Pearson's formulae (xvii) and (xviii),

$$\sigma_q/\tilde{\sigma} = \cdot 229,4863, \quad \sigma_m/\tilde{\sigma} = \cdot 210,5396.$$

Hence $\sigma_q \Big/ \frac{\tilde{\sigma}}{\sqrt{n}} = 1\cdot 3577$ against my 1·3593, and $\sigma_m \Big/ \frac{\tilde{\sigma}}{\sqrt{n}} = 1\cdot 2456$ against my 1·2457.

It follows also that $\qquad r_{mq} = \cdot 57441,$

whereas my formula (28) has given a value ·5760.

* *Biometrika*, Vol. XXIII. pp. 364—390.

There is little doubt that K. Pearson's method is likely to be more accurate than that which I have employed for interpolating between $n = 12$ and the limiting values, but it involves calculations of some length even in the simplest case when $n = 4p + 3$. For the cases $n = 4p + 2$, $n = 4p + 1$ and $n = 4p$ the work involved would be very much longer, and no comparison has been attempted.

8. *Some experimental Sampling Results.*

The following further results have been shown for comparison.

TABLE V.

Size of Sample, n (No. of Samples, N)	4 (1000)	7 (1000)	10 (1000)	15 (1000)	20 (1000)	30 (500)	40 (500)
r_{mq} Experiment	·8011	·5627	·5933	·5290	·6250	·6030	·5538
r_{mq} Theory ...	·7908	·5625	·6322	·5725	·6250	·5951	·6011
$\sigma_{q-m} \Big/ \frac{\tilde{\sigma}}{\sqrt{2n}}$ Experiment	1·0451	1·7496	1·5920	...	1·5290	1·6980	1·6876
$\sigma_{q-m} \Big/ \frac{\tilde{\sigma}}{\sqrt{2n}}$ Theory ...	1·0355	1·6970	1·5275		1·5180	1·6467	1·6244
β_1 for $(q-m)$, Experiment	·5463	·8088	·7907		·4088	·3526	·1637
β_2 " "	3·4345	3·9728	3·7066		3·5659	3·0110	2·9905

The values of σ_{q-m} were calculated directly from the data (except in the case $n = 15$), and it is found that none of them differs from the theoretical values by more than twice the appropriate standard error. r_{mq} was then calculated from equation (29), using the experimental values for σ_q and σ_m tabled in my earlier paper. In the case of $n = 15$, r_{mq} was calculated directly from the data.

It will be seen from the values of β_1 and β_2 that the sampling distribution of $q - m$ is far from normal, even where the samples are large, and as has been pointed out the estimate of $\tilde{\sigma}$ obtainable from this inter-rank distance is not a satisfactory one.

A FURTHER STUDY OF METHODS OF CONSTRUCTING LIFE TABLES WHEN CERTAIN CAUSES OF DEATH ARE ELIMINATED.

By M. NOEL KARN, M.A.

In a recent paper entitled "An Inquiry into Various Death-rates and the Comparative Influence of certain Diseases on the Duration of Life"* D'Alembert's method was applied to the construction of life tables for a population from which cancer and tuberculosis were supposed to be eliminated as causes of death in order to estimate the effect of these diseases in shortening the duration of life. In that paper no reference was made to the work on this problem by Dr Farr towards the end of last century, nor to Louis I. Dublin's work in recent years. In the present paper a comparison has been made of the results of the several methods when applied to the same data in order to determine what, if any, are the practical advantages of employing D'Alembert's formula over others which have been used. As to the theoretical advantages of D'Alembert's formula, I think there can be no question.

Comparison between Results of Farr's Method and D'Alembert's Formula.

Dr Farr's work was published in the *Supplement to the Thirty-Fifth Annual Report of the Registrar-General*, 1875, and entitled "Effect of the Extinction of any single Disease on the Duration of Life†." In this he made passing reference to the previous work on the same kind of problem in connection with the controversy over inoculation, mentioning Daniel Bernoulli and D'Alembert, giving Duvillard's value for the increase in the mean life time which would result from the extinction of small-pox. He then referred to the short method which he had described in the Appendix to the R.-G.'s Fifth Annual Report‡, using it for this particular purpose as being sufficiently exact, and thus saving the labour of constructing and graduating full life tables.

In his short method Farr made use of quinquennial age-groups. The number of deaths in age-group x to $x+5$ years divided by the population of that age-group gave the probability p of living one year in the middle of the period. The fifth power was therefore taken in order to obtain the number of survivors at the end of the period. Thus the chance of living for five years at age x was p^5.

* *Annals of Eugenics*, Vol. IV. pp. 279—326, 1931.

† P. xxxviii. § 21.

‡ P. 352, "A short method of constructing Life Tables."

The method was only applicable for ages after 5 years. The survivors at age 5 would have to be obtained from the known deaths and populations year by year for the ages before 5.

Farr attempted the problem of eliminating a particular disease from the life table population, taking, among other diseases, as examples, cancer and phthisis. To do this in the case of cancer he constructed a new life table on the basis of the mortality from all diseases except cancer, viz. $m_x - c_x$*, using the short method above described. This he assumed gave a life table population as it would be if there were no cancer mortality.

I have first tried to ascertain how closely the numbers of survivors and expectations of life at different ages obtained by this method approximate to the more exact results obtained by D'Alembert's full formula.

The amount of error due to the approximations used in the short method of constructing an ordinary life table was investigated by Farr and is given in Table 58 of the *Supplement*† for the case of the English Life Table for Males. The result is an excess in the short method in the expectation of life, to the amount of ·37 to ·53 of a year in the mid-age groups up to 65, followed by a rapid increase in the excess at the older ages.

Table 60 of the same *Supplement*† gives survivors of a life table based on the deaths for the years 1861—70, and also the survivors for a life table with cancer excluded. I have worked out, and show in Table I, the expectation of life for both these tables for comparison with the results tabulated in Table VI of my previous paper‡.

TABLE I.

Life Tables for Males (calculated from the Facts recorded during 1861—70).

Age x	To die of all Diseases		Cancer excluded		Increase in $\mathring{e}_x$
	l_x	$\mathring{e}_x$	l_x	$\mathring{e}_x$	
0	510 622	40·55	510 622	40·75	·20
5	367 817	50·32	367 835	50·65	·33
10	353 129	47·31	353 165	47·65	·34
15	345 341	43·32	345 393	43·61	·29
20	334 867	39·60	334 951	39·90	·30
25	321 013	36·20	321 126	36·51	·31
35	290 755	29·45	291 032	29·76	·31
45	254 138	22·97	254 890	23·27	·30
55	209 825	16·77	211 503	17·02	·25
65	150 844	11·37	153 945	11·52	·15
75	77 409	7·41	80 501	7·46	·05
85	17 826	5·45	18 970	5·44	·01
95	789	5·11	823	5·11	·00
105	9	—	9	—	—

* See Table 59, p. clxix, *Supplement to Thirty-Fifth Annual Report of R.-G.*, 1861—70.
† *Ibid.* ‡ *Annals of Eugenics*, Vol. IV. p. 309.

The order of the differences in the last column of this table is never greater than about a third of a year for the period 1861—70. Whatever method is used, the results show that these differences have increased to nearly one and a half years in mid-life for the period 1919—23.

I turned next to the modern data and have found the effect of the elimination of cancer mortality from Life Table 9, applying Farr's short method instead of the formula of D'Alembert.

In Table II are the results obtained by Farr's short method from the data to which the full formula of D'Alembert was applied in my original paper.

TABLE II.

Comparison of the Increases in Expectation of Life due to the Elimination of Cancer Mortality, calculated by two Methods.

Age	$\mathring{e}_x$ for Life Table 9	$\mathring{e}_x$ for Life Table with Cancer eliminated by use of D'Alembert's formula	Increase in $\mathring{e}_x$ by D'Alembert's formula	$\mathring{e}_x$ for Life Table with Cancer eliminated by use of Farr's short method	Increase in $\mathring{e}_x$ by Farr's short method	Excess by Farr's method	Percentage excess on increase in expectation of life due to using Farr's method
0	55·62	56·89	1·27	56·97	1·35	·08	6
5	58·81	60·25	1·44	60·35	1·54	·10	7
10	54·64	56·10	1·46	56·19	1·55	·09	6
15	50·12	51·59	1·47	51·68	1·56	·09	6
20	45·78	47·26	1·48	47·37	1·59	·11	7
25	41·60	43·09	1·49	43·21	1·61	·12	8
30	37·40	38·91	1·51	39·03	1·63	·12	8
35	33·25	34·77	1·52	34·88	1·63	·11	7
40	29·19	30·71	1·52	30·84	1·65	·13	9
45	25·22	26·72	1·50	26·85	1·63	·13	9
50	21·36	22·79	1·43	22·93	1·57	·14	10
55	17·73	19·02	1·29	19·17	1·44	·15	12
60	14·36	15·45	1·09	15·63	1·27	·18	17
65	11·36	12·20	·84	12·38	1·02	·18	21
70	8·75	9·34	·59	9·53	·78	·19	32
75	6·59	6·96	·37	7·20	·61	·24	65
80	4·93	5·14	·21	5·40	·47	·26	124

The difference due to using the short method ranges from ·08 to ·26 of a year in excess, and would increase the additional expectation of life by the elimination of cancer from a maximum value of 1·52 years at 40 to 1·65 years, that is, by 9 per cent.

An estimate of the amount of error permissible by using a short method is given by Dr Snow* as ·08 of a year, and if this is to be taken as a criterion the evaluation of the increase in the expectation of life in this special problem would not be

* "An Elementary Rapid Method of Constructing an Abridged Life Table," E. C. Snow. *Supplement to the Seventy-Fifth Annual Report of the R.-G.*, Part II.

TABLE III.

Comparison of the Increases in Expectation of Life due to the Elimination of Pulmonary Tuberculosis Mortality, calculated by two Methods.

Age	$\mathring{e}_x$ for Life Table 9	$\mathring{e}_x$ for Life Table with Pulmonary Tuberculosis eliminated by use of D'Alembert's formula	Increase in $\mathring{e}_x$ by D'Alembert's formula	$\mathring{e}_x$ for Life Table with Pulmonary Tuberculosis eliminated by use of Farr's short method	Increase in $\mathring{e}_x$ by Farr's short method	Excess by Farr's method	Percentage excess on increase in expectation of life due to using Farr's method
0	55·62	57·40	1·78	57·48	1·86	·08	5
5	58·81	60·79	1·98	60·89	2·08	·10	5
10	54·64	56·63	1·99	56·72	2·08	·09	5
15	50·12	52·07	1·95	52·17	2·05	·10	5
20	45·78	47·58	1·80	47·70	1·92	·12	7
25	41·60	43·13	1·53	43·24	1·64	·11	7
30	37·40	38·67	1·27	38·79	1·39	·12	9
35	33·25	34·28	1·03	34·41	1·16	·13	13
40	29·19	29·98	·79	30·12	·93	·14	18
45	25·22	25·80	·58	25·94	·72	·14	24
50	21·36	21·76	·40	21·91	·55	·15	38
55	17·73	17·99	·26	18·15	·42	·16	62
60	14·36	14·51	·15	14·71	·35	·20	133
65	11·36	11·44	·08	11·64	·28	·20	
70	8·75	8·78	·03	9·01	·26	·23	
75	6·59	6·60	·01	6·91	·32	·31	
80	4·93	4·93	·00	5·31	·38	·38	...

TABLE IV.

Comparison of the Increases in Expectation of Life due to the Elimination of Heart Diseases Mortality, calculated by two Methods.

Age	$\mathring{e}_x$ for Life Table 9	$\mathring{e}_x$ for Life Table with Heart Diseases eliminated by use of D'Alembert's formula	Increase in $\mathring{e}_x$ by D'Alembert's formula	$\mathring{e}_x$ for Life Table with Heart Diseases eliminated by use of Farr's short method	Increase in $\mathring{e}_x$ by Farr's short method	Excess by Farr's method	Percentage excess on increase in expectation of life due to using Farr's method
0	55·62	57·32	1·70	57·41	1·79	·09	5
5	58·81	60·74	1·93	60·85	2·04	·11	6
10	54·64	56·58	1·94	56·67	2·03	·09	5
15	50·12	52·03	1·91	52·12	2·00	·09	5
20	45·78	47·67	1·89	47·78	2·00	·11	6
25	41·60	43·45	1·85	43·57	1·97	·12	7
30	37·40	39·20	1·80	39·32	1·92	·12	7
35	33·25	35·01	1·76	35·11	1·86	·10	6
40	29·19	30·90	1·71	31·02	1·83	·12	7
45	25·22	26·89	1·67	27·00	1·78	·11	7
50	21·36	22·97	1·61	23·10	1·74	·13	8
55	17·73	19·26	1·53	19·39	1·66	·13	9
60	14·36	15·77	1·41	15·94	1·58	·17	12
65	11·36	12·61	1·25	12·78	1·42	·17	14
70	8·75	9·77	1·02	9·96	1·21	·19	19
75	6·59	7·39	·80	7·62	1·03	·23	29
80	4·93	5·50	·57	5·76	·83	·26	46

sufficiently accurately worked by the short method of Farr, especially when a comparison is to be made of the alteration in the expectation of life due to the elimination of cancer mortality for two or three past decades.

Farr's short method applied for the elimination of Pulmonary Tuberculosis and Heart Diseases separately from the ordinary life table shows differences in expectation of life ranging again from ·08 of a year at birth to ·38 in the case of Pulmonary Tuberculosis, and from ·09 to ·26 in the case of Heart Diseases when compared with the fuller method, as shown in Tables III and IV.

These differences become large in the later ages when regarded as percentage excess on the increase in expectation of life, especially in the Pulmonary Tuberculosis investigation. This shows that the method of Farr is not accurate enough for this purpose.

Re-calculation in five-yearly Groups of the Data on which Life Table 9 is based.

In estimating the additional expectation of life resulting from eliminating a cause of death by the shorter method, the calculation should perhaps be made not from the standard life table, but from one based on the same data but calculated also by the same method. The comparative values of the normal expectation of life calculated on this basis are set out in Table V.

TABLE V.

Expectation of Life.

Age	For Life Table 9	For Life Table calculated in five-yearly periods	Excess by Farr's method
0	55·62	55·71	·09
5	58·81	58·91	·10
10	54·64	54·74	·10
15	50·12	50·21	·09
20	45·78	45·89	·11
25	41·60	41·72	·12
30	37·40	37·52	·12
35	33·25	33·37	·12
40	29·19	29·32	·13
45	25·22	25·36	·14
50	21·36	21·51	·15
55	17·73	17·89	·16
60	14·36	14·56	·20
65	11·36	11·57	·21
70	8·75	8·98	·23
75	6·59	6·90	·31
80	4·93	5·31	·38
85	3·72	4·14	·42

In Table VI the increases in the expectation of life due to the elimination of the several diseases considered are shown, the results by Farr's method being compared with the standard life table re-calculated by the same method. The percentage

difference on the increase is now diminished, as compared with the results in Tables II, III and IV, to an amount less than one, throughout the table in the case of Pulmonary Tuberculosis, and to age 60 in the table with Cancer eliminated.

TABLE VI.

Increase in Expectation of Life.

Age	With Cancer eliminated			With Pulmonary Tuberculosis eliminated			With Heart Diseases eliminated		
	By Farr's method compared with re-calculated Life Table	As in Table II	By D'Alembert's formula compared with Life Table 9	By Farr's method compared with re-calculated Life Table	As in Table III	By D'Alembert's formula compared with Life Table 9	By Farr's method compared with re-calculated Life Table	As in Table IV	By D'Alembert's formula compared with Life Table 9
0	1·26	1·35	1·27	1·77	1·86	1·78	1·70	1·79	1·70
5	1·44	1·54	1·44	1·98	2·08	1·98	1·94	2·04	1·93
10	1·45	1·55	1·46	1·98	2·08	1·99	1·93	2·03	1·94
15	1·47	1·56	1·47	1·96	2·05	1·95	1·91	2·00	1·91
20	1·48	1·59	1·48	1·81	1·92	1·80	1·89	2·00	1·89
25	1·49	1·61	1·49	1·62	1·64	1·53	1·85	1·97	1·85
30	1·51	1·63	1·51	1·27	1·39	1·27	1·80	1·92	1·80
35	1·51	1·63	1·52	1·04	1·16	1·03	1·74	1·86	1·76
40	1·52	1·65	1·52	·80	·93	·79	1·70	1·83	1·71
45	1·49	1·63	1·50	·58	·72	·58	1·64	1·78	1·67
50	1·42	1·57	1·43	·40	·55	·40	1·59	1·74	1·61
55	1·28	1·44	1·29	·26	·42	·26	1·50	1·66	1·53
60	1·07	1·27	1·09	·15	·35	·16	1·38	1·58	1·41
65	·81	1·02	·84	·07	·28	·08	1·21	1·42	1·26
70	·55	·78	·59	·03	·26	·03	·98	1·21	1·02
75	·30	·61	·37	·01	·32	·01	·72	1·03	·80
80	·09	·47	·21	·00	·38	·00	·45	·83	·57

Table VII shows the percentage difference for some of the later ages in the tables with Cancer or Heart Diseases eliminated, Farr's method now giving a defect.

TABLE VII.

Age	With Cancer eliminated		With Heart Diseases eliminated	
	Defect in increase in expectation of life due to Farr's method	Percentage defect	Defect in increase in expectation of life due to Farr's method	Percentage defect
45	·01	1	·03	2
50	·01	1	·02	1
55	·01	1	·03	2
60	·02	2	·03	2
65	·03	4	·04	3
70	·04	7	·04	4
75	·07	19	·08	10
80	·12	57	·12	21

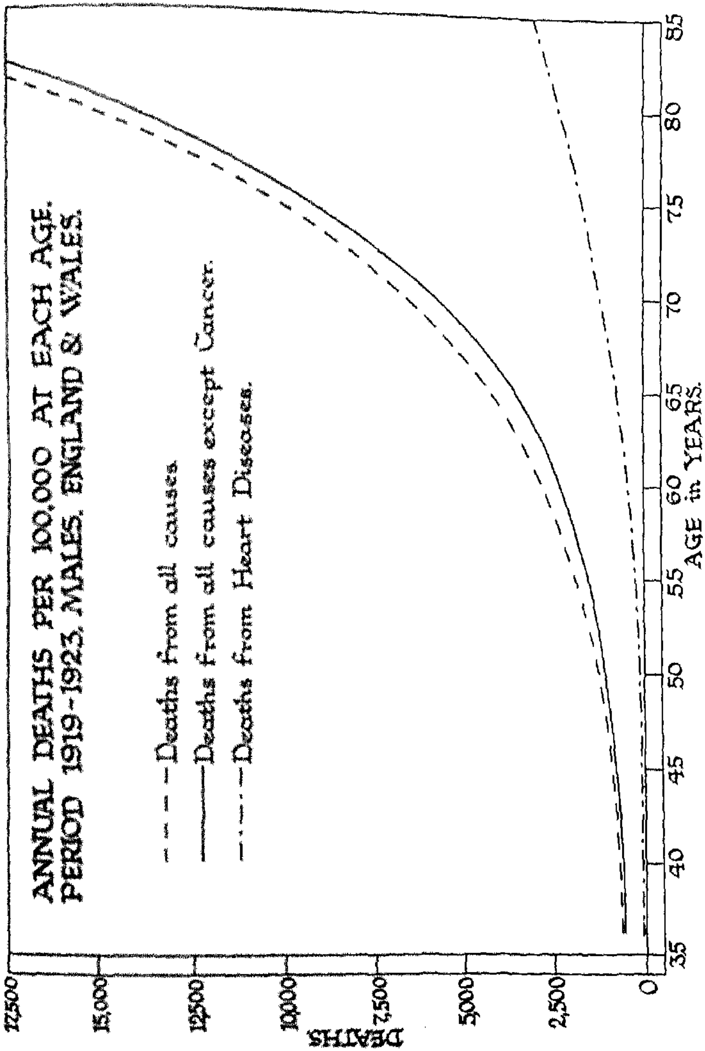

ANNUAL DEATHS PER 100,000 AT EACH AGE.
PERIOD 1919-1923. MALES. ENGLAND & WALES.
Deaths from all causes
Deaths from all causes except Cancer.
Deaths from Heart Diseases.
DEATHS
17,500
15,000
12,500
10,000
7,500
5,000
2,500
0
35
40
45
50
55
60
65
70
75
80
85
AGE in YEARS.

Fig. 1.

In deciding whether the short method is sufficiently accurate when a particular disease is to be eliminated the form of the curve of mortality rates of the disease must be considered. In Fig. 1 on the previous page are given:

The curves of annual mortality rates per 100,000:

(1) for deaths from all causes,

(2) for deaths from heart diseases,

(3) for deaths from all causes except cancer in five-yearly groups are given for the period of the data under consideration.

Similar curves have already been given for cancer and for pulmonary tuberculosis in the former paper.

The curve for deaths from all causes rises rapidly towards the end of life, the curve for heart diseases rises in the same way but at a more gradual slope, and the curve for all deaths except cancer follows a course similar to that for all deaths

The curve of mortality for pulmonary tuberculosis is different in form. The slope is generally gradual whether it is ascending or descending.

Provided then that one starts from a life table constructed in the same way, the short method of Farr is seen to lead to tolerably accurate differences in life expectation due to elimination of the diseases which have been under consideration in this memoir, except in the cases of cancer and heart diseases for ages after 60 when the inaccuracy increases. The method however involves the additional work of first re-computing the standard life table on the basis of five-yearly instead of yearly age-groups.

The error introduced in the short method arises from the fact that the numbers saved from the disease in a particular five-yearly period are regarded as exempt from risk of death from other diseases for the whole of the five years. This is not the case, for those saved from the particular disease will be subjected to the death-rate from other diseases from the moment in which they would have died of the special disease, which connotes on the average for half the period. This error does not occur when infinitesimal periods are used. In the use of D'Alembert's formula there is the additional advantage that a series of annual cancer (or other) mortality rates is obtained which is of interest and value in itself, and which proved to be of use in other problems dealt with in my former paper.

Some Work on the same Problem from Data of the Metropolitan Life Insurance Company of New York.

Some work on the same problem has been published in the *Statistical Bulletin** of the Metropolitan Life Insurance Company of New York, under the title "Effect of Cancer upon the Length of Life" and "Loss in Expectation of Life on account of organic Heart Diseases," for data of the Industrial Population 1911—1916. The tables for males, white, are reproduced as Table VIII.

* *Stat. Bull. M.L.I. Co.*, Oct. 1920, Vol. I. No. 10, p. 5; *Ibid.* Feb. 1921, Vol. II. No. 2, p. 6.

TABLE VIII.

Age	Number of years of Average After Lifetime Lost on Account of Cancer. (All Forms.) White, Males	Average Number of Years of After Lifetime Lost on Account of Organic Diseases of the Heart. White, Males
0	0·62	1·67
1	·70	1·86
2	·72	1·91
3	·73	1·94
4	·73	1·95
5	·72	1·95
15	·73	1·93
25	·75	1·89
35	·79	1·88
45	·80	1·87
55	·73	1·81
65	·53	1·56
75	·39	1·10
85	·30	·96
95	·17	·41

The column referring to cancer gives smaller figures than those for English data of about the same period, viz. 1909—1913, as given in Table VI of my former paper, but the results resemble those of English data of the last three decades in rising to a maximum loss of years between 40 and 50 years.

The results for heart diseases are very similar to those obtained from the English data 1919—1923, given in Table V a of my original paper.

The formula used to obtain the results given in Table VIII has been communicated to me by Messrs Dublin and Lotka as

$$q_x^{(-i)} = \frac{q_x - q_x^{(i)}}{1 - q_x^{(i)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{A}$$

where q_x denotes the usual life table function when all causes of death are effective, $q_x^{(i)}$ denotes the corresponding life table function when only cause i of death is effective, and $q_x^{(-i)}$ denotes the corresponding life table function when all causes except i of death are effective.

The formula A may be obtained as follows:

The probability of living for one year at age x at risk of death from all causes is equal to the product of the probability of living for one year when only cause i of death is effective and the probability of living for one year when all causes except i are effective, that is

$$p_x = p_x^{(i)} \cdot p_x^{(-i)};$$

$$\therefore 1 - q_x = (1 - q_x^{(i)})(1 - q_x^{(-i)});$$

$$\therefore q_x^{(-i)} = \frac{q_x - q_x^{(i)}}{1 - q_x^{(i)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{A}$$

p_x denoting the usual life table function when all causes of death are effective,

$p_x^{(i)}$ denoting the corresponding function when only cause i of death is effective, and

$p_x^{(-i)}$ denoting the corresponding function when all causes of death except i are effective.

I have applied the formula A to the data in hand in yearly periods and find that it gives a very near approximation to the formula of D'Alembert, the survivors in the life table excluding a special disease being the same to within a few units in 100,000 starting life together, at all ages through life, whether the disease considered is Cancer, Pulmonary Tuberculosis, or Heart Diseases. The expectations of life are exactly the same for the two methods.

Comparison of Formula A with Farr's Method.

The formula which Farr used would be

$$m_x - m_x^{(i)} = m_x^{(-i)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{B}$$

where $m_x = \dfrac{2q_x}{2 - q_x}$, and $m_x^{(i)}$, $m_x^{(-i)}$ have similar meanings.

Substituting these values for the m's Formula B gives on reduction

$$q_x^{(-i)} = \frac{q_x - q_x^{(i)}}{1 - q_x^{(i)} + \dfrac{q_x q_x^{(i)}}{4}},$$

which differs from formula A in the denominator of the right-hand side.

The amount of the difference in the expectation of life given by Formula B as compared with that given by D'Alembert's formula, or by Formula A, has already been shown.

It may be of interest to apply Formula A to the cancer data in five-yearly periods in order to compare the expectation of life with the results found in yearly periods. The results are given in Table IX.

This table shows that there is no sensible error involved in computing the additional expectation of life resulting from elimination of cancer as a cause of death by using five-yearly periods, so long as the life table with which comparison is made is computed in similar periods.

In conclusion, the difference in the methods used lies in the evaluation of q_x^i. D'Alembert's formula giving instantaneous values, Dublin's formula values at yearly intervals, and Farr's values at quinquennial periods.

For rapidity of calculation, combined with accuracy, the formula giving yearly values has some advantage over that giving instantaneous values.

In either case the results are arbitrary to some extent, as the original figures are usually only obtainable in quinquennial age-groups.

TABLE IX.

Expectation of Life and Increase in that Expectation over that of a Standard Life Table in a Population excluding Cancer.

Age	Calculated in yearly periods by formula A $\mathring{e}_x$	Increase in expectation of life over Life Table 9	Calculated in 5-yearly periods by formula A $\mathring{e}_x$	Standard Life Table (see Table V) $\mathring{e}_x$	Increase in expectation of life over Standard Life Table
0	56·98	1·27	56·99	55·71	1·28
5	60·25	1·44	60·37	58·91	1·46
10	56·10	1·46	56·21	54·74	1·47
15	51·59	1·47	51·69	50·21	1·48
20	47·26	1·48	47·39	45·89	1·50
25	43·09	1·49	43·23	41·72	1·51
30	38·91	1·51	39·04	37·52	1·52
35	34·77	1·52	34·90	33·37	1·53
40	30·71	1·52	30·86	29·32	1·54
45	26·72	1·50	26·87	25·36	1·51
50	22·79	1·43	22·96	21·51	1·45
55	19·02	1·29	19·20	17·89	1·31
60	15·45	1·09	15·66	14·56	1·10
65	12·20	·84	12·42	11·57	·85
70	9·34	·59	9·57	8·98	·59
75	6·96	·37	7·26	6·90	·36
80	5·14	·21	5·51	5·31	·20

BIBLIOGRAPHY.

(1) KARN, M. N. An Inquiry into Various Death-Rates and the Comparative Influence of Certain Diseases on the Duration of Life. *Annals of Eugenics*, Vol. IV. Parts III and IV, May, 1931.

(2) *Registrar-General's Fifth Annual Report.* Appendix.

(3) *Statistical Bulletin Metropolitan Life Insurance Company of New York.* Oct. 1920, Vol. I. No. 10. Feb. 1921, Vol. II. No. 2.

(4) *Supplement to the Registrar-General's 35th Report.*

(5) *Supplement to the Registrar-General's 75th Report*, Part II.

A TEST OF THE SIGNIFICANCE OF THE DIFFERENCE OF THE CORRELATION COEFFICIENTS IN NORMAL BIVARIATE SAMPLES.

By FRED A. BRANDNER, State University of Iowa.

I. *Introduction.*

THE problem of testing the significance of the difference between correlation coefficients, r_1 and r_2, found in two independent samples of size n_1 and n_2 may be considered as that of testing the hypothesis that the samples have been drawn from populations in which the coefficients of correlation between the variables have some common, but unspecified value, ρ. A method of procedure commonly used, which is adequate if the samples are large and ρ not too near either $+1$ or -1, is to compare the difference, $r_1 - r_2$, with an estimate of its standard error, $\sigma_{r_1-r_2}$. But if these conditions are not satisfied, we are at once faced with certain difficulties: (a) the value of $\sigma_{r_1-r_2}$ is very sensitive to the particular estimate of ρ chosen; (b) the sampling distribution of $r_1 - r_2$ will be asymmetrical, difficult to calculate and again dependent on the estimate of ρ.

To meet this difficulty, R. A. Fisher has suggested the use of the transformation*

$$z = \tfrac{1}{2}\{\log_e(1+r) - \log_e(1-r)\} \quad\text{.........................(1)},$$

because it then follows that if the two samples have been drawn from normal populations with a common ρ, $z_1 - z_2$ will be distributed in repeated samples, approximately normally about zero with a standard error given by

$$\sigma_{z_1-z_2} = \sqrt{\frac{1}{n_1-3} + \frac{1}{n_2-3}} \quad\text{.........................(2)}.$$

That is to say, by adopting this transformation, a test is obtained which is both easy to apply and, but for a certain approximation†, completely independent of the unknown value of ρ.

In a problem of this nature, it is evident that an indefinite number of criteria might be found to use in testing the hypothesis of a common ρ. Owing to the nearly invariant form of its sampling distribution, the criterion $z_1 - z_2$ might be chosen on intuitive grounds as one of the most efficient (as well as most convenient in application), but it is nevertheless of some interest to examine the logical basis

* *Metron* I. 4, pp. 12—18. *Statistical Methods for Research Workers*, Section 35.

† The nature of the approximation involved has been examined at various times. See for example *Biometrika* XXI. pp. 357—360; *Journal of the American Statistical Association*, June 1932, pp. 127—128.

for the choice between criteria. In a series of recent papers* J. Neyman and E. S. Pearson have discussed how, when the hypothesis to be tested and the set of admissible alternatives have been defined, the appropriate criterion may be deduced from certain fundamental principles, without any preconceived notion of what the form of the criterion ought to be. Making use of what has been termed the likelihood ratio, they have shown how a number of existing tests and certain new ones are brought into conformity, and my purpose is to consider the application of this method to the present problem.

II. *The Likelihood Ratio.*

The problem may be stated as follows. Two samples, Σ_1 and Σ_2 of size n_1 and n_2, have been randomly drawn from normally distributed bivariate populations, Π_1 and Π_2 (variables x and y). The two means, two standard deviations and product-moment correlation coefficient are defined as follows:

For $\quad \Pi_t\ (t=1,2),$

$\quad a_t,\ a_t';\ \ \sigma_t,\ \sigma_t';\ \ \rho_t.$

For $\quad \Sigma_t\ (t=1,2),$

$\quad \bar{x}_t,\ \bar{y}_t;\ \ s_t,\ s_t';\ \ r_t.$

The admissible hypotheses concern the set Ω of all possible pairs of bivariate normal populations. The hypothesis we shall test is not that Π_1 and Π_2 are identical, but merely that their correlation coefficients have the same value, or that

$$\rho_1=\rho_2=\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

while the relations

$$a_1=a_2,\quad a_1'=a_2',\quad \sigma_1=\sigma_2,\quad \sigma_1'=\sigma_2'$$

will not necessarily be satisfied. The population pairs (Π_1, Π_2) for which (3) is true, form a subset ω of the set Ω. The likelihood ratio, λ, is to be obtained by choosing from Ω and ω, respectively, the two population pairs which make the chance of the observed sampling result a maximum.

The chance of obtaining Σ_1 from Π_1 with character values falling in the ranges $(x_i \pm \frac{1}{2}h,\ y_i \pm \frac{1}{2}k)$ $(i=1, 2, \ldots, n_1)$ will be asymptotic to

$$C_1=\left[\frac{1}{2\pi\sigma_1\sigma_1'(1-\rho_1^2)^{\frac{1}{2}}}\right]^{n_1} e^{-\frac{1}{2(1-\rho_1^2)}\sum_{i=1}^{n_1}\left[\left(\frac{x_i-a_1}{\sigma_1}\right)^2+\left(\frac{y_i-a_1'}{\sigma_1'}\right)^2-\frac{2\rho_1(x_i-a_1)(y_i-a_1')}{\sigma_1\sigma_1'}\right]}(hk)^{n_1} \quad \ldots\ldots(4)$$

as h and k approach zero.

Likewise for the sample Σ_2 of n_2 observations, the chance is given by

$$C_2=\left[\frac{1}{2\pi\sigma_2\sigma_2'(1-\rho_2^2)^{\frac{1}{2}}}\right]^{n_2} e^{-\frac{1}{2(1-\rho_2^2)}\sum_{i=1}^{n_2}\left[\left(\frac{x_i-a_2}{\sigma_2}\right)^2+\left(\frac{y_i-a_2'}{\sigma_2'}\right)^2-\frac{2\rho_2(x_i-a_2)(y_i-a_2')}{\sigma_2\sigma_2'}\right]}(hk)^{n_2} \quad \ldots\ldots(5).$$

* See *Biometrika* xx^A. pp. 175, 264. *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1930, p. 73.

The combined probability of the occurrence is given by the product

$$C(\Omega) = C_1 C_2 \quad \text{.................................(6).}$$

If the values (4) and (5) are substituted in equation (6) and the sums simplified, the result may be written

$$C(\Omega) = K \prod_{i=1}^{2} \left[\frac{1}{\sigma_i^2 \sigma_i'^2 (1-\rho_i^2)}\right]^{\frac{n_i}{2}}$$
$$\times e^{-\sum_{i=1}^{2} \frac{n_i}{2(1-\rho_i^2)}\left[\frac{(\bar{x}_i - a_i)^2 + s_i^2}{\sigma_i^2} + \frac{(\bar{y}_i - a_i')^2 + s_i'^2}{\sigma_i'^2} - \frac{2\rho_i\{(\bar{x}_i - a_i)(\bar{y}_i - a_i') + r_i s_i s_i'\}}{\sigma_i \sigma_i'}\right]} (hk)^N \ldots (7),$$

where $N = n_1 + n_2$ and $K = (2\pi)^{-N}$.

Finding the values of a_i, a_i', etc. ($i = 1, 2$) to make $C(\Omega)$ a maximum gives

$$a_i = \bar{x}_i, \quad a_i' = \bar{y}_i, \quad \sigma_i = s_i, \quad \sigma_i' = s_i', \quad \rho_i = r_i \quad \text{...............(8).}$$

This gives the pair of populations of maximum likelihood. By substituting relations (8) in (7) the chance of the joint occurrence becomes

$$C(\Omega_{\max}) = K\left[\frac{1}{s_1 s_1'(1-r_1^2)^{\frac{1}{2}}}\right]^{n_1}\left[\frac{1}{s_2 s_2'(1-r_2^2)^{\frac{1}{2}}}\right]^{n_2} e^{-N}(hk)^N \text{.........(9).}$$

Now consider the chance of the samples belonging to the subset ω defined above. The total probability is given by

$$C(\omega) = K \prod_{i=1}^{2}\left[\frac{1}{\sigma_i \sigma_i'}\right]^{n_i}\left[\frac{1}{1-\rho^2}\right]^{\frac{N}{2}}$$
$$\times e^{-\frac{1}{2(1-\rho^2)}\sum_{i=1}^{2}\left[\frac{(\bar{x}_i - a_i)^2 + s_i^2}{\sigma_i^2} + \frac{(\bar{y}_i - a_i')^2 + s_i'^2}{\sigma_i'^2} - \frac{2\rho\{(\bar{x}_i - a_i)(\bar{y}_i - a_i') + r_i s_i s_i'\}}{\sigma_i\sigma_i'}\right]} (hk)^N \ldots (10).$$

Again, if $C(\omega)$ be maximized in respect to a_i, a_i', etc. ($i = 1, 2$) it gives

$$\left.\begin{array}{ll} \text{(A)} & a_i = \bar{x}_i, \quad a_i' = \bar{y}_i \\ \text{(B)} & \dfrac{s_i^2}{\sigma_i^2} = \dfrac{1-\rho^2}{1-r_i\rho} = \dfrac{s_i'^2}{\sigma_i'^2} \\ \text{(C)} & (n_1 r_2 + n_2 r_1)\rho^2 - N(1 + r_1 r_2)\rho + n_1 r_1 + n_2 r_2 = 0 \end{array}\right\} \text{............(11).}$$

Upon solving equation (C) of (11),

$$\rho_M = \frac{N(1 + r_1 r_2) - \sqrt{N^2(1 - r_1 r_2)^2 - 4n_1 n_2 (r_1 - r_2)^2}}{2(n_1 r_2 + n_2 r_1)} \quad \text{.........(12).}$$

This gives as a maximum chance

$$C(\omega_{\max}) = K\left[\frac{(1-\rho_M^2)^{\frac{1}{2}}}{s_1 s_1'(1-\rho_M r_1)}\right]^{n_1}\left[\frac{(1-\rho_M^2)^{\frac{1}{2}}}{s_2 s_2'(1-\rho_M r_2)}\right]^{n_2} e^{-N}(hk)^N \quad \text{...(13).}$$

Thus we may obtain the likelihood ratio

$$\lambda = \frac{C(\omega_{\max})}{C(\Omega_{\max})} \quad \text{...............................(14).}$$

III. *The Case when* $n_1 = n_2 = n$.

The above ratio cannot in general be expressed simply, but when

$$n_1 = n_2 = n \quad \text{..(15)},$$

by putting

$$r_1 = \tanh z_1, \quad r_2 = \tanh z_2, \quad \rho = \tanh z \quad \text{..................(16)},$$

the result

$$z = \frac{z_1 + z_2}{2} \quad \text{..(17)}$$

is easily obtained from equation (11 C). Also, from equation (13),

$$C(\omega_{\max}) = K\left[\frac{\cosh z_1 \cosh z_2}{s_1 s_1' s_2 s_2' \cosh^2 \frac{z_1 - z_2}{2}}\right]^n e^{-N} (hk)^N \quad \text{...........(18)}.$$

If the same substitution is made in equation (9), it gives

$$C(\Omega_{\max}) = K\left[\frac{\cosh z_1 \cosh z_2}{s_1 s_1' s_2 s_2'}\right]^n e^{-N} (hk)^N \quad \text{..................(19)}.$$

Finding the ratio of the likelihoods and extracting the $(2n)$th root gives

$$\lambda^{\frac{1}{2n}} = \operatorname{sech} \frac{z_1 - z_2}{2} \quad \text{..............................(20)}.$$

It follows that if the criterion used is

$$z_1 - z_2 = 2 \operatorname{sech}^{-1} \lambda^{\frac{1}{2n}} \quad \text{..............................(21)},$$

it will be identical to that of R. A. Fisher referred to above. In other words, the contours of constant λ's correspond exactly to the contours of constant values of $(z_1 - z_2)$. As the likelihood of the hypothesis decreases $z_1 - z_2$ increases, and the method already described can be used to determine the significance of the difference of the observed correlations of two normally correlated bivariate samples for which nothing is known as to the values of the parameters involved.

IV. *The Case when* $n_1 \neq n_2$.

The case of n_1 not equal to n_2 may now be considered. Without loss of generality we may assume $r_1 > r_2$. In order to express λ in simple terms it is here found necessary to introduce an approximation for the value of ρ to be inserted in (13). Equation (11 C) may be written in the form

$$\left[\frac{N}{2}(r_1 + r_2) - \frac{n_1 - n_2}{2}(r_1 - r_2)\right] \rho^2 - N(1 + r_1 r_2)\rho + \frac{N}{2}(r_1 + r_2) + \frac{n_1 - n_2}{2}(r_1 - r_2) = 0 \quad \text{........(22)}.$$

Then by substitutions (16),

$$[\tanh(z_1 + z_2) - \epsilon]\rho^2 - 2\rho + [\tanh(z_1 + z_2) + \epsilon] = 0 \quad \text{...........(23)},$$

where

$$\epsilon = \frac{n_1 - n_2}{N} \frac{r_1 - r_2}{1 + r_1 r_2} \quad \text{..............................(24)}.$$

From (23)

$$\rho = \tanh z = \frac{1 - \sqrt{\operatorname{sech}^2(z_1 + z_2) + \epsilon^2}}{\tanh(z_1 + z_2) - \epsilon}$$

$$= \frac{1 - \sqrt{1 - [\tanh^2(z_1 + z_2) - \epsilon^2]}}{\tanh(z_1 + z_2) - \epsilon}$$

$$= \frac{1 - [1 - \frac{1}{2}\{\tanh^2(z_1 + z_2) - \epsilon^2\} - \frac{1}{8}\{\tanh^2(z_1 + z_2) - \epsilon^2\}^2 \dots\dots]}{\tanh(z_1 + z_2) - \epsilon} \quad \dots\dots(25).$$

If all powers of $\{\tanh^2(z_1 + z_2) - \epsilon^2\}$ greater than the second, which in most cases will be small, are neglected in (25) the approximate value

$$\rho = \frac{\frac{1}{2}[\tanh^2(z_1 + z_2) - \epsilon^2] + \frac{1}{8}[\tanh^2(z_1 + z_2) - \epsilon^2]^2}{\tanh(z_1 + z_2) - \epsilon} \quad \dots\dots\dots(26)$$

will result. The above approximation tends to slightly decrease the value for ρ.

Thus
$$\frac{e^{2z} - 1}{e^{2z} + 1} = \frac{A}{2} + \frac{A^2B}{8} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(27),$$

where
$$A = \tanh(z_1 + z_2) + \epsilon \text{ and } B = \tanh(z_1 + z_2) - \epsilon \quad \dots\dots\dots\dots(28).$$

Solving for e^{2z} gives
$$e^{2z} = \frac{1 + \frac{A}{2} + \frac{A^2B}{8}}{1 - \frac{A}{2} - \frac{A^2B}{8}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(29),$$

from which

$$2z = \log\left(1 + \frac{A}{2} + \frac{A^2B}{8}\right) - \log\left(1 - \frac{A}{2} - \frac{A^2B}{8}\right)$$

$$= \frac{A}{2} + \frac{A^2B}{8} - \frac{1}{2}\left(\frac{A}{2} + \frac{A^2B}{8}\right)^2 + \dots\dots + \left(\frac{A}{2} + \frac{A^2B}{8}\right) + \frac{1}{2}\left(\frac{A}{2} + \frac{A^2B}{8}\right)^2 + \dots\dots \quad \dots\dots\dots(30).$$

Again, by neglecting all terms in A and B of the second order and higher powers, another slight decrease will be made in the value for ρ. This gives the approximation

$$z = \tfrac{1}{2}[\tanh(z_1 + z_2) + \epsilon]$$

$$= \frac{1}{2}\left[\frac{r_1 + r_2}{1 + r_1 r_2} + \frac{n_1 - n_2}{N} \cdot \frac{r_1 - r_2}{1 + r_1 r_2}\right]$$

$$= \frac{n_1 r_1 + n_2 r_2}{N(1 + r_1 r_2)} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(31).$$

These slight decreases in the value of ρ, and consequently the value of z as noted above, are almost exactly counterbalanced by again approximating in (31). By expanding each of the values for r_i $(i = 1, 2)$ in terms of z_i, the numerator is slightly increased and the denominator is decreased by discarding powers of z_i of second degree and higher. The value

$$z = \tanh^{-1}\rho = \frac{n_1 z_1 + n_2 z_2}{N} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(32)$$

is thus obtained as an approximation for the true value of z.

The accuracy of this approximation for ρ for widely varying values of r_1 and r_2 is shown in Tables I and II, where the exact maximum likelihood value from equation (12) is denoted by ρ_M, and the value obtained from equation (32) by ρ_a. Both ρ_M and ρ_a depend only on the ratio of n_1 to n_2, and the cases examined are for $n_1 = 2n_2$ and $n_1 = \frac{1}{2}n_2$ (Table I), and $n_1 = 6n_2$ and $n_1 = \frac{1}{6}n_2$ (Table II), where we have taken the higher of the two sample correlations as r_1. It will be noticed that when $n_1 > n_2$, $\rho_a \leqslant \rho_M$ and when $n_1 < n_2$, $\rho_a \geqslant \rho_M$. This shows that the sample containing more observations carries greater weight in determining the maximum likelihood z, than is allowed for in the weighted arithmetic mean of z_1 and z_2.

Needless to say for either $n_1 = n_2$ or $r_1 = r_2$, ρ_M and ρ_a will agree exactly.

TABLE I.

				Case $n_1 = 2n_2$			Case $n_1 = \frac{1}{2}n_2$			
z_1	z_2	r_1	r_2	ρ_M	ρ_a	$\rho'(n_1=20)$	ρ_M	ρ_a	$\rho'(n_1=10)$	$\frac{z_1-z_2}{\sqrt{\frac{1}{17}+\frac{1}{7}}}$
0·2	0·0	·1974	·0000	·133	·133	·141	·067	·067	·058	0·46
0·4	0·0	·3800	·0000	·262	·261	·276	·131	·133	·116	0·89
0·7	0·0	·6044	·0000	·442	·430	·450	·221	·220	·201	1·56
0·4	0·2	·3800	·1974	·322	·322	·320	·200	·201	·253	0·45
0·7	0·2	·6044	·1974	·460	·488	·504	·348	·351	·334	1·11
1·1	0·2	·8005	·1974	·674	·664	·684	·448	·462	·432	2·00
1·5	0·2	·9052	·1974	·808	·788	·808	·523	·560	·522	2·89
0·7	0·4	·6044	·3800	·538	·537	·546	·402	·402	·452	0·67
1·1	0·4	·8005	·3800	·704	·700	·714	·555	·560	·540	1·56
1·5	0·4	·9052	·3800	·823	·812	·827	·625	·645	·617	2·45
2·3	0·4	·9801	·3800	·949	·931	·941	·706	·775	·742	4·23
1·1	0·7	·8005	·6044	·748	·747	·755	·681	·682	·673	0·89
1·5	0·7	·9052	·6044	·847	·844	·853	·741	·747	·732	1·78
2·3	0·7	·9801	·6044	·963	·943	·950	·814	·844	·823	3·56
1·5	1·1	·9052	·8005	·878	·878	·882	·843	·844	·839	0·89
2·3	1·1	·9801	·8005	·960	·956	·960	·898	·905	·890	2·67
2·3	1·5	·9801	·9052	·967	·961	·968	·942	·943	·939	1·78

It should be noted that the estimate of ρ suggested by R. A. Fisher* is based on a weighting of z_1 and z_2 inversely proportional to the approximate values of their sampling variances, namely

$$\rho' = \tanh\left\{\frac{(n_1-3)\,z_1 + (n_2-3)\,z_2}{N-6}\right\} \quad \text{.................(33).}$$

This result does not correspond either to the maximum likelihood value of (12) or to the approximation (32). ρ' will depend upon the actual values of n_1 and n_2, and has been computed for the cases

$$\left.\begin{array}{l} n_1 = 20,\ n_2 = 10;\ \text{and}\ n_1 = 10,\ n_2 = 20\ \text{(Table I)} \\ n_1 = 60,\ n_2 = 10;\ \text{and}\ n_1 = 10,\ n_2 = 60\ \text{(Table II)} \end{array}\right\} \quad \text{.........(34).}$$

* *Metron* I. 4, p. 18.

TABLE II.

				Case $n_1 = 6n_2$			Case $n_1 = \frac{1}{6}n_2$			$\frac{z_1 - z_2}{\sqrt{\frac{1}{57} + \frac{1}{7}}}$
z_1	z_2	r_1	r_2	ρ_M	ρ_a	$\rho' (n_1 = 60)$	ρ_M	ρ_a	$\rho' (n_1 = 10)$	
0·2	0·0	·1974	·0000	·170	·170	·176	·028	·029	·022	0·49
0·7	0·0	·6044	·0000	·544	·537	·554	·091	·100	·076	1·75
0·7	0·2	·6044	·1974	·559	·557	·568	·262	·265	·249	1·25
1·5	0·2	·9052	·1974	·877	·865	·876	·326	·368	·329	3·25
1·1	0·4	·8005	·3800	·765	·762	·771	·455	·462	·443	1·75
2·3	0·4	·9801	·3800	·973	·966	·970	·507	·586	·543	4·74
1·5	0·7	·9052	·6044	·885	·882	·888	·665	·672	·657	2·00
2·3	1·1	·9801	·8005	·974	·972	·974	·843	·854	·843	3·00

It will be seen that, although ρ_a is generally a better approximation to ρ_M than is ρ' (for the range of cases taken), this is not always so; some light is thrown on the position by calculating the ratio

$$\frac{z_1 - z_2}{\sigma_{z_1 - z_2}} = \frac{z_1 - z_2}{\sqrt{\dfrac{1}{n_1 - 3} + \dfrac{1}{n_2 - 3}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35)$$

for the values of n_1 and n_2 given in (34). These ratios are entered in the last column of each table, and provide a measure of the significance of the differences between the pairs of sample values r_1 and r_2. It will be seen that ρ_a is closer to ρ_M than is ρ' so long as $(z_1 - z_2)/\sigma_{z_1 - z_2} < 2{\cdot}00$, but that when the ratio exceeds 2·00, the position is reversed. Of course, as n_1 and n_2 increase, $\rho' \to \rho_a$; but the approach of ρ_a to ρ_M will depend on the ratio of n_1 to n_2.

Finally, if the approximation (32) be accepted as adequate, and ρ_a substituted for ρ_M in (13), we obtain from this equation, and from (9),

$$C(\Omega_{\max}) = K \left[\frac{\cosh z_1}{s_1 s_1'}\right]^{n_1} \left[\frac{\cosh z_2}{s_2 s_2'}\right]^{n_2} e^{-N} (hk)^N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36)$$

and

$$C(\omega_{\max}) = K \left[\frac{\cosh z_1}{s_1 s_1' \cosh (z - z_1)}\right]^{n_1} \left[\frac{\cosh z_2}{s_2 s_2' \cosh (z - z_2)}\right]^{n_2} e^{-N} (hk)^N \ldots(37).$$

Taking the ratio of these likelihoods gives

$$\lambda = \frac{C(\omega_{\max})}{C(\Omega_{\max})} = [\operatorname{sech}(z - z_1)]^{n_1} [\operatorname{sech}(z - z_2)]^{n_2}$$

$$= \left[\operatorname{sech} \frac{n_2}{N}(z_1 - z_2)\right]^{n_1} \left[\operatorname{sech} \frac{n_1}{N}(z_1 - z_2)\right]^{n_2} \ldots\ldots\ldots(38).$$

Thus with approximation the contours of constant λ again agree with R. A. Fisher's contours for $(z_1 - z_2)$.

The following example will illustrate the use of the test. Suppose

$$n_1 = 60, \quad n_2 = 10; \quad r_1 = {\cdot}6044, \quad r_2 = {\cdot}1974 \text{ (see Table II).}$$

Here $z_1 = 0·7, \quad z_2 = 0·5$

and $(z_1 - z_2)/\sigma_{z_1 - z_2} = (0·7 - 0·2)/\sqrt{\frac{1}{17} + \frac{1}{7}} = 1·25.$

If we refer this value to the normal probability scale we find $\frac{1}{2}(1 + \alpha) = ·894$.

We may now reason as follows: since the alternative to the hypothesis tested ($\rho_1 = \rho_2 = \rho$) which naturally demands our first attention is that $\rho_1 > \rho_2$, we ask what is the chance that z_1 would exceed z_2 by 0·5 or more, were $\rho_1 = \rho_2$? This chance is $\frac{1}{2}(1 - \alpha) = ·106$, and we should conclude that there was no clear call to reject the hypothesis, $\rho_1 = \rho_2$, in favour of an alternative $\rho_1 > \rho_2$. Evidently there would be far less reason still to reject it in favour of some alternative, $\rho_1 < \rho_2$.

V. Conclusion.

The problem discussed is that of testing the hypothesis that two samples have been drawn from populations in which the coefficient of correlation has some common but unspecified value.

It is assumed that the variation in the populations sampled is normal. Following the method of Pearson and Neyman for testing what they have termed a composite hypothesis*, a criterion, λ, has been obtained which for $n_1 = n_2$ is exactly, and for other cases closely, related to the criterion $z_1 - z_2$ suggested by R. A. Fisher. This result is analogous to others which have been reached by the two writers in the papers referred to above, in so far as it shows that by employing the method of likelihood to determine the appropriate criterion to use in testing a statistical hypothesis, we are lead to certain standard tests. In this particular case Fisher's z-test was first reached from a quite different line of approach.

* It may of course be argued that a hypothesis of this type cannot be tested by calculation of a single criterion with simple probability character, and that we should determine separately the significance of the differences between r_1 and r_2 and σ_1 and σ_2 [illegible]

PLURAL BIRTHS WITH A NEW PEDIGREE.

By JULIA BELL, M.A., M.R.C.P.

(1) Some two years ago, Dr E. A. Barton, working in connection with the Obstetric Department of University College Hospital, came across an interesting history of plural births in three generations of a family, transmitted, in the particular case under his observation, through a male, III. 10, who was believed himself to have been a single birth. Mr Herbert Spencer warns us* that statistics of patients treated in lying-in-wards are alone reliable in respect of multiple births, as one of a twinship often dies and the fact of a twin pregnancy may not be mentioned to the patient. However, their history of multiple births has been of interest, not unaccompanied by anxiety, in the family whose pedigree is given under Fig. 1, and so far as his mother and his grandmother were aware, III. 10 was born at a single birth.

It will be seen from Fig. 1 of the adjoining plate that one quadruplet, probably two triplets and seven pairs of twins have been born in five sibships.

Dr Barton kindly sent the mother, III. 3, to see us after the birth of her second pair of twins, IV. 12 and 13; she is an intelligent woman, with no history of twins or multiple births in her own family; she and her husband provided most of the facts shown in the pedigree. There was some difference of opinion in the family regarding the triplet of III. 18—20; the mother, II. 12, believing she had no triplet, whilst her mother, I. 4, declared that this triplet was born in Middlesex Hospital, and resulted from a shock to the mother on hearing that her elder son, III. 10, was to have a diseased bone removed from his face. This experience fully supports Mr Herbert Spencer's warning. Dr E. A. Cockayne kindly obtained verification of the fact from the Registrar of Middlesex Hospital. It is of interest to note that though both parents, II. 11 and 12, had siblings belonging to multiple births, they themselves apparently produced eight children at single births in addition to two pairs of twins and one triplet. There is some uncertainty about the triplet II. 4—6; its occurrence is a family tradition, but nothing is now known regarding the sex of the children or whether any of them survived birth.

II. 12, who is still living, is sure that her mother had only one multiple birth in a large family; this consisted in a quadruplet of four boys named Matthew, Mark, Luke and John; all the boys lived for some time, but died before the age of 20 years. Of the fifteen children of II. 11 and 12, only four are now living; the

* In a personal letter to Professor Karl Pearson.

triplet of this sibship was born prematurely and all its members died. One brother of this family, II. 21, was killed in the war. One twin sister, III. 14, is still living; she is married and has had two confinements, with three daughters; one of the twin daughters died young, of diphtheria.

III. 3, the wife of III. 10, has had six children at four confinements; each pair of twins were of like sex, but their mother says the children are not at all alike; one of the first pair of twins died aged 13 months from pneumonia.

III. 3 is a young woman, aged 30 at the birth of her last twins; she suffers from asthma, and, with a husband unemployed for nearly two years, considers anxiously the possibility of further multiple births; she remarked that she had indeed married into a dangerous family.

(2) Remarkable examples of multiple births have been reported from time to time since the days of Aristotle. These include: (*a*) Individual cases of large numbers of children born at a single birth. (*b*) Histories of successive plural births to the same individual; of some special interest are a number of histories, some of them undoubtedly authentic, recording a sequence to the same individual of the same type of multiple birth, the mother producing, say, always twins, or always triplets. (*c*) Family histories recording plural births in a number of individuals of the same stock; the more interesting of these refer to cases in which the liability to the occurrence is transmitted through the male, as may be seen in our Fig. 1; also in Figs. 2 and 3.

(*a*) Writing in 1850 (*Notes and Queries*, p. 459), Richard Owen says that the largest number at a birth of which any authentic record appears is five; there was at this time, in the Royal College of Surgeons, a specimen jar containing five foetuses from one birth, of which three were still-born, the two born alive survived for a short time only*.

Garthshore, writing in the *Phil. Trans.* 1787†, recorded the occurrence of this particular case; he states that he had employed various friends at Petersburg, Berlin, Vienna, Lyons, Paris and Ghent to collect for him well authenticated cases of this kind, and that he had not yet been able to procure any. Garthshore collected and published at that date a number of records of multiple births, but expressed the opinion that, when we advance further than five at a birth, we get into the region of tradition and improbability.

Foy‡, who in 1890 again collected all records he could find of plural births, including quadruplets and upwards, expresses the opinion that we may err by an excess of incredulity, and that we need to be very cautious before rejecting definite statements made by medical men recording their personal experiences.

* The specimen may still be seen in the Museum of the College, together with several more recent examples.

† Bibl. No. 7.

‡ Bibl. No. 12.

Ambroise Paré* expresses no doubt in quoting the statement of Martin Cromer, the Polish historian, that "one Margaret, a woman sprung from a noble and ancient familie near Cracovia, and wife to Count Verboslaus, brought forth at one birth thirtie five live children†, upon the twentieth date of January in the year 1296"; Paré was a contemporary of the historian and presumably might have satisfied himself of the evidence on which the statement was made. We also learn from this distinguished surgeon that "Franciscus Picus Mirandula writeth that one Dorothie, an Italian, had twentie children at two births; *at the first nine, and at the second eleven,* and that shee was so big that shee was forced to bear up her bellie, which laie upon her knees, with a broad and large scarf tied about her neck, as you may see by this figure"—an illustration of the said Dorothy is given, with her scarf in position. At a later date, in 1684, Dr Seignette writes of an "Accouchement Surprenant" which he had seen at Rochefort, where "une femme de Xaintonge étoit accouchée *de neuf enfans*, tous bien formés, et ausquels on distinguoit le sexe; et que cette même femme l'année précédente avoit accouché *de onze*‡." It is difficult to believe in or to deny the possibility of any of these occurrences.

Was Martin Cromer (1512—1589) a reliable historian? He reports that Mathias Golancevius, Bishop of Vladislavia in Poland, was the only survivor of twelve sons delivered at one birth, the rest dying as soon as they were born§. The name here recalls that of Margarita Gonçalez of Valencia, whose remarkable fecundity was noted by Henrique Cock when travelling in Spain with Philip II in 1585. "The midwives and several doctors of Valencia are witnesses, counting the children that her two husbands had the good fortune to have. They found one hundred and forty-four males and fourteen females. Amongst them there were baptised forty-nine sons and three females‖." This case is obviously either a fabrication, or some misprint has occurred, or the midwives and doctors were unable to count, for we are told that this Margarita had thirty-three parturitions between the ages of 15 years and 35; the woman, however, may have provided an example of remarkable fecundity, and my attention was drawn to it by the similarity of the name to that of the Polish report. Margarita Gonçalez was said to be the daughter of a Basque father and a Parisian mother; she married first a Neapolitan and secondly a Basque, so was cosmopolitan with regard to her connections. She was again pregnant when the travellers left Valencia.

If Ambroise Paré was ready to be credulous, Owen and Garthshore would appear to have understated the case. There are certainly numerous authentic accounts of six at a birth, and there is what surely one must accept as definite evidence of a septuplet, from a memorial stone on a house at Hameln a. Weser;

* Bibl. No. 1.

† In editions of Cromer's work, published in German in 1562 and in Latin in 1589, I find the number given as thirty-six live children. The oft repeated story of the Gräfin von Henneberg and her 365 children at a birth is undoubtedly a myth.

‡ Bibl. No. 6. § Bibl. No. 2, p. 197. ‖ Bibl. No. 3.

kneeling parents and seven babies in swaddling clothes are represented on the stone below a crucifix, and the following inscription is given:

> Allhier ein Bürger Thiele Römer genannt
> Seine Hausfrau Anna Breyers wohlbekannt.
> Als man zählte 1600 Jahr
> Den 9 Januarius des Morgens 3 Uhr war
> Von ihr zwei Knäbelein und fünf Mägdelein
> Auf eine Zeit geboren sein.
> Haben auch die heiligen Tauf erworben
> Folgends den 20ten 12 Uhr Seelig gestorben.
> Gott wolle ihn [-en] geben die Seeligkeit,
> Die allen Gläubigen ist bereit.

Obiges original Denkmal hat durch die Güte der Herrn Bürgermeister Domeier, der jetzige Besitzer dieses damahls Römerschen Hauses Gerichtsschreiber Hoppe wieder erhalten und aufgestellt im Jahr 1818.

A photograph of this stone is given by Barfurth*, whose account, in 1895, is the first reference to it which is known to me. We must conclude that seven at one birth is the greatest number of which we have an authentic account, but we have no reason to believe that this number has never been exceeded.

(*b*) With regard to successive plural births in the same individual, the most astonishing report is one whose first account I find in *The Gentleman's Magazine* (Vol. LIII. p. 753, London, 1783): "In an original letter now before me, dated St Petersburg, Aug. 13, 1782, O.S., Feodor Wassilief, aged 75, a peasant, said to be now alive and in perfect health, in the Government of Moscow, has had—.

By his first wife:	By his second wife:
$4 \times 4 = 16$	$6 \times 2 = 12$
$7 \times 3 = 21$	$2 \times 3 = 6$
$16 \times 2 = 32$	8 births 18 children.
27 births 69 children.	

In all 35 births, 87 children, of which 84 are living and only three buried....The above relation, however astonishing, may be depended upon, as it came directly from an English merchant at St Petersburg to his relatives in England, who added that the peasant was to be introduced to the Empress."

This history was published independently—with a caution—by Hermann†, writing on Statistics of the Russian population in 1790. My impulse was to reject the case as unworthy of serious consideration, as apparently the cautious Garthshore did in 1787. However, from a statement in the *Lancet* of 1878 (Vol. I. p. 290), we learn that a few years earlier the French Academy of Science had endeavoured to obtain verification of the occurrence; they appealed to M. Khanikoff of the Imperial Academy of St Petersburg for advice as to the means they should pursue, but were told by him that all investigation was superfluous, that members of the family still lived in Moscow and that they had been the object of favours from the Government. Are we to accept this case as an established record?

* Bibl. No. 14.

† Bibl. No. 8.

What then of the following report*: "In the year 1755 a Muscovite peasant, named James Kyrloff, and his wife were presented to the Empress of Russia. This peasant had been twice married and was then 70 years of age. His first wife was brought to bed twenty-one times, four times with four children each time, seven times of three, and ten times of two, making in all fifty-seven children who were then alive. His second wife, who accompanied him, had already been delivered seven times, once of three children, six times of twins." Surely both these Russian cases must be regarded as under suspicion; other similar cases have been reported, one of which referred to a handworker in Lille who had 82 children by two wives. There is, however, this point of interest about all these and other reports—they may exaggerate the details, but they each suggest the probable occurrence of a rather remarkable sequence of multiple births in the two wives of one man, and thus indicate in these cases examples of the probably inherited liability to produce multiple births, transmitted by the male.

At a much earlier date Aristotle writes (*History of Animals*, Lib. VII. Cap. 4): "As a rule and in most countries women have but one child at a birth; yet frequently and in many districts they bear twins, particularly in the land of Egypt. But even three and four occur at a birth, and this quite frequently in certain places as it has been stated above. A woman does not bear more than five at a birth, and this has been observed to happen several times. Indeed a certain woman bore twenty children in four parturitions, five at each, and most of them were reared." This cannot be regarded as proven evidence of the occurrence of a sequence of four quintuplets, but the reference is of great historic interest, and there is some measure of probability regarding its accuracy.

To return now to Ambroise Paré†. He writes: "In our time, between Sarte and Main, in the parish of Seaux, nor far from Chambellay, there is a familie and noble hous called Maldemeure; the wife of the Lord of Maldemeure, the first year shee was married brought forth twins, the second year shee had three children, the third year four, the fourth year five, the fifth year six, and of that birth she died; of those six one is yet alive, and is Lord of Maldemeure." We must receive with caution Paré's recital of cases taken from other writers, but this statement regarding a noble family in his own country, which could be refuted any day if untrue, cannot easily be rejected. It is of some interest to note the increasing number recorded at each birth in this history, in view of the fact that there is a good deal of evidence in favour of the statement that on the whole plural births tend to occur later in life than single births; the average age of the mother at the birth of twins being older than that at the age of single births, and the average age of the mother at the birth of triplets being greater than that at the age of twins. So that mistrust of the case, which has been suggested on the grounds of the improbability of the regularly increasing number at birth, loses perhaps some of its justification.

* Bibl. No. 5. † Bibl. No. 1.

None of these remarkable cases really carries conviction; we cannot refuse to believe in the possibility of their occurrence, but if we were able to obtain verification of one extreme case, other accounts would assuredly receive a measure of support. I have tried unsuccessfully to get into touch with a more recently published case; in 1886 the Naples correspondent of the *Paris Register* writes*: "About twenty-five miles from here, and by rail two or three stations beyond Pompeii, is the historical city of Nocera. In it lives Maddalena Granata, aged 47 who was married at the age of 28 to a peasant just nineteen years ago. Maddalena Granata has given birth to, either dead or living, fifty-two children, forty-nine of whom were males. She enjoys florid health, is robust, and twenty-four hours after her last accouchement was ready to go out to her accustomed labour in the field.... Her physician, Dr de Sanctis of Nocera, says that there is not the least exaggeration in these statements....She has had triplets fifteen times." It seems almost incredible that anybody should have triplets fifteen times. There should be no difficulty in finding out whether there is any truth in this report; a letter to the village Syndicato has brought no reply; probably a visit to Nocera, by a traveller in those parts, would be the most satisfactory way of getting into touch with a descendant of Maddalena Granata or of Dr de Sanctis.

A similar history, published in the *Gazette Médicale de Lyons* (Oct. 1, 1863)†, reports that the wife of a medical man at Fuentemajor in Spain, aged 43, had just been delivered of three girls; it was the thirteenth time she had been confined of triplets.

A number of authentic cases of a sequence of twin births are on record; the mother of the famous Dr Lettsom had twins seven times, all of whom were males‡; Dr Lettsom and his twin brother were the last children borne by her, and were the only two who survived. A patient of Mr Herbert Spencer's told him that she herself had had six pairs of twins and no other pregnancy.

A history due to Peiper (Fig. 2 of our pedigree plate) shows a sequence of nine twin births; each pair of twins were of unlike sex. Is there a tendency for a sequence confined to one type of plural birth to be uniform with regard to sex? The only case of such a sequence known to me refers to three triplets consisting of nine boys; there was an interval of 18 months between each birth; all the boys are living. Uniformity of the male sex may well have occurred in the triplets of Maddalena Granata, if we can trust that history, though she did not have exclusively triplets and we are only told that forty-nine of her fifty-two children were males.

An epitaph of some interest is referred to by Hakewill§ in 1635: "Neither can I call to minde any example in all antiquity parallel to that of a woman buried in the Church at Dunstable who (as her epitaph testifies) bore at three severall times three children at a birth and five at a birth two other times." This epitaph

* See *Medical Press and Circular*, Vol. I. for 1886, p. 57. London, 1886.

† Reference in *The Lancet*, Vol. II. for 1863, p. 466. London, 1863.

‡ Bibl. No. 9. § Bibl. No. 4.

is given by Francis Thynne in his collection (Cleopatra, c. 3, p. 114) from the MS. in the Cottonian Library; it was copied by him in Sept. 1583, and is given in the appendix to Hearne's 1733 edition of *Chronicon sive Annales Prioratus de Dunstable.* Several reproductions have been published with some variations in spelling. Francis Thynne's manuscript is not easy to decipher by the unskilled, and we are greatly indebted to Professor H. E. Butler for copying the epitaph from the script for us; it runs as follows:

Hic William Mulso sibi quam sociault et Alice
Marmore sub duro conclusit nex (?) generalis
Ter tres bis quinos natos hace fertur habere
Per sponsos binos, Deus his clemens miserere.

We are also much indebted to Mr T. W. Bagshawe, of the Dunstable Library and Museum, for a description of the memorial stone (taken from Derbyshire's *History of Dunstable*, 1872). "In the middle aisle, opposite the pulpit is a large slab, beneath which is buried a woman who had nineteen children at five births; viz. three several times three children at a birth, and twice five at other times. It is not strange that this account is frequently disbelieved;...but if tradition the most straightforward can be accredited, it is a literal fact. Upon the slab were the figures of a man and woman in brass, both dressed in gowns, with their hands in the attitude of prayer, at their feet was the inscription. Beneath the latter were two groups, one of boys and the other of girls, with the types of the evangelists at the corners." Derbyshire, writing in the nineteenth century, quotes the inscription from Hearne's edition of the Chronicles; he does not state where he found the description of the slab, which is not, so far as I have discovered, to be found in Hearne's volumes.

Hakewill's interpretation of the epitaph has been repeatedly quoted; indeed, I must confess to having myself been ready to present the case as an authentic account of a sequence of three triplets and two quintuplets. Professor Butler, however, is of the opinion that the epitaph by no means justifies this conclusion; he considers that all we can accept from the epitaph is that Alice had nineteen children by two husbands, nineteen being expressed as "Ter tres bis quinos" to meet the needs of the verse. The date of the stone is not given, all we know is that it was prior to 1583. Possibly further information was available to Hakewill; in the absence of any reference to it we must reject the case from consideration as an undoubted example, which would have been of interest for our purposes, and express gratitude to Professor Butler for his caution.

(c) I will now add a few illustrative examples showing the occurrence of plural births in the same family; these cases are all relatively recent and the histories are probably accurate as far as they go. Peiper, in 1923*, Fig. 2, publishes an extremely interesting history of a woman, II. 4, married to a man who had a twin sister; they had a sequence of nine pairs of twins; each pair included a male and a female; all the eighteen children died, and one is reminded of the statements of

* Bibl. No. 17.

Fig. 6. Roberts

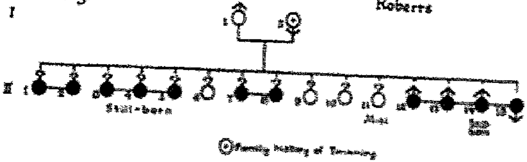
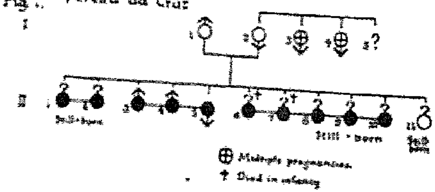


Fig. 7. Pereira da Cruz



Aristotle* and Pliny regarding the high mortality of twins of unlike sex—such a pedigree as this in Aristotle's day might readily lead to a tradition, which of course would have no general application. It is however of interest to remember the mortality of Dr Lettsom's sibship. The mother of the twins in Peiper's case had nine siblings and so far as was known no twins had been born in her family; this woman married a second husband and had six single births, four daughters and two sons; the first of these children, III. 19, died aged three months; III. 20—23 all died at birth; the last child, III. 24, delivered by Caesarian section, was the only one of the mother's twenty-four children who remained alive; moreover III. 24 was not without difficulties as he had to be operated on for spasm of the pylorus. The interesting points of this pedigree are the marked inheritance of twinning, through the father; also the long sequence of twins of unlike sex, without any triplets or single births. The high mortality rate would appear probably to be due to the mother rather than to her twins, since five of her six single births by a second husband shared the same fate.

Another interesting small pedigree of twinning, due to Strassmann†, is shown in Fig. 3; it includes six pairs of twins of whom four pairs were known to be of like sex, one pair included both sexes; four of the cases had arisen through the father. The only female of the family to be married, II. 4, was herself a twin; she had only single births; the male twin of II. 4 was twice married and had twin children by both wives; the only male of a single birth to marry had twin offspring. No triplets or higher multiple births are noted to have occurred in this family. Fig. 4, due to Grigg‡, provides a marked contrast to the two previous cases in that it presents a history of multiple births including twins and triplets, transmitted through the female in every case, so far as we can judge. The family history is very incomplete; it was given to the recorder by his patient, IV. 7; she said that her mother, III. 7, had two brothers who were married, neither of them had any children; we are not told whether these brothers belonged to the triplet of that generation, or were twins. Did these brothers abstain from parenthood, for perhaps economic reasons, or were they infertile on physiological grounds? Apart from this reference we have no knowledge of any male of the family having married, nor are we told how many males had been born in the various sibships. IV. 7, aged 44 at the time of observation, had had sixteen children, triplets twice with ten single births; she had seven daughters living; we are not told whether the remaining nine children were males or females or at what age they died. The eldest daughter of IV. 7 had four children, of whom three were born at one birth; the second daughter of IV. 7 was also married and was pregnant at the time of the record. A maternal great-aunt of IV. 7, aged 90 and still living, said that her grandmother had told her that triplets had occurred in the family as far back as any record could be obtained. We have tried to get in touch with this exceedingly interesting family, but have not succeeded in doing so. It is very unfortunate that

* *De Animalibus*, Liber VII. Cap. IV. Aristotle, after stating that mixed sexes in a litter do not affect surviving in the case of animals, adds: "but in the case of men few survive of twins if one is female and the other male."

† Bibl. No. 16.

‡ Bibl. No. 13.

the only two males of the family of whom we have any record married, but had no children.

Three further cases showing the liability to multiple births transmitted, so far as we can judge, through the mother seem worth putting on record. *The Lancet* (Vol. I. for 1889, p. 392) reports a case which was communicated to the Lisbon Medical Society by Señor Pereira da Cruz, of Aveiro. A woman had had four confinements in eight years; first, twins were still-born or died soon after birth: the following year a triplet, including two boys and one girl, was born; five years later a quintuplet was born, of which the first child lived 50 days, the second lived 28 hours, the remaining three were still-born; two years later the woman had a single birth of a still-born child. The mother of this sibship had two sisters and an aunt who were said to have been the subjects of similar multiple pregnancies; there is no mention of plural births among the children of any male of the family, nor are we told of any plural births in the mother's sibship; no information is given as to whether the aunt who had plural births was on the maternal or on the paternal side. The case is thus very imperfectly described, but the history of the quintuplet in association with other plural births, of the heavy mortality and of the similar history on the mother's side of the family provide positive information of considerable interest. In examining this case and the following we must remember Mr Spencer's warning that when one member of a twinship dies the fact often remains unrecorded; we should not perhaps regard it as demonstrated that the mother of either of these sibships was certainly born alone.

In 1905, Dr Roberts, of New South Wales, described a case of his experience*. A woman, I. 2 of Fig. 6, aged 32, had been married twelve years; she had a family history of twinning on her maternal side; she had a personal history of seven confinements at the first of which twins were delivered, at the second a triplet which was born prematurely at seven months and did not survive; there followed a single birth; the fourth confinement produced twins; the fifth and sixth were two single births; then there was a miscarriage and eighteen months later a quadruplet consisting of three males and a female of whom one male was still-born. Ten of this woman's fourteen children were living. At the birth of the quadruplet four adherent placentas were removed, they were distinct and separate; four separate bags of membranes are described.

Finally, it is perhaps worth while to include here the case of Dr de Leon, reported by Foy†. A woman aged 26 had given birth to fourteen children by two husbands; by her first husband she had one pair of twins and four single births; by her second husband she had eight children in three years—two pairs of twins and a quadruplet; all members of the quadruplet were living, the weight at birth of the children was 6 lbs., 5 lbs., 4½ lbs., and 4 lbs. respectively.

This short account of a few selected cases and references is extremely limited in its scope; the purpose of it is primarily to suggest through a few illustrative

* Bibl. No. 15. † Bibl. No. 12.

examples the varied problems which arise in any consideration of the inheritance of multiple births. The historic extreme cases, which we must assuredly receive in a sceptical and doubting spirit, cannot for the most part be disproved, but unfortunately under modern social conditions it is difficult to believe that the possibility of some of them could ever again be demonstrated, at least in European populations. It is indeed astonishing that in 1886 the report concerning Maddalena Granata did not attract more interest and comment and obtain verification; if this be an accurate history, it certainly justifies some belief in many other reports which have been discarded as too improbable for consideration. In our own country it is almost equally surprising that the family history due to Grigg was allowed to be lost sight of and remain without any adequate description. How very incurious most of us appear to be with regard to remarkable occurrences which do not touch upon our own immediate needs and occupations.

I would call attention to one further point of some importance and interest. Dr R. A. Fisher, from his very valuable study of Triplet Children*, concludes that "The triplet data indicate that the paternal influence is only exerted in the production of diembryony." Now my very small collection of a few exceptional cases cannot be used to *prove* anything at all, but Figs. 1 and 2 do suggest that Dr Fisher's experience is not of universal application. In Fig. 1, the twin births of IV. 10—13 are presumably due to their father's influence, each pair is of like sex, but their mother had no hesitation in her statement that they were very unlike in other respects. Again, in Fig. 2, due to Peiper, the long sequence of twins of unlike sex must be attributed to their father's influence.

Should the production of multiple births be regarded as a mark of exceptional vitality, or is it a sign of degeneracy with lack of control, or can we deduce nothing from the evidence available? There appears to be a good deal of evidence supporting the view that on the whole the older mothers tend to produce multiple births, but we are not justified in accepting this as a sign of weakness or decadence on her part; so far as I know, it is not necessarily the mothers who are worn out by repeated child-births who have twins, and we need to be very cautious in drawing any general conclusions from the mere fact of the age of the mother at the birth of her offspring.

For obvious reasons the infant death rate is greatly augmented at multiple births, and thus it would appear to be a wasteful form of reproduction. As long ago as 1820, Merriman† gave figures from the Dublin Lying-in-Hospital referring to the years 1787—1793, showing that of 368 twin children born to 184 mothers, 17·1 °/₀ died in hospital or were still-born, whereas 10,199 uniparous women only lost 7 °/₀ of their children. Mitchell gives worse figures for the same hospital, pointing again to the excessive mortality of twin babies in 1862. Strassmann quotes Prussian statistics since 1871‡ indicating that the numbers of still-born children (*a*) at single births, (*b*) among twins, and (*c*) among triplets are as 3·3 : 5·8 : 12·1. Mitchell§ formed the impression, and gives some figures to support

* Bibl. No. 18. † Bibl. No. 10. ‡ Bibl. No. 16. § Bibl. No. 11.

it, that twins and triplets occurred with rather a marked frequency amongst idiots or in the families of idiots, but I have no knowledge that his suggestion has been confirmed by an adequate investigation.

Dr Fisher found no evidence suggesting that at a fixed age the surviving members of triplets were undersized as compared with children of single births. It is of interest to hear from Mr Herbert Spencer of a woman patient of his over six feet in height, who was one of a quadruplet; her three brothers were also over six feet in height and were serving in the same regiment in India.

BIBLIOGRAPHY OF WORKS CITED.

1. Paré, Ambroise: *The Works of.* Translated by T. Johnson. London, 1649. [References on pp. 654, 655.]
2. Cromer, Martin: *De origine et rebus gestis Polonorum libri XXX.* Coloniae Agrippinae, 1589. [References on pp. 164, 197.]
3. Cock, Henrique: *Relación del viaje hecho por Felipe II, en 1585, a Zaragoza, Barcelona y Valencia, escrita por Henrique Cock.* Madrid, 1876. [References on pp. 248, 249.]
4. Hakewill, G.: *An Apologie or Declaration of the Power and Providence of God in the Government of the World.* London, 1635. [Reference on p. 253.]
5. Wanley, N.: *The Wonders of the Little World: or a General History of Man.* London, 1678. [Revised and enlarged, 1806. Reference on p. 81.]
6. Seignette: Accouchement surprenant. *Le Journal des Sçavans,* T. for 1684, p. 92. Paris, 1684.
7. Garthshore, M.: A remarkable Case of Numerous Births, with Observations. *Phil. Trans.* Vol. LXXVII, pp. 344—358. London, 1787.
8. Hermann, B. F. J.: *Statistische Schilderung von Russland.* St Petersburg, 1790.
9. Pettigrew, T. J.: *Memoirs of the Life and Writings of the late John Coakley Lettsom, M.D., F.R.S.* London, 1817. [Reference in Vol. I. on p. 6.]
10. Merriman, S.: *A Synopsis of the various kinds of Difficult Parturition.* London, 1820.
11. Mitchell, A.: Plural Births in Connexion with Idiocy. *Dublin Medical Press,* Vol. II. for 1862, pp. 526—527. Dublin, 1862.
12. Foy, G.: Plural Births. *Medical Press and Circular,* Vol. II. for 1890, pp. 304—308. London, 1890.
13. Grigg, W. C.: Heredity as to Triplets. *British Medical Journal,* Vol. I. for 1890, p. 541. London, 1890.
14. Barfurth, D.: Ein Zeugnis für eine Geburt von Siebenlingen beim Menschen. *Anatomischer Anzeiger,* Bd. X. S. 330—332. Jena, 1895.
15. Roberts, L. W.: A Case of Quadruplets. *British Medical Journal,* Vol. II. for 1905, p. 629. London, 1905.
16. Strassmann, P.: Die Anthropologische Bedeutung der Mehrlinge. *Zeitschrift f. Ethnologie,* Bd. XL. S. 362—382. Berlin, 1908.
17. Peiper, A.: Zur Vererbung der Zwillingsschwangerschaft durch den Mann. *Klinische Wochenschrift,* Bd. II. 1932, S. 1651. Berlin, 1923.
18. Fisher, R. A.: Triplet Children in Great Britain and Ireland. *Proc. R. S.,* Vol. CII. B. pp. 286—311. London, 1928.

ON CORRELATION FUNCTIONS OF TYPE III.

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1. As originally shown by Helmert* the distribution of the second order moments in samples of N individuals, chosen at random from an infinitely large supply (parent population) will be of the form generally called Type III if the supply be normally distributed and the moments are taken around the true mean (mean of the supply). Helmert also has shown that the same form of error distribution will be obtained if the sample moments are taken around the respective sample means, a result which has later on been rediscovered by several writers.

It will be the main purpose of this paper to study the correlation surface obtained for the second order moments in samples of x and y, which are taken at random from a normally distributed bivariate supply (supposed to be infinitely large). As it is clear that the marginal distributions of this surface will both be of Type III, we have here an interesting object of investigation, i.e. a solid Type III distribution.

The most convenient way to study problems of this kind seems to be by the aid of the so-called *reciprocal* or *characteristic functions* of the distributions. These functions are defined in the following way:

Univariate case: $$U(t) = \int dx\, f(x)\, e^{xt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Bivariate case: $$U(t_1, t_2) = \iint dx\, dy\, f(x, y)\, e^{xt_1 + yt_2} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Here f denotes the frequency function of the distribution in question and U its characteristic function. The integrations are to be extended over the whole range of applicability of f.

Evidently $U(t, 0)$ and $U(0, t)$ are the respective characteristic functions of the marginal distributions of the correlation $f(x, y)$.

If the Napierian logarithm of U is developed into powers of t (or t_1 and t_2) the coefficients of this expansion correspond to the so-called seminvariants λ (of Thiele) in the following way:

Univariate case: $$\log U(t) = \sum_{k=1}^{\infty} \frac{\lambda_k}{k!} t^k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Bivariate case: $$\log U(t_1, t_2) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\lambda_{kl}}{k!\, l!} t_1^k t_2^l \quad \ldots\ldots\ldots\ldots\ldots (4).$$

* See *Biometrika*, Vol. XXIII. 1931, p. 416. "Historical Note on the Distribution of the Standard Deviations of Samples of any Size drawn from an Indefinitely large Normal Parent Population." Editorial.

If the characteristic function of a distribution has been found the theorem of Fourier gives the following solutions of the integral equations (1) and (2):

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dw\,U(wi)\,e^{-xwi}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

and

$$f(x,y)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dw_1\,dw_2\,U(w_1i,\,w_2i)\,e^{-xw_1i-yw_2i}\quad\ldots\ldots\ldots(6).$$

We shall not here go into the questions of convergency of this inversion as it is quite evident that no troubles of this kind will arise in the applications contained in this paper.

Furthermore, as is easily seen, the characteristic function of the distribution of *a function of* x, $\{z=g(x)\}$, is given by the equation

$$U(t)=\int dx\,f(x)\,e^{g(x)t}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and the characteristic function of the bivariate distribution of two different functions of x and y, $\{z_1=g(x,y),\ z_2=h(x,y)\}$, is given by the equation

$$U(t_1,t_2)=\iint dx\,dy\,f(x,y)\,e^{g(x,y)t_1+h(x,y)t_2}\quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Putting $h\equiv 0$ in this equation we obtain the characteristic function of the distribution of a function $g(x,y)$ of a pair of correlated variables x and y.

Finally, it is well known and easily demonstrated that the characteristic function of the distribution of the sum of a number of independent variables is equal to the product of the characteristic functions of the distributions of the respective variables taken alone.

This we may state in the following form: If $U_i(t)$ is the characteristic function of the distribution of x_i, then the characteristic function of the distribution of

$$z=x_1+x_2+x_3+\ldots+x_N$$

is

$$U(t)=U_1(t)\,.\,U_2(t)\,.\,U_3(t)\ldots U_N(t)\quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

if the variables $x_1, x_2, x_3, \ldots x_N$ are distributed independently of each other.

Similarly: If $U_i(t_1,t_2)$ is the characteristic function of the correlation of any pair x_i, y_i, then

$$U(t_1,t_2)=U_1(t_1,t_2)\,.\,U_2(t_1,t_2)\,.\,U_3(t_1,t_2)\ldots U_N(t_1,t_2)$$

is the characteristic function of the correlation of $z_1=x_1+x_2+\ldots+x_N$ and $z_2=y_1+y_2+\ldots+y_N$, if the different pairs x_i, y_i are chosen independently of each other.

2. I shall begin by demonstrating Helmert's proposition with the aid of the characteristic function.

Assuming the frequency function of x in the supply to be

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{x^2}{2\sigma^2}},$$

the characteristic function of the distribution of $\frac{x^2}{N}$ is, according to (7), equal to

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2\sigma^2}+\frac{x^2 t}{N}} = \left(1-\frac{2\sigma^2}{N}t\right)^{-\frac{1}{2}} \quad \text{..................(10).}$$

Consequently the characteristic function of the distribution of the second order moment about the "true" mean, $z = \frac{1}{N}\Sigma x^2$, in samples of N will, according to (9), be

$$U(t) = \left(1-\frac{2\sigma^2}{N}t\right)^{-\frac{N}{2}} \quad \text{..........................(11).}$$

On account of (5) we finally get the frequency function of z from the equation

$$f_N(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw\, e^{-zwi}\left(1-\frac{2\sigma^2}{N}wi\right)^{-\frac{N}{2}} \quad \text{..................(12).}$$

On evaluation the integral (12) gives the Type III distribution spoken of. This will most easily be seen by the transformation

$$\frac{2\sigma^2}{N}w = \tau,$$

$$\frac{zN}{2\sigma^2} = \zeta,$$

which gives $$f_N(z) = \frac{Ne^{-\zeta}}{2\sigma^2 2\pi}\int_{-\infty}^{\infty} d\tau\,(1-\tau i)^{-\frac{N}{2}} e^{(1-\tau i)\zeta} \quad \text{..................(13),}$$

or, putting $(1-\tau i)\zeta = \xi$,

$$\frac{2\sigma^2}{N} f_N(z) = e^{-\zeta}\zeta^{\frac{N}{2}-1}\frac{1}{2\pi}\int_l i\xi^{-\frac{N}{2}} e^{\xi} d\xi \quad \text{..................(14),}$$

ξ being a complex variable and the integration taking place along a line running at the distance $+\zeta$, parallel to the imaginary axis. As the integrand vanishes at the extreme ends of this line the value of the integral will not be affected by a variation in ζ. Hence, as the integral is clearly convergent, it is constant and independent of ζ, and it follows that we must have

$$f_N(z) = k\frac{N}{2\sigma^2}\zeta^{\frac{N}{2}-1}e^{-\zeta} \quad \text{..........................(15).}$$

An actual evaluation of the integral will show that we have $k = \frac{1}{\Gamma(\frac{1}{2}N)}$, which also follows from the equation

$$\int_0^{\infty} f(z)\,dz = 1,$$

which must here necessarily be fulfilled.

Thus we find the Type III distribution for z,

$$f_N(z) = \frac{1}{\Gamma(\frac{1}{2}N)}\left(\frac{N}{2\sigma^2}\right)^{\frac{N}{2}} z^{\frac{N}{2}-1} e^{-\frac{N}{2\sigma^2}z} \quad \text{.....................(16).}$$

Q.E.D.

By taking the logarithm of (11) and expanding in powers of t we get the following well-known formula for the seminvariants of Type III as given in (16),

$$\lambda_r = (r-1)!\left(\frac{2\sigma^2}{N}\right)^{r-1}\sigma^2 \quad \dots\dots\dots\dots\dots\dots\dots\dots(17).$$

3. *A Generalised Form of Type III* will be at hand in the error function of the second order moment, taken around a fixed point at a given distance a from the mean. To derive this function we have, in analogy to (10), the characteristic function of the distribution of $\frac{(x-a)^2}{N}$ equal to

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2\sigma^2}+\frac{(x-a)^2}{N}t} = \left(1-\frac{2\sigma^2}{N}t\right)^{-\frac{1}{2}} e^{\frac{a^2 t}{N-2\sigma^2 t}} \quad \dots\dots\dots(10^*),$$

which gives for the frequency function of $z=\frac{1}{N}\Sigma(x-a)^2$ in samples of N the following integral form (cp. (12)):

$$F_N(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw\left(1-\frac{2\sigma^2}{N}wi\right)^{-\frac{N}{2}} e^{\frac{\frac{a^2 wi}{2\sigma^2}}{1-\frac{2\sigma^2}{N}wi}-zwi} \quad \dots\dots\dots(12^*).$$

This function, which passes over into the function (12) and the ordinary Type III curve when $a\to 0$, generally* cannot be expressed in elementary functions, except as an infinite series. If, for instance, the factor

$$e^{\frac{\frac{a^2 wi}{2\sigma^2}}{1-\frac{2\sigma^2}{N}wi}}$$

is developed in powers of the exponent, and it is noticed that according to (12) we have, when $n<\frac{N}{2}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dw\,(iw)^n\left(1-\frac{2\sigma^2}{N}wi\right)^{-\frac{N}{2}} e^{-zwi} = (-1)^n\frac{d^n}{dz^n}f_N(z) \quad \dots\dots(13^*),$$

we find the following serial expression:

$$F_N(z) = f_N(z) - a^2 f'_{N+2}(z) + \frac{a^4}{2!}f''_{N+4}(z) - \frac{a^6}{3!}f'''_{N+6}(z) + \dots \quad \dots(16^{**}),$$

which series may readily be reduced to a form fit for numerical applications. As a matter of fact we may write

$$f^{(n)}{}_{N+2n}(z) = (-1)^n f_N(z)\,P_n(z),$$

where $P_n(z)$ is a polynomial of the nth degree in z. Its general form is, if we put $\frac{N}{2\sigma^2}z=\zeta$,

$$P_n(z) = \left(\frac{N}{2\sigma^2}\right)^n \Gamma(\tfrac{1}{2}N)$$
$$\times\left[\frac{\zeta^n}{\Gamma(\frac{1}{2}N)+n} - \binom{n}{1}\frac{\zeta^{n-1}}{\Gamma(\frac{1}{2}N+n-1)} + \binom{n}{2}\frac{\zeta^{n-2}}{\Gamma(\frac{1}{2}N)+n-2} - \dots + (-1)^n\right].$$

* When $N=1$ (12*) must be integrable as it is easily shown that we must have

$$F_1(z) = Kz^{-\frac{1}{2}}e^{-\frac{a(+\sqrt{z})^2}{2\sigma^2}}.$$

It will be seen that (16**) is a special case of the development of a generalized Pearson Type III, given by Romanowsky (*Biometrika*, XVI. pp. 114—116). The objections advanced by Professor Pearson against Romanowsky's generalisations do not apply to the special form here arrived at, as (16**) is, by the nature of the problem of which it gives the solution, an analytically well-defined probability function with a limited number of arbitrary parameters (one more than in the ordinary Type III)*.

Taking the logarithm of (10*), multiplying by N and expanding in powers of t, we get, on comparing with formula (3), the following simple general formula for the seminvariants of the generalised Type III, as defined by formula (12*),

$$\lambda_r = (r-1)!\left(\frac{2\sigma^2}{N}\right)^{r-1}(\sigma^2 + ra^2) \quad \text{.....................(17*)}.$$

Forming the standardised seminvariants

$$\gamma_r = \frac{\lambda_r}{\lambda_2^{r/2}} = (r-1)!\left(\frac{2\sigma^2}{N}\right)^{\frac{r}{2}-1} \frac{\sigma^2 + ra^2}{(\sigma^2 + a^2)^{r/2}},$$

we see that our generalised Type III will pass over into the normal frequency function (of which $\gamma_r = 0$, except for $r = 2$) not only when N grows, but also with a growing value of a^2.

As a frequency function to be fitted to a given set of data the generalised Type III has four arbitrary constants, i.e. in the notation here used, the parameters a^2, $\frac{2\sigma^2}{N}$ and N, and the position, on the scale in which the variate is measured, of the origin (starting point) of the curve. These constants may be determined from the mean and higher moments in the following way: We first have

$$\lambda_2 = \nu_2; \quad \lambda_3 = \nu_3; \quad \lambda_4 = \nu_4 - 3\nu_2^2,$$

if ν_k denotes the central moment of the kth order. We further put

$$s = \sqrt{9\lambda_3^2 - 6\lambda_2\lambda_4}.$$

This quantity must be real as a first condition for applicability. If $s = 0$, we have the ordinary Type III curve†. We now get

$$a^2 = \frac{s}{6\lambda_4^2}(3\lambda_3 + s)^2; \quad \frac{2\sigma^2}{N} = \frac{1}{6\lambda_2}(3\lambda_3 - s); \quad \tfrac{1}{2}N = \frac{\lambda_3 - s}{2\lambda_4^3}(3\lambda_3 + s)^3.$$

If Z_0 is the starting point of the curve, we finally have, m denoting the mean of z,

$$Z_0 = m - \sigma^2 - a^2 = m - \frac{1}{6\lambda_4^2}(3\lambda_3 - 2s)(3\lambda_3 + s)^2.$$

* As pointed out by Dr E. S. Pearson the function $F_N(z)$ must also be identical with a function studied by R. A. Fisher in connection with his investigations of the "General Sampling Distribution of the Multiple Correlation Coefficient" (*Proc. R. Soc.* A.V. 121, 1928). Fisher uses another form of development, which is obtained if the series (16**) is rearranged according to powers of ζ.

† In Pearson's well-known notation we have

$$s = \nu_2\sqrt{3\nu_2(6 + 3\beta_1 - 2\beta_2)}.$$

It is seen that we must have on the one side (in order that the coefficients should be real)

$$3\lambda_3^2 > 2\lambda_2\lambda_4,$$

and on the other side (in order that $\frac{1}{2}N > 0$)

$$\lambda_3 > s,$$

which gives
$$4\lambda_3^2 < 3\lambda_2\lambda_4.$$

It follows that λ_4 must be positive (positive excess). The seminvariant λ_3 can always be considered as positive, as this only requires that the positive direction on the z-axis is appropriately chosen.

Thus the above inequalities require that we must have

$$8\lambda_2\lambda_4 < 12\lambda_3^2 < 9\lambda_2\lambda_4,$$

or, in Pearson's notation, $\quad 8\beta_2 - 24 < 12\beta_1 < 9\beta_2 - 27.$

In the β_1, β_2 diagram this inequality defines the area between two straight lines, intersecting in the point $\beta_1 = 0$, $\beta_2 = 3$, one of the lines being the line on which the ordinary Type III is strictly valid. The angle between the lines is rather narrow, but one has the impression that it must embrace an important part of the combinations of β_1 and β_2 occurring in practice.

4. We shall now treat the following:

Problem 1. From an infinitely large normal bivariate supply samples of N pairs of x and y are taken at random. What will be the bivariate distribution of $z_1 = \frac{1}{N}\Sigma x^2$, and $z_2 = \frac{1}{N}\Sigma y^2$, if x and y are reckoned from the true means?

Putting
$$f(x, y) = \frac{1}{\sigma_1\sigma_2 2\pi\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - 2r\frac{xy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)} \quad \text{.........(18)},$$

we find from (8) that the characteristic function of the bivariate distribution of

$$g = \frac{x^2}{N} \text{ and } h = \frac{y^2}{N}$$

is
$$\left[\left(1 - \frac{2\sigma_1^2}{N}t_1\right)\left(1 - \frac{2\sigma_2^2}{N}t_2\right) - r^2\frac{4\sigma_1^2\sigma_2^2}{N^2}t_1t_2\right]^{-\frac{1}{2}} \quad \text{...............(19)}.$$

Consequently the characteristic function of the bivariate distribution of z_1 and z_2 will be

$$U(t_1, t_2) = \left[\left(1 - \frac{2\sigma_1^2}{N}t_1\right)\left(1 - \frac{2\sigma_2^2}{N}t_2\right) - r^2\frac{4\sigma_1^2\sigma_2^2}{N^2}t_1t_2\right]^{-\frac{N}{2}} \quad \text{......(20)},$$

and the correlation function of z_1 and z_2 will be given by the double integral

$$F(z_1, z_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dw_1 dw_2$$

$$\times \left[\left(1 - \frac{2\sigma_1^2}{N}w_1 i\right)\left(1 - \frac{2\sigma_2^2}{N}w_2 i\right) + r^2\frac{4\sigma_1^2\sigma_2^2}{N^2}w_1 w_2\right]^{-\frac{N}{2}} e^{-z_1 w_1 i - z_2 w_2 i} \quad \text{...(21)}.$$

Introducing the transformation

$$\zeta_1 = \frac{z_1 N}{2\sigma_1^2}; \quad \zeta_2 = \frac{z_2 N}{2\sigma_2^2},$$

and putting

$$\frac{2\sigma_1^2}{N} w_1 = \tau_1; \quad \frac{2\sigma_2^2}{N} w_2 = \tau_2,$$

we find for the correlation function of ζ_1 and ζ_2 the somewhat simpler formula

$$f(\zeta_1, \zeta_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau_1 d\tau_2 [(1-\tau_1 i)(1-\tau_2 i) + r^2 \tau_1 \tau_2]^{-\frac{N}{2}} e^{-\zeta_1 \tau_1 i - \zeta_2 \tau_2 i} \quad \ldots (22).$$

It is immediately seen that if $r=0$ this formula reduces to the product of two integrals of the form (13), and $F(z_1, z_2)$ will then be the product of two Type III functions of the form (16). It is also easily seen that for any value of r the marginal distributions,

$$f_1(\zeta_1) = \int_0^{\infty} d\zeta_2 f(\zeta_1, \zeta_2); \quad f_2(\zeta_2) = \int_0^{\infty} d\zeta_1 f(\zeta_1, \zeta_2) \quad \ldots\ldots\ldots\ldots (23),$$

take the form (13). As a matter of fact the characteristic functions of the marginal distributions are $U(t_1, 0)$ and $U(0, t_2)$, respectively. The correlation function (21) or (22) is thus of Type III in both its marginals*.

Now, as our function has only one more arbitrary constant than the Bravais function (18), it must be a rather special form of correlation function. As a matter of fact it will be found on inspection that both the marginals of (21) have the same skewness or, which is the same, that the marginals of the function (22) are identical in form when reduced to scales in which the standard deviations are equal. Furthermore, the regression may be shown to be strictly linear, which, in a way, is a serious restriction to the applicability of a skew correlation function. But, on the other side, the coefficient of correlation is free and not a function of the marginal constants†.

The linearity of regression will most conveniently be shown by finding the seminvariants of (22). We evidently have

$$-\tfrac{1}{2} N \log [(1-t_1)(1-t_2) - r^2 t_1 t_2] = \Sigma\Sigma \frac{\lambda_{kl}}{k!\, l!} t_1^k t_2^l \quad \ldots\ldots\ldots\ldots (24).$$

Hence we find (besides $\lambda_{00} = 0$),

$$\lambda_{k0} = \lambda_{0k} = \tfrac{1}{2} N (k-1)! \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25),$$

which gives

$$\left.\begin{aligned} \lambda_{10} = \lambda_{01} = \tfrac{1}{2} N; \quad \lambda_{20} = \lambda_{02} = \tfrac{1}{2} N, \\ \lambda_{30} = \lambda_{03} = N; \quad \lambda_{40} = \lambda_{04} = 3N. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (26)$$

* Of course, the same would have been the case if we had taken the correlation surface of the second order moments about the sample means instead of, as here has been done, around the means of the parent population. In *Biometrika*, Vol. XVII, 1928, Professor Pearson has studied the error surface of the sample standard deviations σ_1 and σ_2. Deriving from this the surface of $z_1 = \sigma_1^2$ and $z_2 = \sigma_2^2$ a formula very similar to (21) and (22) will be obtained.

† Cp. Karl Pearson: "Notes on Skew Frequency Surfaces," *Biometrika*, Vol. XV, pp. 222 f.

Further, we easily derive the general formula

$$\lambda_{k1} = \lambda_{1k} = \tfrac{1}{2} N\, k!\, r^2 \quad \text{.........................(27)},$$

which gives

$$\lambda_{11} = \tfrac{1}{2} N r^2,$$
$$\lambda_{21} = \lambda_{12} = N r^2,$$
$$\lambda_{31} = \lambda_{13} = 3 N r^2.$$

For λ_{22} we find the particular value

$$\lambda_{22} = N (2r^2 + r^4).$$

As the general criterion for linear regression is *

$$\lambda_{k1}\lambda_{20} = \lambda_{k+1,0}\lambda_{11}, \quad \lambda_{1k}\lambda_{02} = \lambda_{0,k+1}\lambda_{11} \quad \text{.................(28)},$$

respectively, for the two regressions ζ_2 on ζ_1, and ζ_1 on ζ_2, it is seen that both the regressions of (22) are linear. The coefficient of correlation of ζ_1 and ζ_2 (as well as of z_1 and z_2) is

$$\rho = \frac{\lambda_{11}}{\sqrt{\lambda_{20}\lambda_{02}}} \quad \text{.................................(29)},$$

whence we see that we have

$$\rho = r^2 \quad \text{.....................................(30)}.$$

The regression lines in the plane of ζ_1, ζ_2 are consequently given by the equations

$$\left.\begin{aligned} \bar{\zeta}_2 &= \tfrac{1}{2} N + r^2 (\zeta_1 - \tfrac{1}{2} N) \\ \bar{\zeta}_1 &= \tfrac{1}{2} N + r^2 (\zeta_2 - \tfrac{1}{2} N) \end{aligned}\right\} \quad \text{.........................(31)}.$$

In the distribution of z_1 and z_2 the seminvariants are clearly

$$\lambda_{k0} = \tfrac{1}{2} N (k-1)! \left(\frac{2\sigma_1^2}{N}\right)^k \quad \text{.........................(32)},$$

$$\lambda_{k1} = \tfrac{1}{2} N\, k!\, r^2 \left(\frac{2\sigma_1^2}{N}\right)^k \frac{2\sigma_2}{N} \quad \text{.........................(33)}.$$

Hence we have

$$\lambda_{10} = \sigma_1^2; \quad \lambda_{01} = \sigma_2^2,$$

$$\lambda_{20} = \frac{2\sigma_1^4}{N}; \quad \lambda_{11} = \frac{2\sigma_1^2\sigma_2^2 r^2}{N}; \quad \lambda_{02} = \frac{2\sigma_2^4}{N};$$

$$\lambda_{30} = \frac{8\sigma_1^6}{N^2}; \quad \lambda_{21} = \frac{8\sigma_1^4\sigma_2^2}{N^2} r^2; \quad \lambda_{12} = \frac{8\sigma_1^2\sigma_2^4}{N^2} r^2; \quad \lambda_{03} = \frac{8\sigma_2^6}{N^2};$$

$$\lambda_{40} = \frac{48\sigma_1^8}{N^3}; \quad \lambda_{31} = \frac{48\sigma_1^6\sigma_2^2}{N^3} r^2; \quad \lambda_{22} = \frac{16\sigma_1^4\sigma_2^4 (2r^2 + r^4)}{N^3};$$

$$\lambda_{13} = \frac{48\sigma_1^2\sigma_2^6}{N^3} r^2; \quad \lambda_{04} = \frac{48\sigma_2^8}{N^3}.$$

The regression equations in the z_1, z_2 plane are thus

$$\left.\begin{aligned} \bar{z}_2 &= \sigma_2^2 + r^2 \frac{\sigma_2^4}{\sigma_1^4} (z_1 - \sigma_1^2) \\ \bar{z}_1 &= \sigma_1^2 + r^2 \frac{\sigma_1^4}{\sigma_2^4} (z_2 - \sigma_2^2) \end{aligned}\right\} \quad \text{.......................(34)}.$$

* This formula may be generally known only when the λ_{kl} denote central *moments*. It may, however, be shown that it is valid also in the case of seminvariants.

The regression of the means may also be deduced in another way from the characteristic function, and this method has the merit to be applicable also in finding the regressions of the higher order moments in the arrays (the scedasticity, the clisy, etc.). Writing any correlation function in the form

$$f(x, y) = f(x)\, p_x(y) \quad \text{.............................(35)},$$

$p_x(y)$ is the relative frequency function of y in an x-array. The moments about a fixed point in the x-arrays we denote by

$$v_n'(x) = \int dy\, y^n p_x(y) \quad \text{.............................(36)}.$$

If now the above form of $f(x, y)$ is inserted in (2) we get

$$U(t_1, t_2) = \int dx\, e^{xt_1} f(x) \int dy\, e^{yt_2} p_x(y) \quad \text{.....................(37)},$$

whence we find that

$$\left[\frac{\partial^n U(t_1, t_2)}{\partial t_2^n}\right]_{t_2=0} = \int dx\, e^{xt_1} f(x)\, v_n'(x) \quad \text{..................(38)}.$$

Using the Fourier theorem we thus get the important formula*

$$f(x)\, v_n'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dw\, e^{-xwi} \left[\frac{\partial^n U(wi, t_2)}{\partial t_2^n}\right]_{t_2=0} \quad \text{...........(39)}.$$

Now we have, in the case here in question,

$$\left[\frac{\partial^n U(\tau_1, \tau_2)}{\partial \tau_2^n}\right]_{\tau_2=0} = \frac{(1-\tau_1+r^2\tau_1)^n}{2^n(1-\tau_1)^{\frac{1}{2}N+n}} N(N+2)(N+4)\ldots(N+2(n-1))\ldots(40).$$

Thus we get, remembering (13*),

$$f_N(\zeta_1)\, v_n'(\zeta_1) = N(N+2)(N+4)\ldots(N+2(n-1))\frac{1}{2^n}\Big[f_N(\zeta_1) - \binom{n}{1} r^2 f'_{N+2}(\zeta_1)$$
$$+ \binom{n}{2} r^4 f''_{N+4}(\zeta_1) + \ldots (-1)^n r^{2n} f^{(n)}_{N+2n}(\zeta_1)\Big] \ldots(41).$$

But it will be easily verified that we always have, as already remarked in § 3,

$$f^{(s)}_{N+2s}(\zeta_1) = (-1)^s f_N(\zeta_1)\, P_s(\zeta_1) \quad \text{.......................(42)},$$

where $P_s(\zeta_1)$ is a polynomial of the nth degree in ζ_1. Hence we find that we have

$$v_n'(\zeta_1) = \frac{1}{2^n} N(N+2)(N+4)\ldots(N+2(n-1)) \sum_{s=0}^{n} \binom{n}{s} r^{2s} P_s(\zeta_1) \ldots(43),$$

and it is seen that the array moment of the nth order, when taken around a fixed point (the total mean for instance), is a whole rational function (of the nth degree) of the independent variable.

Carrying out the development for $n=1$ and $n=2$ we now get for the regression of the mean

$$v_1'(\zeta_1) = \bar{\zeta}_2 = \tfrac{1}{2}N + r^2(\zeta_1 - \tfrac{1}{2}N),$$

* By the aid of this formula many regression problems can be readily solved, even when the correlation function is not explicitly given. In a forthcoming paper I have used the formula in developing regression formulae, depending on the marginal frequency function $f(x)$ and the seminvariants λ_{k1} only, thus allowing us to take a full advantage of the theory of univariate frequency functions, in particular that of Pearson.

in accordance with (31), and for the regression of the variance (scedasticity)

$$v_2(\zeta_1) = v_2'(\zeta_1) - [v_1'(\zeta_1)]^2 = (1 - r^2)[\tfrac{1}{2}N(1 - r^2) + 2r^2\zeta_1].$$

It is consequently seen that the central moment of the second degree (the variance) in an array is a linear function of the independent variate. The scedasticity as measured by the variance is thus linear as well as the regression of the mean.

Finally, it may be pointed out that although the marginals of our correlation function are both of Type III this is, except when $r = 0$, *not* the case with the arrays. It will easily be seen that the distribution of an array must be equal to the sample distribution of the second order moment taken around a point which does not coincide with the mean (except when $r = 0$). Thus this distribution is of the generalised Type III given in § 3 (Eqs. (12*) and (16**)). We may conclude that (21) and (22) generally cannot, except as an infinite series, be expressed by elementary functions. This follows from the fact that a correlation function is always expressible as the product of the marginal and the array distribution.

In order to solve the integrals (21) or (22), thus to be able to compute $f(z_1, z_2)$, we shall therefore have to use expansions of one sort or another. Such expansions have at my request been developed by Mr Tage Larsson. One such expansion is obtained by putting $z = z_1$, $\sigma^2 = \sigma_1^2$ in (16) and $z = z_2$, $\sigma^2 = \sigma_2^2(1 - r^2)$, $a^2 = \dfrac{2\sigma^2}{N} r^2 z_1$ in (16**), and multiplying the two expressions.

From this development it follows, on taking account of (17*), that not only the mean and variance but also the seminvariants of higher order of z_2 in a z_1-array are linear functions of z_1.

Our correlation function thus has the property that all the array-seminvariants have linear regression.

5. The problem treated in the preceding section led to correlation functions of Type III, but of a rather special kind, the number of arbitrary parameters being only one more than for normal correlation, the skewness being the same in both marginals.

In order to increase the number of parameters, in particular to get a surface with different degrees of skewness in the two marginals, we shall give the problem in the following somewhat generalized form.

Problem 2. From a normally correlated supply samples of $n + n_1$ x's and $n + n_2$ y's are taken. The n first sample values are taken from individuals for which both x and y are given (n pairs of x and y are first taken). The remaining n_1 x's are taken from individuals for which y is not observed and the n_2 remaining y's from individuals for which x is not observed. Find the bivariate distribution of the sample moments

$$z_1 = \frac{\Sigma x^2}{n + n_1}, \text{ and } z_2 = \frac{\Sigma y^2}{n + n_2},$$

x and y being reckoned from the mean of the supply.

In this case, as may easily be verified, the characteristic function takes the form

$$U(t_1, t_2) = \left[\left(1 - \frac{2\sigma_1^2}{n+n_1}t_1\right)\left(1 - \frac{2\sigma_2^2}{n+n_2}t_2\right) - r^2\frac{4\sigma_1^2\sigma_2^2}{(n+n_1)(n+n_2)}t_1t_2\right]^{-\frac{n}{2}}$$

$$\times\left[1 - \frac{2\sigma_1^2}{n+n_1}t_1\right]^{-\frac{n_1}{2}}\left[1 - \frac{2\sigma_2^2}{n+n_2}t_2\right]^{-\frac{n_2}{2}} \quad \ldots(44).$$

Introducing the variables

$$\zeta_1 = \frac{n+n_1}{2\sigma_1^2}z_1; \quad \zeta_2 = \frac{n+n_2}{2\sigma_2^2}z_2,$$

and

$$\tau_1 = \frac{2\sigma_1^2 t_1}{n+n_1}; \quad \tau_2 = \frac{2\sigma_2^2 t_2}{n+n_2},$$

we have for the characteristic function of the bivariate distribution of ζ_1 and ζ_2,

$$U(\tau_1, \tau_2) = [(1-\tau_1)(1-\tau_2) - r^2\tau_1\tau_2]^{-\frac{n}{2}}[1-\tau_1]^{-\frac{n_1}{2}}[1-\tau_2]^{-\frac{n_2}{2}} \quad \ldots(45).$$

Thus we have

$$f(\zeta_1, \zeta_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dw_1 dw_2 [(1-w_1 i)(1-w_2 i) + r^2 w_1 w_2]^{-\frac{n}{2}}$$

$$\times [1-w_1 i]^{-\frac{n_1}{2}}[1-w_2 i]^{-\frac{n_2}{2}} e^{-\zeta_1 w_1 i - \zeta_2 w_2 i} \quad \ldots(46).$$

In this function there are three more arbitrary parameters than in the Bravais function and two more arbitrary parameters than in (22).

The seminvariants are here evidently, as far as the marginal seminvariants are concerned, given by the formulae

$$\lambda_{k0} = \frac{n+n_1}{2}(k-1)!; \quad \lambda_{0k} = \frac{n+n_2}{2}(k-1)! \quad \ldots\ldots\ldots\ldots\ldots(47).$$

For the mixed seminvariants we evidently have the same formulae as in § 3, i.e.

$$\lambda_{k1} = \frac{n}{2}k!\,r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48).$$

The criteria for linear regression,

$$\lambda_{k1}\lambda_{20} = \lambda_{k+1,0}\lambda_{11},$$

$$\lambda_{1k}\lambda_{02} = \lambda_{0,k+1}\lambda_{11},$$

are here evidently also fulfilled and thus also here the regression is linear. A further calculation will show that the scedasticity as given by the variance is of the second degree in z_1, except for $n_1 = 0$, when it is linear. The coefficient of correlation between ζ_1 and ζ_2 (and also between z_1 and z_2) is

$$\rho = r^2\sqrt{\frac{n^2}{(n+n_1)(n+n_2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(49).$$

But the skewness is, if n_1 and n_2 are not equal, not the same in both marginals.

We have indeed $$\beta_{10}=\frac{\lambda_{30}^2}{\lambda_{20}^3}=\frac{8}{n+n_1},$$

and $$\beta_{01}=\frac{\lambda_{03}^2}{\lambda_{02}^3}=\frac{8}{n+n_2}.$$

Expansions for the computation of (46) have also been developed by Mr Larsson.

6. Another interesting correlation surface will be obtained if we solve the following *Third Problem*: To find the correlation of $z_1=\frac{1}{N}\Sigma x_i^2$ and $z_2=\frac{1}{N}\Sigma y_i$, if x_i, y_i are N pairs taken at random from a normally distributed supply (x and y being deviations from the mean of the supply). *This surface will evidently be of Type III in one margin and of normal type in the other.* Its characteristic function is easily obtained. Evidently

$$U(t_1, t_2)=\left[\frac{1}{\sigma_1\sigma_2 2\pi\sqrt{1-r^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx\,dy\,e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2}-2r\frac{xy}{\sigma_1\sigma_2}+\frac{y^2}{\sigma_2^2}\right)+t_1\frac{x^2}{N}+t_2\frac{y}{N}}\right]^N \quad\text{......(50)},$$

which gives $$U(t_1, t_2)=\left(1-\frac{2\sigma_1^2}{N}t_1\right)^{-\frac{N}{2}}e^{\frac{t_2^2\sigma_2^2}{2N}+\frac{t_2^2\sigma_2^2}{2N}r^2\frac{2\sigma_1^2}{N}\frac{t_1}{1-\frac{2\sigma_1^2}{N}t_1}} \quad\text{.........(51)}.$$

If we write $$\xi=\frac{1}{2}\frac{N}{\sigma_1^2}z_1;\quad \eta=\frac{Nz_2}{\sigma_2},$$

it is seen that $$U(t_1, t_2)=(1-\tau_1)^{-\frac{1}{2}N}e^{\frac{1}{2}N\tau_2^2+\frac{1}{2}N\tau_2^2r^2\frac{\tau_1}{1-\tau_1}} \quad\text{..................(52)}$$

is the characteristic function of the correlation of ξ and η. Hence we have in this case

$$f(\xi,\eta)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dw_1dw_2(1-w_1i)^{-\frac{N}{2}}e^{-\frac{N}{2}w_2^2\left[1+r^2\frac{w_1i}{1-w_1i}\right]-\xi w_1i-\eta w_2i} \quad\text{......(53)},$$

which function may easily be expanded in a series suited for numerical computation.

So, for instance, we may write the characteristic function in the form

$$U(\tau_1,\tau_2)=e^{\frac{n}{2}\tau_2^2}(1-\tau_1)^{-\frac{N}{2}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{N}{2}r^2\right)^k\tau_2^{2k}(1-\tau_1)^{-k} \quad\text{.........(54)},$$

and it follows that if we put

$$f_N(\xi,\eta)=\frac{1}{\sqrt{2\pi}\,\Gamma\left(\frac{N}{2}\right)}\xi^{\frac{N}{2}-1}e^{-\xi-\frac{\eta^2}{2}} \quad\text{....................(55)},$$

the correlation function of ξ and η can be expanded in the series

$$f(\xi,\eta)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\tfrac{1}{2}Nr^2\right)^k\frac{\partial^{3k}}{\partial\xi^k\partial\eta^{2k}}f_{N+2k}(\xi,\eta) \quad\text{..................(56)},$$

which is a combined Charlier and Romanowsky series (with given coefficients).

Using the same methods as in the previous sections we find the following values of the seminvariants:

$$\left.\begin{aligned}
&\lambda_{k0} = \tfrac{1}{2} N (k-1)! \\
&\lambda_{0l} = 0 \quad (\text{except when } l = 2) \\
&\lambda_{02} = N \\
&\lambda_{k1} = 0 \\
&\lambda_{k2} = N\gamma^2 k! \\
&\lambda_{kl} = 0 \quad (\text{when } l \geqslant 3)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (57).$$

By the same methods as in the previous sections we further easily find that

$$\bar{\eta}_\xi = 0,$$

$$\bar{\xi}_\eta = \tfrac{1}{2} N + \frac{\gamma^2}{2}\left(\frac{\eta^2}{N} - 1\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (58).$$

We thus see that the regression of η on ξ is constant (non-regression), and that the regression of ξ on η is parabolic of the second degree. Similarly, the scedasticity as expressed by the variance of η in the ξ-arrays will be found to be linear and the scedasticity as expressed by the variance of ξ in the η-arrays to be parabolic of the second degree.

ON THE PARENT POPULATION WITH INDEPENDENT VARIATES WHICH GIVES THE MINIMUM VALUE OF ϕ^2 FOR A GIVEN SAMPLE.

BY KARL PEARSON.

(1) THIS paper arises from a very bad blunder made by me in the last issue of *Biometrika*, Vol. XXIV. pp. 461—463. It has probably been noticed by others, and I hasten to correct it, for I recognised my error as soon as the printed Journal was in my hands.

My problem was the following: Given that a sample in the form of a bivariate contingency table has been drawn from a parent population, what is the best form of parent population to take on the assumption that the variates are not correlated in that population?

Clearly we ought to take that form which will cause the mean square contingency in the sample to be a minimum. We will denote this mean squared contingency by $\breve{\phi}^2$. What choice shall we make of the relative frequencies $\breve{p}_s$ and $\breve{q}_t$ of the sth and tth categories of the two variates in the parent population in order that we may have the maximum probability that the two variates in the sample come from an uncorrelated parent population? The fallacious proof referred to professed to show that $\breve{p}_s$ and $\breve{q}_t$ should have the values provided by the marginal totals of the sample.

(2) Let us suppose we have a parent population classed according to two *independent* variates and that p_s is the probability of drawing an individual of the sth category of the first variate, and q_t the chance of drawing an individual of the tth category of the second variate. Then the chance of an individual combining both these categories being drawn will be $p_s q_t$.

Now, if we have a sample of size N, containing n_{st} individuals in the sth-tth category drawn from this population, the mean square contingency ϕ^2 will be given by

$$1+\phi^2 = \underset{s,\,t}{S}\left(\frac{n_{st}^2}{N^2 p_s q_t}\right) \quad \text{.............................(i).}$$

Here ϕ^2 is a measure of deviation from the assumed independent variate population, and it is usual to take $Np_s = n_{s.}$, and $Nq_t = n_{.t}$, where $n_{s.}$ and $n_{.t}$ are the totals of the individuals occurring in the sample with the sth category of the first and the tth category of the second variate. This is to assume that the sample adequately represents the parent population as far as the totals in the various categories reached. But if the parent population be unknown, and we are desirous

of determining whether the variates in the sample are or are not independent, then it would appear that we ought to choose the quantities p_s and q_t so that we have the greatest probability of their independence. In other words we ought to choose p_s and q_t so that ϕ^2 is a minimum.

Let us write $n_{st}/N = u_{st}$, so that u_{st} is a relative frequency in the sample, and

$$\underset{s,t}{S}(u_{st}) = 1 \quad \text{.................................(ii).}$$

Then

$$1 + \phi^2 = \underset{s,t}{S}\left(\frac{u_{st}^2}{p_s q_t}\right) \quad \text{.................................(iii),}$$

while

$$\sum_s (p_s) = 1, \quad \sum_t (q_t) = 1 \quad \text{.............................(iv).}$$

We need to find a minimum value of ϕ^2 in (iii), subject to the conditions (iv). Let $\breve{\phi}^2$ represent this minimum value.

We have in the usual manner

$$\delta\phi^2 = -\sum_s \frac{1}{p_s^2}\sum_t \frac{u_{st}^2}{q_t}\,\delta p_s - \sum_t \frac{1}{q_t^2}\sum_s \frac{u_{st}^2}{p_s}\,\delta q_t,$$

$$\sum_s \delta p_s = 0, \quad \sum_t \delta q_t = 0.$$

Or, by the aid of indeterminate multipliers λ_1 and λ_2, we find

$$\frac{1}{p_s^2}\sum_t \frac{u_{st}^2}{q_t} = \lambda_1, \quad \frac{1}{q_t^2}\sum_s \frac{u_{st}^2}{p_s} = \lambda_2 \quad \text{..........................(v).}$$

Multiplying these respectively by p_s and q_s and then summing respectively for s and t we have, by (iv),

$$\lambda_1 = 1 + \breve{\phi}^2, \quad \lambda_2 = 1 + \breve{\phi}^2,$$

and accordingly, writing $\quad a_s^2 = \sum_t \frac{u_{st}^2}{q_t}, \quad b_t^2 = \sum_s \frac{u_{st}^2}{p_s},$

both right-hand sides being positive quantities,

$$a_s/\sqrt{1+\breve{\phi}^2} = p_s, \quad b_t/\sqrt{1+\breve{\phi}^2} = q_t \quad \text{..................(v } bis\text{).}$$

If the number of the categories of the first variate be m and of the second m', we have $m+m'$ equations, which are, however, owing to the relations (iv) not independent. It is interesting to note that this dependence is illustrated by writing (v) in the form

$$p_s = \sum_t \frac{u_{st}^2}{p_s q_t}\Big/(1+\breve{\phi}^2), \quad q_s = \sum_t \frac{u_{st}^2}{p_s q_t}\Big/(1+\breve{\phi}^2),$$

or

$$p_s = \frac{\text{contribution to } 1+\breve{\phi}^2 \text{ of } s\text{th row}}{\text{sum of contributions to } 1+\breve{\phi}^2 \text{ of all rows}},$$

$$q_t = \frac{\text{contribution to } 1+\breve{\phi}^2 \text{ of } t\text{th column}}{\text{sum of contributions to } 1+\breve{\phi}^2 \text{ of all columns}} \quad \text{...........(vi).}$$

These conditions of the values of p_s and q_t to give the minimum ϕ^2 will not as a rule be satisfied by taking $p_s = \frac{n_{s.}}{N}$ and $q_t = \frac{n_{.t}}{N}$, but they will be satisfied in the case when there is complete independence in the sample itself, i.e. $n_{st} = n_{s.}n_{.t}/N$.

Equations (v) present considerable difficulties in solution in the general case, which it is to be hoped some competent mathematician will overcome. They appear to lead to very high order equations.

We can, however, illustrate the matter on a simple case, that of a fourfold table.

u_{11}	u_{12}	Np_1
u_{21}	u_{22}	Np_2
Nq_1	Nq_2	N

We have from (v)

$$\left.\begin{aligned}
\frac{u_{11}^2}{q_1}+\frac{u_{12}^2}{q_2}&=p_1^2(1+\breve{\phi}^2), \quad\text{or}\quad p_1=\sqrt{\frac{u_{11}^2/q_1+u_{12}^2/q_2}{1+\breve{\phi}^2}}\\
\frac{u_{21}^2}{q_1}+\frac{u_{22}^2}{q_2}&=p_2^2(1+\breve{\phi}^2), \quad\text{or}\quad p_2=\sqrt{\frac{u_{21}^2/q_1+u_{22}^2/q_2}{1+\breve{\phi}^2}}\\
\frac{u_{11}^2}{p_1}+\frac{u_{21}^2}{p_2}&=q_1^2(1+\breve{\phi}^2), \quad\text{or}\quad q_1=\sqrt{\frac{u_{11}^2/p_1+u_{21}^2/p_2}{1+\breve{\phi}^2}}\\
\frac{u_{12}^2}{p_1}+\frac{u_{22}^2}{p_2}&=q_2^2(1+\breve{\phi}^2), \quad\text{or}\quad q_2=\sqrt{\frac{u_{12}^2/p_1+u_{22}^2/p_2}{1+\breve{\phi}^2}}
\end{aligned}\right\}\text{.........(vii).}$$

Accordingly
$$\frac{p_2^2}{p_1^2}=\frac{q_2u_{21}^2+q_1u_{22}^2}{q_2u_{11}^2+q_1u_{12}^2}\text{ ....(viii)}$$

Substituting for q_1 and q_2 from the last two equations on the right, we have

$$\left(\frac{p_2}{p_1}\right)^2=\frac{u_{21}^2\sqrt{\dfrac{u_{12}^2}{p_1}+\dfrac{u_{22}^2}{p_2}}+u_{22}^2\sqrt{\dfrac{u_{11}^2}{p_1}+\dfrac{u_{21}^2}{p_2}}}{u_{11}^2\sqrt{\dfrac{u_{12}^2}{p_1}+\dfrac{u_{22}^2}{p_2}}+u_{12}^2\sqrt{\dfrac{u_{11}^2}{p_1}+\dfrac{u_{21}^2}{p_2}}}\text{ ....(ix).}$$

Writing $\dfrac{p_2}{p_1}=z$, we obtain on rationalising an equation of the 10th order to find z, and since $p_1+p_2=1$, theoretically the problem is solved, for the same process may be repeated on the q_t's.

We can, however, shorten the process by using the b_t's of Equation (v). For

$$\left(\frac{p_2}{p_1}\right)^2=\frac{u_{21}^2b_2+u_{22}^2b_1}{u_{11}^2b_2+u_{12}^2b_1}\text{ ....(ix)}^{bis},$$

where
$$b_1^2=\frac{u_{11}^2}{p_1}+\frac{u_{21}^2}{p_2},\quad b_2^2=\frac{u_{12}^2}{p_1}+\frac{u_{22}^2}{p_2}\text{ ....(x).}$$

Solve these equations for $\dfrac{1}{p_1}$ and $\dfrac{1}{p_2}$. We find

$$\frac{1}{p_1}=\frac{b_1^2u_{22}^2-b_2^2u_{21}^2}{u_{11}^2u_{22}^2-u_{12}^2u_{21}^2},\quad \frac{1}{p_2}=\frac{b_2^2u_{11}^2-b_1^2u_{12}^2}{u_{11}^2u_{22}^2-u_{12}^2u_{21}^2}$$

or
$$\frac{p_2}{p_1}=\frac{b_1^2u_{22}^2-b_2^2u_{21}^2}{b_2^2u_{11}^2-b_1^2u_{12}^2}\text{ ....(xi).}$$

Now write $b_1/b_2 = X$, then

$$\frac{p_2}{p_1} = \frac{1}{p_1} - 1 = \frac{X^2 u_{22}^2 - u_{21}^2}{u_{11}^2 - X^2 u_{12}^2},$$

or

$$p_1 = \frac{u_{11}^2 - X^2 u_{12}^2}{u_{11}^2 - u_{21}^2 - X^2 (u_{12}^2 - u_{22}^2)} \quad \text{.....................(xii).}$$

This gives p_1 when X is known. Now, returning to Equation (viii) and using (x) and (xi), we have

$$\left(\frac{X^2 u_{22}^2 - u_{21}^2}{u_{11}^2 - X^2 u_{12}^2}\right)^2 = \frac{u_{21}^2 + u_{22}^2 X}{u_{11}^2 + u_{12}^2 X},$$

which on expansion leads to

$$\begin{aligned} X^5 u_{12}^2 u_{22}^2 (u_{22}^2 - u_{12}^2) &+ X^4 (u_{22}^4 u_{11}^2 - u_{12}^4 u_{21}^2) \\ &- 2X^3 u_{12}^2 u_{22}^2 (u_{21}^2 - u_{11}^2) - 2X^2 u_{21}^2 u_{11}^2 (u_{22}^2 - u_{12}^2) \\ &+ X (u_{21}^4 u_{12}^2 - u_{11}^4 u_{22}^2) + u_{21}^2 u_{11}^2 (u_{21}^2 - u_{11}^2) \doteq 0 \quad \text{...........(xiii).} \end{aligned}$$

The appropriate root of X being found from (xiii), (xii) will then give p_1 and p_1 being known $p_2 = 1 - p_1$. Hence by (viii) we have the ratio q_2/q_1*, and since $q_2 + q_1 = 1$, we find q_2. Lastly, from p_1, p_2, q_1 and q_2 we can find ϕ^2, and so complete the problem.

(3) *Illustrations from Tetrachoric Tables.*

We may illustrate this first on the following example, showing the relation of Intelligence to Athletic capacity in 1708 schoolboys. The decimals arise from boys placed on the boundary lines.

	"Intelligent" and above	"Slow Intelligent" and below	Totals
Athletic ...	581·25	566·75	1148
Non-Athletic	209·25	350·75	560
Totals	790·5	917·5	1708

Treated as a fourfold table the correlation of Intelligence with Athletic Capacity is ·2035. But this result really measures the correlation within this particular sample. We may inquire what is the probability of this as a sample:

(*a*) From a parent population with independent variates, determined by the marginal totals of the sample itself.

(*b*) From the most probable independent variates' parent population.

To answer these questions we first rewrite the table in terms of u's, thus:

$u_{11} = {}$·3403,1030	$u_{12} = {}$·3318,2085	$u_{1.} = {}$·6721,3115
$u_{21} = {}$·1225,1171	$u_{22} = {}$·2053,5714	$u_{2.} = {}$·3278,6885
$u_{.1} = {}$·4628,2201	$u_{.2} = {}$·5371,7799	Total = 1·0000,0000

* We have $X = b_1/b_2 = q_1/q_2$ by the last two equations of (vii), or

$$q_1 = X/(1+X), \quad q_2 = 1/(1+X) \quad \text{.......................................(xiv).}$$

(a) Here $u_{1.} = p_1$, $u_{2.} = q_2$, $u_{.1} = q_1$, $u_{.2} = q_2$, if we take the independent parent population to have its probabilities determined by the marginal totals of the sample. Call the resulting ϕ^2, ϕ_1^2. Then

$$1 + \phi_1^2 = \frac{u_{11}^2}{u_{.1} u_{1.}} + \frac{u_{12}^2}{u_{.2} u_{1.}} + \frac{u_{21}^2}{u_{.1} u_{2.}} + \frac{u_{22}^2}{u_{.2} u_{2.}}$$
$$= \cdot 3722{,}9068 + \cdot 3049{,}5454 + \cdot 0989{,}1020 + \cdot 2394{,}4251$$
$$= 1\cdot 0155{,}9793.$$

Hence $\phi_1^2 = \cdot 0155{,}9793$ and $\chi^2 = N\phi^2 = 26\cdot 6413$. The only constraint is the total size of the sample, and in a second sample $u_{1.}$ and $u_{2.}$ would differ. Thus we interpolate for our χ^2 with $n' = 4$ in the Table* and we find $P = \cdot 000{,}007$, and accordingly it is highly improbable that such a concentration of intelligence and athletic capacity could have been drawn from a parent population in which the two characters were independent *provided that parent population had the same relative proportions of intelligence and athletic capacity as are shown in the sample.*

But why should we limit possible parent populations having these two characters independent to this particular case? Rather we ought to seek for the most likely population of independent variates from which the sample might have been drawn.

(b) To solve this problem we must solve Equation (xiii) for our particular case, and then use (xii) and (xiv) to find p_1 and q_1 which for the minimum value $\breve{\phi}^2$ we will term $\breve{p}_1$ and $\breve{q}_1$. We have

$$u_{11}^2 = \cdot 1158{,}1110, \quad u_{12}^2 = \cdot 1101{,}0508, \quad u_{21}^2 = \cdot 0150{,}0912, \quad u_{22}^2 = \cdot 0421{,}7156,$$
$$u_{11}^4 = \cdot 0134{,}1221, \quad u_{12}^4 = \cdot 0121{,}2313, \quad u_{21}^4 = \cdot 0002{,}2527, \quad u_{22}^4 = \cdot 0017{,}7844,$$

whence we readily find, on substituting in (xiii),

$$-\cdot 0003{,}15436X^5 + \cdot 0000{,}24005X^4 + \cdot 0009{,}36108X^3 + \cdot 0002{,}36167X^2$$
$$- \cdot 0005{,}40811X - \cdot 0001{,}75216 = 0,$$

or in a more convenient form

$$31\cdot 5436X^5 - 2\cdot 4005X^4 - 93\cdot 6108X^3 - 23\cdot 6167X^2 + 54\cdot 0811X + 17\cdot 5216 = 0.$$

This equation has a positive root between 0 and 1 approaching the latter value. Two approximations by Newton's method gave

$$X = \cdot 864{,}7955,$$

whence $\quad q_1 = \cdot 4637{,}4817, \quad q_2 = \cdot 5362{,}5183,$

since by (xiv) $\quad q_1/q_2 = X$, or $q_1 = X/(1 + X)$.

From (xii) we find

$$p_1 = \cdot 6694{,}0856, \quad p_2 = \cdot 3305{,}9144.$$

Thence we have $\quad 1 + \breve{\phi}^2 = 1\cdot 0155{,}6244,$

and accordingly $\quad \chi^2 = 1708 \times \cdot 0155{,}6244 = 26\cdot 5806.$

This marks a slight increase of probability on the parent population based on the marginal totals, but the increase would have in this case small importance for practical statistics.

* *Tables for Biometricians and Statisticians*, Part I, p. 26.

I will take as a second illustration of the method the following fourfold table giving a tetrachoric correlation between ·3 and ·4.

	A	Not A	Totals
B	270	88	358
Not B	33	33	66
Totals	303	121	424

To obtain an accurate equation for X, ten figures were taken so that the relative frequencies were as follows:

u_{11} ·6367,924,528	u_{12} ·2075,471,698	$u_{1.}$ ·8443,396,226
u_{21} ·0778,301,887	u_{22} ·0778,301,887	$u_{2.}$ ·1556,603,774
$u_{.1}$ ·7146,226,415	$u_{.2}$ ·2853,773,585	1·0000,000,000

Here

$$u_{11}^2 = \cdot 4055,046,279, \quad u_{12}^2 = \cdot 0430,758,277, \quad u_{21}^2 = u_{22}^2 = \cdot 0060,575,383.$$

Whence the ordinary ϕ^2 is given by

$$1 + \phi^2 = 1\cdot 0416,404,766,$$

$$\chi^2 = 424 \times \cdot 0416,404,766 = 17\cdot 66,$$

and

$$P = \cdot 000,531.$$

This results from assuming the parent population has the class probabilities given by the marginal totals of the sample itself. We now leave the marginal frequencies to be found so that they give a minimum χ^2; we need the following values:

$$u_{11}^4 = \cdot 1644,340,032, \quad u_{12}^4 = \cdot 0018,555,269, \quad u_{21}^4 = \cdot 0000,366,931 = u_{22}^4.$$

Hence we obtain the quintic equation for X,

$$9,659,306X^5 - 3,639,581X^4 - 208,459,178X^3 - 181,860,477X^2 - 994,484,656X + 981,190,690 = 0.$$

This equation has a positive root between 2 and 3, and another between 4 and 5. The latter gives a high value of ϕ^2. The former root is

$$X = 2\cdot 442,129,416,$$

and as this equals q_1/q_2, we have

$$q_1 = \cdot 709,482,161, \quad q_2 = \cdot 290,517,839.$$

Proceeding now to find the ratio p_2/p_1, we have from (ix)^{bis}

$$\left(\frac{p_2}{p_1}\right)^2 = \frac{\cdot 0208,508,308}{\cdot 5107,013,738} = \cdot 0408,278,338,$$

which gives $\quad p_1 = \cdot 831,905,931, \quad p_2 = \cdot 168,094,069.$

These values of the p's and q's provide

$$1 + \breve{\phi}^2 = 1\cdot 040,104,809, \text{ and } \chi^2 = 424\breve{\phi}^2 = 17\cdot 00,$$

giving $\quad P = \cdot 000,707.$

The change, as in the previous case, increases the value of P, but only slightly. Thus here again the parent population of maximum probability seems scarcely worth the labour of computing.

Lastly, I took a third case, in which the tetrachoric correlation was very low, namely the distribution of Boys and Girls by their hair colour, in two categories, Dark and Light.

The table is as follows:

Hair Colour.

Sex	Dark	Light	Totals
Girls	529	176	705
Boys	442	148	590
Totals	971	324	1295

This gives us for the relative frequencies:

u_{11} $\cdot 4084,9420,84$	u_{12} $\cdot 1359,0733,59$	$u_{1.}$ $\cdot 5444,0154,43$
u_{21} $\cdot 3413,1274,12$	u_{22} $\cdot 1142,8571,43$	$u_{2.}$ $\cdot 4555,9845,57$
$u_{.1}$ $\cdot 7498,0695,06$	$u_{.2}$ $\cdot 2501,9304,94$	$1\cdot 0000,0000,00$

Whence with marginal totals we find

$$1 + \phi^2 = 1\cdot 0000,0190,8 \quad \text{or} \quad \chi^2 = \cdot 0000,019,08 \times 1295 = \cdot 0024,7086,$$

which gives a high probability that the hair colour in these groups of boys and girls is independent of sex.

We now turn to the most probable parent population. It is necessary to work to a large number of decimal places, because the population of maximum likelihood is so close to that of the marginal totals. We have

$$u_{11}^2 = \cdot 1668,6751,80, \qquad u_{11}^4 = \cdot 0278,4476,86,$$
$$u_{21}^2 = \cdot 1164,9438,72, \qquad u_{21}^4 = \cdot 0135,7094,22,$$
$$u_{12}^2 = \cdot 0184,7080,40, \qquad u_{12}^4 = \cdot 0003,4117,08,$$
$$u_{22}^2 = \cdot 0130,6122,44, \qquad u_{22}^4 = \cdot 0001,7059,56.$$

From these values we deduce the quintic

$$130{\cdot}506{,}820X^5 + 1127{\cdot}761{,}551X^4 - 2430{\cdot}516{,}818X^3 - 2{,}1031{\cdot}503{,}410X^2 + 1{,}1302{\cdot}055{,}758X + 9{,}7920{\cdot}980{,}050 = 0.$$

Localising a root near three, we find by a double Newtonian approximation

$$X = 2{\cdot}9969{,}2335.$$

It may be doubted whether we can approximate closer to the root without carrying the coefficients of the quintic to more places of figures.

From the relation $q_1/q_2 = X$ we find

$$q_1 = {\cdot}7498{,}0756, \qquad q_2 = {\cdot}2501{,}9244.$$

Then from Equation (viii) we deduce

$$p_1 = {\cdot}5444{,}0146, \qquad p_2 = {\cdot}4555{,}9854.$$

We see from these results how little the p's and q's have been modified by seeking the most probable parent population.

Substituting in the expression for $1 + \breve{\phi}^2$, we have

$$1 + \breve{\phi}^2 = 1{\cdot}0000{,}0190{,}8$$

exactly as before, or $\chi^2 = {\cdot}0024{,}7086.$

Probably the last figure in $1 + \breve{\phi}^2$ is untrustworthy, and it is not possible without more labour than the matter is worth to distinguish between the marginal totals' parent population and the most probable parent populations.

To judge by the results of the three tables here discussed, it is not likely that, with the restricted freedom of a fourfold table, we shall obtain a substantially higher degree of the probability that two series come from the same parent population, when we take that population to be the most probable population rather than base it on the marginal totals.

(4) *Theory for* $2 \times n$ *Tables.*

In the case of Biserial Tables, where the two series are both supposed to have totals due only to random sampling, the equations become harder of solution; and thus far I have only reached such solution by a method of approximation. We will suppose the tables reduced to the form:

u_{11}	u_{12}	...	u_{1s}	...	u_{1t}	...	u_{1n}	p_1
u_{21}	u_{22}	...	u_{2s}	...	u_{2t}	...	u_{2n}	p_2
q_1	q_2	...	q_s	...	q_t	...	q_n	1

where $p_1, p_2, q_1, q_2 \ldots q_s \ldots q_t \ldots q_n$ are the chances deduced from the parent population which we intend to choose so as to give the minimum ϕ^2 to the table. Usually they are found as we know from the sums of the rows and columns. We have supposed the cell frequencies reduced to relative frequencies, their total being unity. $\breve{\phi}^2$ is the minimum mean square contingency.

By the second equation of (v) we have for the minimum

$$\left.\begin{aligned} q_s &= \sqrt{\frac{u_{1s}^2}{p_1}+\frac{u_{2s}^2}{p_2}}\frac{1}{\sqrt{1+\breve{\phi}^2}} \\ q_t &= \sqrt{\frac{u_{1t}^2}{p_1}+\frac{u_{2t}^2}{p_2}}\frac{1}{\sqrt{1+\breve{\phi}^2}} \end{aligned}\right\} \quad \text{.........................(xv).}$$

It follows therefore that $$q_s = \frac{\sqrt{u_{1s}^2+\frac{p_1}{p_2}u_{2s}^2}}{\overset{n}{\underset{t=1}{S}}\sqrt{u_{1t}^2+\frac{p_1}{p_2}u_{2t}^2}} \quad \text{.............................(xvi),}$$

since $\overset{n}{\underset{s=1}{S}}(q_s)=1$.

But from the first equation of (v)

$$p_1^2(1+\breve{\phi}^2) = \overset{n}{\underset{t=1}{S}}\left(\frac{u_{1t}^2}{q_t}\right),$$

$$p_2^2(1+\breve{\phi}^2) = \overset{n}{\underset{t=1}{S}}\left(\frac{u_{2t}^2}{q_t}\right).$$

Let us write $p_1/p_2 = Y$; then

$$Y^2 = \overset{n}{\underset{s=1}{S}}\left(\frac{u_{1s}^2}{q_s}\right)\Big/\overset{n}{\underset{t=1}{S}}\left(\frac{u_{2s}^2}{q_s}\right).$$

Now substitute the value of q_s in (xvi) and the denominator will divide out and we have

$$Y^2 = \frac{\overset{n}{\underset{s=1}{S}}\dfrac{u_{1s}^2}{\sqrt{u_{1s}^2+Yu_{2s}^2}}}{\overset{n}{\underset{s=1}{S}}\dfrac{u_{2s}^2}{\sqrt{u_{1s}^2+Yu_{2s}^2}}} \quad \text{.........................(xvii).}$$

This is the equation to find Y and on its solution we shall have p_1 and p_2 known, and the q's may then be found from (xvi).

(xvii) may be written $$0 = \overset{n}{\underset{s=1}{S}}\frac{u_{1s}^2 - Y^2u_{2s}^2}{\sqrt{u_{1s}^2+Yu_{2s}^2}} \quad \text{.........................(xviii).}$$

Now assume Y_0 is an approximate value of Y, and take $Y = Y_0(1+\epsilon)$, using Newton's rule to determine an approach to ϵ. We have

$$\epsilon = \frac{2\overset{n}{\underset{s=1}{S}}\dfrac{u_{1s}^2 - Y_0^2u_{2s}^2}{\sqrt{u_{1s}^2+Y_0u_{2s}^2}}}{4\overset{n}{\underset{s=1}{S}}\dfrac{u_{2s}^2Y_0^2}{\sqrt{u_{1s}^2+Y_0u_{2s}^2}}+\overset{n}{\underset{s=1}{S}}\dfrac{Y_0u_{2s}^2(u_{1s}^2-Y_0^2u_{2s}^2)}{(u_{1s}^2+Y_0u_{2s}^2)^{\frac{3}{2}}}}.$$

Hence after some transformation we have

$$\epsilon = \frac{2\overset{n}{\underset{s=1}{S}}\dfrac{u_{1s}^2 - Y_0^2u_{2s}^2}{\sqrt{u_{1s}^2+Y_0u_{2s}^2}}}{\overset{n}{\underset{s=1}{S}}\dfrac{u_{2s}^2u_{1s}^2Y_0(1+Y_0)}{(u_{1s}^2+Y_0u_{2s}^2)^{\frac{3}{2}}}+\overset{s=n}{\underset{s=1}{S}}\dfrac{3Y_0^2u_{2s}^2}{(u_{1s}^2+Y_0u_{2s}^2)^{\frac{1}{2}}}}.$$

Now suppose Y_0 to be the value due to $\bar{p}_1$ and $\bar{p}_2$, or $Y=\frac{\bar{p}_1}{\bar{p}_2}(1+\epsilon)$, then substituting for Y_0 we find after some reductions

$$\epsilon=\frac{\overset{s=n}{\underset{s=1}{S}}\left(\frac{u_{1s}^2}{\bar{p}_1^2}-\frac{u_{2s}^2}{\bar{p}_2^2}\right)\Big/\left(\frac{u_{1s}^2}{\bar{p}_1}+\frac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{1}{2}}}{\overset{s=n}{\underset{s=1}{S}}\frac{\frac{u_{1s}^2}{\bar{p}_1^2}\frac{u_{2s}^2}{\bar{p}_2^2}}{\left(\frac{u_{1s}^2}{\bar{p}_1}+\frac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{3}{2}}}+3\overset{s=n}{\underset{s=1}{S}}\frac{\frac{u_{2s}^2}{\bar{p}_2^2}}{\left(\frac{u_{1s}^2}{\bar{p}_1}+\frac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{1}{2}}}} \quad\text{.................(xix).}$$

Accordingly we take suitable values for $\bar{p}_1$ and $\bar{p}_2$, say those of the actual marginal totals, from these we compute $u_{1s}^2/\bar{p}_1$, $u_{1s}^2/\bar{p}_1^2$, $u_{2s}^2/\bar{p}_2$ and $u_{2s}^2/\bar{p}_2^2$, and then form the sums indicated in (xix). Knowing ϵ we determine another p_1 and p_2 from

$$p_1/p_2=(1+\epsilon)\,\bar{p}_1/\bar{p}_2,$$

together with $p_1+p_2=1$. If ϵ does not come out adequately small we must now repeat the process using p_1, p_2 for $\bar{p}_1$, $\bar{p}_2$. Next the q's are found from (xvi), which may be written

$$q_s=\frac{\sqrt{\frac{u_{1s}^2}{p_1}+\frac{u_{2s}^2}{p_2}}}{\overset{s=n}{\underset{s=1}{S}}\left(\frac{u_{1t}^2}{p_1}+\frac{u_{2t}^2}{p_2}\right)^{\frac{1}{2}}} \quad\text{..........................(xvi)}^{bis},$$

where the values of $\frac{u_{1t}^2}{p_1}$ and $\frac{u_{2t}^2}{p_2}$ will already have been calculated.

We can proceed to find $\breve{\phi}^2$ from (xvi)^{bis} directly without determining the q's.

Let
$$T=\overset{s=n}{\underset{s=1}{S}}\left(\frac{u_{1t}^2}{p_1}+\frac{u_{2t}^2}{p_2}\right)^{\frac{1}{2}},$$

then clearly
$$T^2 q_s=\frac{u_{1s}^2}{p_1 q_s}+\frac{u_{2s}^2}{p_2 q_s};$$

hence summing for s we have since $\overset{n}{\underset{1}{S}}(q_s)=1$,

$$T^2 q_s=1+\breve{\phi}^2,$$

or
$$\breve{\phi}^2=T^2-1 \quad\text{..................................(xx),}$$

leading to $\breve{\chi}^2=N(T^2-1)$, where N is the total frequency of the table.

We may note that this result can be generalised. By (v) we have

$$q_t=\left(\Sigma_s\frac{u_{st}^2}{p_s}\right)^{\frac{1}{2}}\frac{1}{\sqrt{1+\breve{\phi}^2}},$$

hence
$$\frac{q_1}{\left(\Sigma_s\left(\frac{u_{st}^2}{p_s}\right)\right)^{\frac{1}{2}}}=\frac{q_2}{\left(\Sigma_s\left(\frac{u_{s2}^2}{p_s}\right)\right)^{\frac{1}{2}}}=\dots=\frac{q_n}{\left(\Sigma_s\left(\frac{u_{sn}^2}{p_s}\right)\right)^{\frac{1}{2}}}$$

$$=\frac{1}{\left(\Sigma_s\left(\frac{u_{s1}^2}{p_s}\right)\right)^{\frac{1}{2}}+\left(\Sigma_s\left(\frac{u_{s2}^2}{p_s}\right)\right)^{\frac{1}{2}}+\dots+\left(\Sigma_s\left(\frac{u_{sn}^2}{p_s}\right)\right)^{\frac{1}{2}}}.$$

Thus
$$q_t = \frac{\left(\Sigma_s\left(\frac{u_{st}^2}{p_s}\right)\right)^{\frac{1}{2}}}{T} \quad \text{.............................(xxi)},$$

where
$$T = \left(\Sigma_s\left(\frac{u_{s1}^2}{p_s}\right)\right)^{\frac{1}{2}} + \left(\Sigma_s\left(\frac{u_{s2}^2}{p_s}\right)\right)^{\frac{1}{2}} + \dots + \left(\Sigma_s\left(\frac{u_{sn}^2}{p_s}\right)\right)^{\frac{1}{2}}$$
$$= S_t\left(\Sigma_s\left(\frac{u_{st}^2}{p_s}\right)\right)^{\frac{1}{2}}.$$

Thus
$$q_t = \Sigma_s\left(\frac{u_{st}^2}{p_s q_t}\right)\Big/T^2,$$

and
$$1 = (1 + \breve{\phi}^2)/T^2,$$

by summing for t. Accordingly, as before,
$$\breve{\phi}^2 = T^2 - 1 \quad \text{...............................(xxii)}.$$

It is thus only necessary to find the p's (or the q's) in order to determine the minimum $\breve{\phi}^2$.

(5) *Illustration.*

I take the following data for Anaemia in the boys of a small L.C.C. School—one of the poorest schools of London—see *Report of Medical Officer*, 1909. The ages of the boys were 7 to 13.

Ages.

	7	8	9	10	11	12	13	Totals
Non-Anaemic	19	23	19	26	14	13	17	131
Anaemic ...	34	40	28	40	33	34	27	236
Totals	53	63	47	66	47	47	44	367

We take as our hypothesis the supposition that there is no relation of Anaemia to the age of the boys. Now this is a sample from a population of boys frequenting schools in districts where lack of employment, improvidence and drink are widespread. Assuming an indefinitely large parent population, we do not know the proportion of anaemic and non-anaemic boys in that population, nor do we know what is the exact distribution of ages. Hitherto we have assumed these to be given by the marginal totals.

Let us take p_1 to be the chance of drawing a non-anaemic boy, p_2 of drawing an anaemic boy from the parent population. Let the chance of drawing boys of ages 7, 8, 9, ... 13 be $q_1, q_2, q_3, \dots q_7$, respectively. Then the deviation of our sample from the parent population of supposed independent categories will be measured by ϕ^2, where
$$1 + \phi^2 = \underset{s,\,t}{S}\left(\frac{n_{st}^2}{N^2 p_s q_t}\right),$$

where s is 1 or 2, giving the blood class, and t ranges from 1 to 7. n_{st} is the cell frequency and N = size of sample = 367. It is convenient to write $u_{st} = n_{st}/N$, so that

$$\phi^2 = \underset{s,\,t}{S}\left(\frac{u_{st}^2}{p_s q_t}\right) \quad \text{.............................(xxiii).}$$

If as usual we put for p_s and q_t the values deduced from the sample we find the table may be written:

	7	8	9	10	11	12	13	p
	·0517,7112 ·0926,4305	·0626,7030 ·1089,9183	·0517,7112 ·0762,9428	·0708,4460 ·1089,9182	·0381,0714 ·0899,1826	·0354,2234 ·0926,4305	·0463,2153 ·0735,6948	·3569,4823 ·6430,5177
q	·1444,1417	·1716,6213	·1280,6540	·1798,3651	·1280,6540	·1280,6540	·1198,9101	1·0000,0000

Using the p's and q's of this table, hereafter to be written $\bar{p}$ and $\bar{q}$, we find

$$1 + \bar{\phi}^2 = 1{\cdot}0083{,}7311,$$

and

$$\chi^2 = \bar{\phi}^2 \times 367 = 3{\cdot}0729.$$

We now proceed to compute ϵ from formula (xix). As we have to find

$$\frac{u_{1s}^2}{\bar{p}_1} + \frac{u_{2s}^2}{\bar{p}_2}, \quad \left(\frac{u_{1s}^2}{\bar{p}_1} + \frac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{1}{2}} \left(\frac{u_{1s}^2}{\bar{p}_1} + \frac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{3}{2}},$$

and $\dfrac{u_{1s}^2}{\bar{p}_1^2}, \dfrac{u_{2s}^2}{\bar{p}_2^2}$, for each column, the process is somewhat laborious.

I give here merely the final sums, in case any reader cares to check the arithmetic. We have

$$\underset{s}{S}\left[\frac{\left(\dfrac{u_{1s}^2}{\bar{p}_1^2} - \dfrac{u_{2s}^2}{\bar{p}_2^2}\right)}{\left(\dfrac{u_{1s}^2}{\bar{p}_1} + \dfrac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{1}{2}}}\right] = {\cdot}0101{,}7747,$$

$$\underset{s}{S}\left[\frac{\dfrac{u_{1s}^2}{\bar{p}_1^2}\,\dfrac{u_{2s}^2}{\bar{p}_2^2}}{\left(\dfrac{u_{1s}^2}{\bar{p}_1} + \dfrac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{3}{2}}}\right] = {\cdot}9730{,}6644,$$

$$\underset{s}{S}\left[\frac{\dfrac{u_{2s}^2}{\bar{p}_2^2}}{\left(\dfrac{u_{1s}^2}{\bar{p}_1} + \dfrac{u_{2s}^2}{\bar{p}_2}\right)^{\frac{1}{2}}}\right] = 1{\cdot}0000{,}3172.$$

Hence

$$\epsilon = 2 \times \frac{{\cdot}0101{,}7747}{4{\cdot}9731{,}6160} = {\cdot}0040{,}9296.$$

Or, if we use $\breve{p}_1$ and $\breve{p}_2$ to represent the p's which approximate to the p's of the parent population of greatest probability, we have

$$\frac{\breve{p}_1}{\breve{p}_2} = \frac{\bar{p}_1}{\bar{p}_2}(1+\epsilon) = \cdot 5573{,}5669,$$

leading to $\breve{p}_1 = \cdot 3578{,}8634, \quad \breve{p}_2 = \cdot 6421{,}1366.$

We now proceed to find $\left(\frac{u_{1s}^2}{\breve{p}_1} + \frac{u_{2s}^2}{\breve{p}_2}\right)^{\frac{1}{2}}$ in order to determine $\breve{q}_s$. We have:

s	$\left(\frac{u_{1s}^2}{\breve{p}_1} + \frac{u_{2s}^2}{\breve{p}_2}\right)^{\frac{1}{2}}$ and accordingly	$\breve{q}_s$
1	·1444,1429	·1438,1539
2	·1716,8145	·1709,6948
3	·1286,6311	·1281,2953
4	·1803,4440	·1795,9650
5	·1290,6507	·1285,2983
6	·1298,9364	·1293,5497
7	·1201,0237	·1196,0430
$T = \Sigma =$	1·0041,6433	

We have for the approximate minimum ϕ^2,

$$1 + \breve{\phi}^2 = T^2 = 1{\cdot}0083{,}4600,$$

or $$\breve{\phi}^2 = \cdot 0083{,}4600,$$

and $$\breve{\chi}^2 = 3{\cdot}0630.$$

In this particular case we certainly have made a reduction in the ϕ^2 and χ^2, but not one of any practical importance. The ϵ is already so small that it does not seem that we should gain much by a second approximation.

It cannot, however, be asserted that in other cases the parent population of maximum probability may not differ much more considerably than the present does from the parent population as represented by the marginal totals.

We want far more experience of what differences can arise between the two ways of approaching the subject. In particular we need the aid of a more competent mathematician than the present writer to deal with the general case of ϕ^2 for an $n \times n'$ table. As far as the present very limited data for two very special types of contingency tables reach, we might conclude, but hardly safely, that we shall not make large errors if we replace the most probable independent variates' population by a marginal totals' parent population.

THE SKULLS FROM EXCAVATIONS AT DUNSTABLE, BEDFORDSHIRE.

BY DORIS DINGWALL, M.A., B.Sc., AND MATTHEW YOUNG, D.Sc., M.D.

Introduction.

With the kind permission of the Conservators of the Dunstable Downs and the Stint Holders, the University College and Hospital Anthropological Society, in collaboration with T. W. Bagshawe, Esq., F.S.A., undertook the excavation of the most northerly of a group of bell barrows known as the Five Knolls, situated three-quarters of a mile west of Dunstable, Bedfordshire. The work was begun in 1925 and was carried on during the four succeeding seasons 1925—1929*. The skulls are preserved in the Institute of Anatomy, University College, London.

The barrow is from 50 to 60 ft. in diameter and the primary burial was found to be nearly central. It consisted of the skeleton of a woman in the crouched position lying on the right side in an oval cist cut in the chalk. This was surrounded by a flat-bottomed ditch possibly used for ceremonial purposes and belongs most probably to the Early Bronze Age.

The secondary burials were of cremated bones covered by an inverted urn of the Middle Bronze Age type, and a collection of burnt bones in a shallow hollow.

The tertiary burials were scattered without any clearly defined plan over the southern half of the barrow and the adjacent surface of the downs. It is with these tertiary burials that this paper is concerned.

The dating of the skeletons is difficult because they were so close to the surface of the mound that it is unsafe to attribute them to the same date as that of the objects found in the same layer. The stratum contained a brooch of La Tène III type (probably 100—50 B.C.) and a gilded bronze buckle with iron tang of probably the 5th century A.D. Besides these there were various objects of Roman, post-Roman and Saxon date.

Mr Dunning and Dr Wheeler† have discussed the archaeological evidence in detail and have come to the conclusion that the "objects associated with the burials suggest the 5th or 6th century A.D. for the date of the whole group." Whether their conjecture that the series of skeletons "represents part of a Saxon raiding party which had been worsted by the local inhabitants and summarily executed" is a correct one will be discussed when the analysis of the cranial measurements has been presented. During the excavation there were clear

* For a full report see the *Archaeol. Journ.* Vol. LXXXVIII. 1931, pp. 193—217.

† *Op. cit.* pp. 205—210.

TABLE I.

Showing the Means and Variabilities of the Characters in the Male series of Skulls.

Characters	No.	Means	σ	V
C*	25	1524·80 ± 21·32†	106·59 ± 15·07†	6·99 ± 0·99†
L	52	184·92 ± 0·81	5·84 ± 0·57	3·16 ± 0·31
F	51	182·70 ± 0·91	6·50 ± 0·64	3·56 ± 0·35
B	52	145·42 ± 0·73	5·28 ± 0·52	3·63 ± 0·36
B'	50	98·66 ± 0·72	5·07 ± 0·51	5·14 ± 0·51
H'	50	135·58 ± 0·65	4·63 ± 0·46	3·41 ± 0·34
OH	46	115·25 ± 0·49	3·32 ± 0·35	2·88 ± 0·30
LB	48	102·67 ± 0·63	4·34 ± 0·44	4·23 ± 0·43
Q	44	317·30 ± 1·44	9·54 ± 1·02	3·01 ± 0·32
Q'	44	321·25 ± 1·40	9·91 ± 1·06	3·08 ± 0·33
S	50	374·10 ± 1·97	13·91 ± 1·39	3·72 ± 0·37
S_1	50	128·86 ± 0·67	4·76 ± 0·48	3·69 ± 0·37
S_2	51	127·24 ± 1·35	9·63 ± 0·95	7·57 ± 0·75
S_3	52	117·82 ± 0·93	6·71 ± 0·66	5·69 ± 0·56
S_1'	50	112·18 ± 0·58	4·09 ± 0·41	3·65 ± 0·37
S_2'	51	112·74 ± 0·99	7·06 ± 0·70	6·26 ± 0·62
S_3'	52	97·27 ± 0·76	5·48 ± 0·54	5·63 ± 0·55
U	51	527·10 ± 1·98	14·13 ± 1·40	2·68 ± 0·27
GH	32	118·03 ± 0·93	5·24 ± 0·66	4·44 ± 0·56
$G'H$	34	70·97 ± 0·64	3·74 ± 0·45	5·27 ± 0·64
GB	33	96·73 ± 0·78	4·49 ± 0·55	4·64 ± 0·57
J	19	138·11 ± 1·06	4·61 ± 0·75	3·34 ± 0·54
NH'	32	50·69 ± 0·53	2·98 ± 0·37	5·88 ± 0·74
NH, R	34	51·65 ± 0·43	2·50 ± 0·30	4·84 ± 0·59
NH, L	34	51·35 ± 0·46	2·71 ± 0·33	5·28 ± 0·64
NB	34	25·24 ± 0·26	1·54 ± 0·19	6·10 ± 0·74
O_1, R	31	42·81 ± 0·28	1·57 ± 0·20	3·67 ± 0·47
O_1, L	34	42·44 ± 0·28	1·65 ± 0·20	3·89 ± 0·47
O_2, R	31	33·42 ± 0·33	1·86 ± 0·24	5·57 ± 0·71
O_2, L	34	33·47 ± 0·33	1·93 ± 0·23	5·77 ± 0·70
O_1', R	27	40·74 ± 0·25	1·32 ± 0·18	3·24 ± 0·44
G_1	29	51·10 ± 0·51	2·77 ± 0·36	5·42 ± 0·71
G_1'	27	46·52 ± 0·47	2·44 ± 0·33	5·25 ± 0·71
G_2	36	40·94 ± 0·43	2·56 ± 0·30	6·25 ± 0·74
GL	36	97·81 ± 0·79	4·74 ± 0·56	4·85 ± 0·57
fml	48	36·71 ± 0·33	2·28 ± 0·23	6·21 ± 0·63
fmb	43	30·40 ± 0·32	2·10 ± 0·23	6·91 ± 0·75
$100\ B/L$	52	78·69 ± 0·47	3·40 ± 0·33	4·32 ± 0·42
$100\ B/F$	51	79·83 ± 0·51	3·66 ± 0·36	4·58 ± 0·45
$100\ H'/L$	50	73·40 ± 0·43	3·03 ± 0·30	4·13 ± 0·41
$100\ B/H'$	50	107·28 ± 0·71	5·05 ± 0·51	4·71 ± 0·47
$100\ G'H/GB$	32	73·48 ± 0·69	3·90 ± 0·49	5·31 ± 0·66
$100\ G'H/J$	18	51·65 ± 0·58	2·46 ± 0·41	4·76 ± 0·79
$100\ NB/NH$, R	34	48·95 ± 0·56	3·26 ± 0·40	6·66 ± 0·81
$100\ NB/NH$, L	34	49·25 ± 0·60	3·48 ± 0·42	7·07 ± 0·86
$100\ NB/NH'$	32	50·03 ± 0·70	3·95 ± 0·49	7·90 ± 0·99
$100\ O_2/O_1$, R	31	78·15 ± 0·87	4·86 ± 0·62	6·22 ± 0·79
$100\ O_2/O_1$, L	34	78·86 ± 0·90	5·27 ± 0·64	6·68 ± 0·81
$100\ O_2/O_1'$, R	27	81·57 ± 0·92	4·78 ± 0·65	5·86 ± 0·80
$100\ fmb/fml$	43	82·94 ± 0·98	6·40 ± 0·69	7·72 ± 0·83
$100\ G_2/G_1$	29	80·44 ± 0·91	4·91 ± 0·64	6·10 ± 0·80
$P\angle$	26	88°·04 ± 0·65	3°·30 ± 0·46	3°·75 ± 0·52
$N\angle$	33	64°·17 ± 0·58	3°·32 ± 0·41	5°·17 ± 0·64
$A\angle$	33	74°·62 ± 0·47	2°·72 ± 0·33	3°·65 ± 0·45
$B\angle$	33	41°·21 ± 0·42	2°·40 ± 0·30	5°·82 ± 0·72
$\theta_1\angle$	26	27°·44 ± 0·71	3°·62 ± 0·50	13°·19 ± 1·83
$\theta_2\angle$	26	13°·50 ± 0·75	3°·81 ± 0·53	28°·22 ± 3·91
$Oc.\ I.$	50	59·30 ± 0·35	2·50 ± 0·25	4·22 ± 0·42

* *Note on Cubic Capacity.* The mean cubic capacity of 1524·8 c.c. based on 25 skulls was estimated by filling the skulls with mustard seed and measuring the amount contained in the graduated cylinder. The mean cubic content was also estimated by transforming the weight in grams of firmly packed mustard seed which each skull could contain into cubic centimetres, the volume of seed when tightly packed which corresponded to 1000 grams having previously been determined by means of the *crâne étalon* (Macdonell's method). The mean capacity thus found was 1495 c.c. The slight difference in the mean estimates by the two methods is probably to be explained by the fact that owing to the fragile condition of some of the skulls it was not possible in these to use such firm pressure in packing the seed as in the *crâne étalon*. The mean capacity of the 25 skulls computed from the mean length, breadth and height by Miss Hooke's formula is 1533 ± 9·2 c.c.

† Standard errors.

indications that the burials had not all taken place at the same time, and the haphazard nature of most of them was emphasised by disturbance and overlapping. A certain number were in trench graves, and in one of these an iron buckle was found in close association with a skeleton. It is of a type common in Roman and later times. There is nothing, however, to show that the trench graves and superficial burials belong to different periods although they probably took place at slightly different times.

An interesting feature of the group is that about one-tenth of the skeletons had their arms crossed either behind their backs or before their chests. Such a posture suggests that they had been bound at death and this, coupled with the casual nature of the burials, points to the conclusion that the site is not a formal cemetery.

The Crania: Means and Variabilities.

From the total series of approximately 100 skeletons, about 64 skulls, 52 male and 12 female were, after a certain degree of reconstruction, in a suitable condition for obtaining most of the principal measurements. The sexing of these skulls can be relied upon, as it was in almost all cases verified from an examination of the pelvic bones. The means and variabilities of the cranial characters of the male series are shown in Table I and the means of some of the principal characters of the short female series in Table II. As the variabilities shown in the length, breadth, height and cephalic index of the male skulls are not greater than those shown by the corresponding characters in the Farringdon Street series of 17th Century Londoners which were taken from one graveyard, or in the contemporary Whitechapel series collected from a single pit, there is no reason to regard the present series as other than fairly homogeneous. The methods of measurement of

TABLE II.

Showing the Mean Values of some of the Principal Characters in the Short Female Series.

Characters	No.	Means
L	12	179·1
F	12	178·3
B	12	142·9
B'	12	96·3
H'	11	126·9
OH	11	110·5
LB	11	95·1
U	11	516·2
S	11	368·3
Q'	11	312·2
$100\ B/L$	12	79·9
$100\ H'/L$	11	71·1
$100\ B/H'$	11	112·7

the several cranial characters practised in the Biometric Laboratory were closely adhered to, and the symbols by which the characters are represented in the tables are those described and used in the various craniological memoirs published in *Biometrika**, to which reference may be made.

The Affinities of the Skulls: Coefficients of Racial Likeness.

The archaeological evidence being very fragmentary and not at all convincing, the first point to be determined was the racial type to which the skulls as a group most closely conformed. For this purpose Professor Pearson's Coefficient of Racial Likeness† was used. The available evidence seemed to suggest that the skulls were those of Anglo-Saxons (*vide* Dunning and Wheeler), but a mean cephalic index of approximately 79 for males and for females did not support the view that the skulls conformed to the type shown by well-authenticated Anglo-Saxon skulls such as those preserved in the London Museums which have been described by Morant‡, the Bidford-on-Avon collection described by Brash § or the Burwell ‖ collection in the Anatomy Department at Cambridge.

TABLE III.

Coefficients of Racial Likeness between the Dunstable Group and other Groups of Male Crania.

Dunstable (39·8).

	Crude coefficients		Reduced coefficients ($n_1 = n_2 = 100$)	
	All characters	Indices and angles	All characters	Indices and angles
English Bronze Age (27·2)	3·86 ± ·18** [29]	3·12 ± ·29** [11]	11·94 ± ·50**	13·16 ± 1·22**
Anglo-Saxon ... (36·2)	5·18 ± ·17 [30]	7·07 ± ·28 [12]	13·08 ± ·45	19·77 ± 0·78
Hythe ... ... (101·8)	7·18 ± ·17 [30]	8·80 ± ·28 [12]	12·55 ± ·30	16·07 ± 0·51
British Iron Age (50·8)	8·40 ± ·20 [22]	13·20 ± ·30 [7]	18·82 ± ·45	31·37 ± 0·86
Whitechapel ... (92·3)	11·50 ± ·18 [29]	14·76 ± ·20 [11]	20·68 ± ·32	28·31 ± 0·50
British Neolithic (28·9)	17·93 ± ·18 [27]	26·28 ± ·30 [10]	53·55 ± ·54	113·65 ± 1·30

* See more especially Vols. I. pp. 416—418; III. pp. 199—207 and XIV. pp. 200—201.

† *Biometrika*, Vol. XVI. 1924, pp. 11—14 and Vol. XVIII. 1926, pp. 105—117. The standard deviations used in Table III to form the C. R. L.'s are those for the long Egyptian Series E of the 26th—30th Dynasties, provided by Davin and Pearson in *Biometrika*, Vol. XVI. 1924, p. 338.

‡ *Ibid.* Vol. XVIII. 1926, pp. 55—98.

§ *Archaeologia*, Vol. LXXIII. 1922—23, p. 106 (Appendix I).

‖ This collection has been measured by Doris Dingwall, but the mean measurements have not yet been published.

** Throughout this table the quantities following the ± sign are "probable" not "standard" errors.

The male skulls were thus compared not only with an Anglo-Saxon* series but also with the English Bronze Age series, the British Iron Age* series, the British Neolithic* series, the Hythe crania† and the 17th Century Londoners from Whitechapel‡. For the comparable figures for these several types we are indebted to the authors of the memoirs containing the data relating thereto which have been published in *Biometrika*. The crude coefficients of racial likeness between the Dunstable male group and these other groups are shown in Table III. The reduced coefficients which result from an adjustment that is made to allow for the variation in numbers of skulls available in the different groups are also included in the table. These coefficients are relatively comparable. As the mean numbers of skulls available for comparison in the Bronze Age and Anglo-Saxon groups are not very different, the crude coefficients of racial likeness between the Dunstable skulls and these groups may in the first place be compared directly. Such a comparison suggests that the Dunstable skulls are more closely related to the Bronze Age type than to the Anglo-Saxon type. For shape characters (indices and angles) alone the coefficient in the former case is 3·1 and in the latter 7·1. For all characters the difference is not so evident as shown by coefficients of 3·9 and 5·2. When due allowance is made for the difference in number of Bronze Age and Anglo-Saxon skulls available for comparison, the difference in relative closeness of relationship just described is confirmed. In general form or shape the Dunstable skulls clearly resemble Bronze Age more nearly than Anglo-Saxon skulls. The reduced coefficient of racial likeness for all characters between the Dunstable and Bronze Age type is still smaller than that between the Dunstable and Anglo-Saxon types, suggesting a closer affinity with the former, and as the difference is about 2·5 times its probable error it may probably be considered significant. In general form the Dunstable group is also more similar to the Hythe group than to the Anglo-Saxon group, but, when all characters are considered, though the coefficient between the Dunstable and Hythe groups is still smaller than that between the Dunstable and the Anglo-Saxon, the difference is not so great that it can be considered really significant. The Dunstable group is distinctly more divergent from the Iron Age group than from the Anglo-Saxon group. It shows a divergence of about the same order from the type of the Whitechapel 17th Century Londoners which, as is well known, closely resembles that of the Iron Age. The Neolithic type is, as might be expected, very different from that found at Dunstable.

Comparison of Individual Cranial Characters: Values of α.

The characters in which the Dunstable type resembles most closely, or differs most notably from, the Anglo-Saxon and Bronze Age types may be seen readily by reference to Table IV in which the values of $\alpha = \frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s \sim M_s'}{\sigma_s}\right)^2$ are tabulated. The most pronounced differences between the Dunstable and Anglo-Saxon skull

* *Biometrika*, Vol. xx^B. 1928, pp. 301—375.

† *Ibid.* Vol. xxiv. 1932, pp. 185—202.

‡ *Ibid.* Vol. iii. 1904, pp. 191—245.

TABLE IV.

Values of $\alpha = \frac{n_s n_s'}{n_s + n_s'} \left(\frac{M_s \sim M_s'}{\sigma_s}\right)^2$ *between the Dunstable and other British Male Series.*

Characters	Anglo-Saxon	English Bronze Age	Hythe	British Iron Age	White-chapel	British Neolithic
100 B/L	57·92	15·66	75·20	59·44	100·34	196·86
100 H'/L	9·33	0·17	16·00	21·49	47·21	27·37
100 B/H'	8·56	12·01	9·05	0·59	0·94	20·40
$Oc. I.$	2·29	—	0·79	—	—	—
100 $G'H/GB$	2·42	4·47	0·00	—	7·30	0·90
100 $NB/NH, R$	—	—	0·02	—	3·07	—
100 NB/NH'	8·83	1·79	—	13·36	—	18·82
100 $O_2/O_1, R$	—	0·03	0·09	—	0·21	—
100 $O_2/O_1', R$	1·93	—	—	4·22	—	0·54
100 fmb/fml	0·14	4·31	3·69	—	2·37	0·09
100 G_2/G_1	0·29	0·60	2·41	—	0·84	—
$P\angle$	0·10	5·77	8·43	—	6·33	7·30
$N\angle$	4·34	0·09	0·35	0·12	2·04	0·44
$A\angle$	0·73	0·40	1·63	0·20	2·89	0·01
L	27·23	0·12	53·19	5·36	20·32	62·13
B	20·88	29·34	2·65	24·32	36·60	68·95
B'	3·23	1·41	0·02	0·71	1·08	0·00
OH	—	—	0·02	—	20·70	—
H'	0·12	0·32	—	8·74	—	0·01
LB	2·34	0·04	10·32	2·15	2·63	0·33
Q'	4·18	0·77	1·71	0·75	33·85	6·61
U	3·80	11·93	13·99	0·23	1·52	16·39
S	5·83	1·11	15·96	2·76	2·14	27·14
$G'H$	0·38	3·34	1·73	3·34	0·87	0·04
J	13·45	0·10	10·97	38·03	40·38	30·86
NH, R	0·39	—	1·50	—	0·70	—
NH'	—	1·51	—	0·03	—	2·51
NB	2·40	0·20	1·32	16·20	5·92	13·89
O_1, R	0·04	4·98	4·26	—	0·31	—
O_1', R	—	—	—	3·37	—	0·09
O_2, R	0·16	0·67	1·06	0·23	0·00	2·53
G_1	1·07	6·16	0·18	—	14·44	—
G_2	0·36	2·55	1·95	—	4·14	0·42
fml	1·48	0·83	6·65	1·19	3·57	0·33
fmb	1·35	1·12	0·27	—	0·07	0·00

types are shown in the maximum length and maximum breadth and the indices involving these absolute measurements, namely, $100 \times B/L$, $100 \times B/H'$ and $100 \times H'/L$. The Anglo-Saxon skull is on the average about 5·5 mm. longer and about 3·5 mm. narrower at its broadest part than the Dunstable skull although of about equal height. It has also a longer sagittal arc, a narrower face (bizygomatic diameter) and a relatively narrower nasal aperture. The Bronze Age skull is on the average 4·5 mm. broader than the Dunstable skull, but is practically equivalent to it in mean length and mean height. The indices involving the maximum breadth, $100 \times B/L$ and $100 \times B/H'$, are thus higher on the average in the Bronze

Age than in the Dunstable specimens. The Bronze Age skull has also a larger horizontal circumference, a longer palate, a wider orbit and a smaller profile angle ($\angle P$); the mean sagittal arc and the mean bizygomatic breadth, however, correspond closely in the two types.

The Dunstable skull differs from the Hythe skull principally in maximum length. On the average it is about 7 mm. longer. The indices involving length, $100 \times B/L$ and $100 \times H'/L$, are thus higher in the Hythe skull. The Hythe skull has also a shorter cranio-facial base (basi-nasal length), a shorter sagittal arc, a less extensive horizontal circumference, a smaller profile angle ($\angle P$), a narrower face (bizygomatic breadth), and a shorter foramen magnum.

The Female Skulls.

As there are only 12 female skulls, any detailed comparison of the average measurements of this group with those of other types is not justifiable. So far as an average based upon such a small number of specimens can be relied upon, however, there is a strong suggestion that the female skull is definitely broader than the female Anglo-Saxon skulls described by Morant. Its mean maximum breadth is 142·9 mm. as compared with 135·6 mm. in the female Anglo-Saxon. As the mean lengths in the two groups are equivalent, the Dunstable skull is thus relatively broader than the Anglo-Saxon; the cephalic index is 79·9 as compared with 74·4. The mean $100 \times B/H'$ index is in the Dunstable skulls 112·7 as compared with 105·7 in the Anglo-Saxons in the London Museums. Comparison of these characters and of the mean measurements of others in the two groups seems to indicate that the female skulls conform generally in type to the male skulls and differ considerably from the female Anglo-Saxons.

Discussion.

It has been suggested by Dunning and Wheeler* that the skeletal remains found in the most northerly knoll at Dunstable are probably those of a Saxon raiding party which was exterminated in the 6th century A.D. by the local inhabitants. The detailed analysis of the cranial measurements of the group and the survey of its cranial affinities with other racial types, including Anglo-Saxon, Bronze Age and Hythe crania, would appear to suggest that the skulls cannot be considered those of Anglo-Saxons. It might be regarded as more probable, if the dating of the burials can be accepted as approximately correct, that the skulls are those of a colony of the local inhabitants who were summarily disposed of possibly by a raiding party of Anglo-Saxons. This view receives support from the circumstance that almost 20 per cent. of the more or less complete skulls that could be measured are those of women and that these female skulls conform in their general shape to the male skulls

It is not improbable that a definite broad-headed element derived largely from the Bronze Age population may have persisted in certain localities such as Dunstable. On the other hand, the existence of a "moderate" degree of association

* *loc. cit.*

between the Dunstable and the Hythe crania is of interest in view of Stoessiger and Morant's* conclusion that the latter are probably for the most part the descendants of a Roman colony with a large central European element. The Five Knolls at Dunstable are in the vicinity of Watling Street and the Icknield Way, and it is not impossible that the broad-headed tendency which is a feature of the relatively homogeneous Dunstable series may be evidence of a similar European element in the inhabitants, introduced at the time of, and persisting to a notable degree after, the Roman occupation. The various objects of Roman date found with the skeletal remains at least provide distinct evidence of contact with this people.

The view that the skulls represent a survival of Roman invaders is, perhaps, on the whole more probable than that they indicate the persistence of the Bronze Age population. The C.R.L. between the Dunstable skulls and an Etruscan series has been found by Dr Morant to be lower than that between the Dunstable and Bronze Age groups, and he thinks it not unlikely that the Pompeians and possibly a few other European series would also resemble the Dunstable skulls more closely than the Bronze Age type does.

While some reasonable doubt may still be held as to the origin of the skulls, all the evidence that can be derived from the crania themselves seems to indicate quite unequivocally that they are not those of Anglo-Saxons of a recognised type.

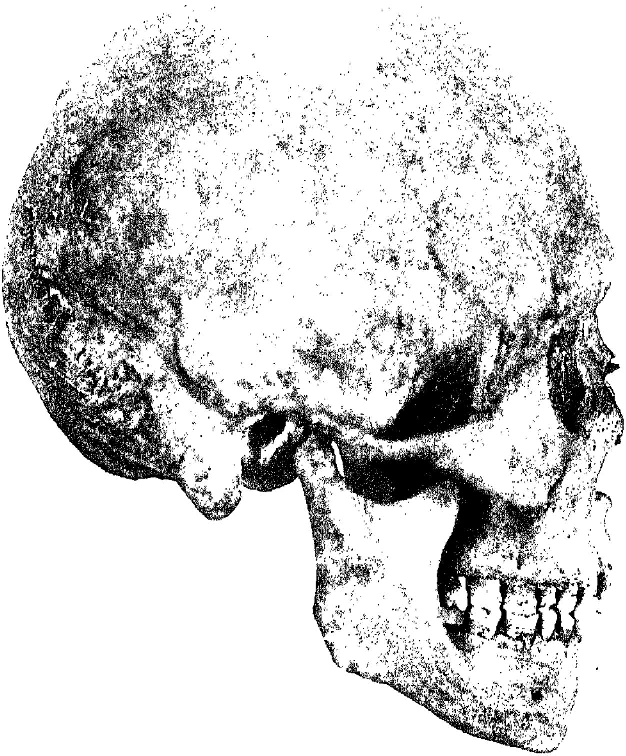
Measurements of the Mandibles.

Comparatively little is yet known about the detailed comparison of measurements of the mandibular characters in adequate series of specimens belonging to different types of skulls. Now that a comprehensive scheme for the measurement of this bone has been devised and described by Morant† it is probable that suitable and adequate data for the comparison of different types will soon be available. The records of mean values of measurements of numerous characters, using the technique described by this author for series of mandibles associated with Anglo-Saxon‡, 17th Century London§, Badari Egyptian‖, Tibetan A¶, Nepalese¶, Tamil**, Fukien**, and Hylam** skulls which have already been published in memoirs in *Biometrika*, indicate that definite progress is being made in the provision of material for a proper study of the racial characters of an important part of the skull which has hitherto, largely through unavoidable circumstances, received but little attention.

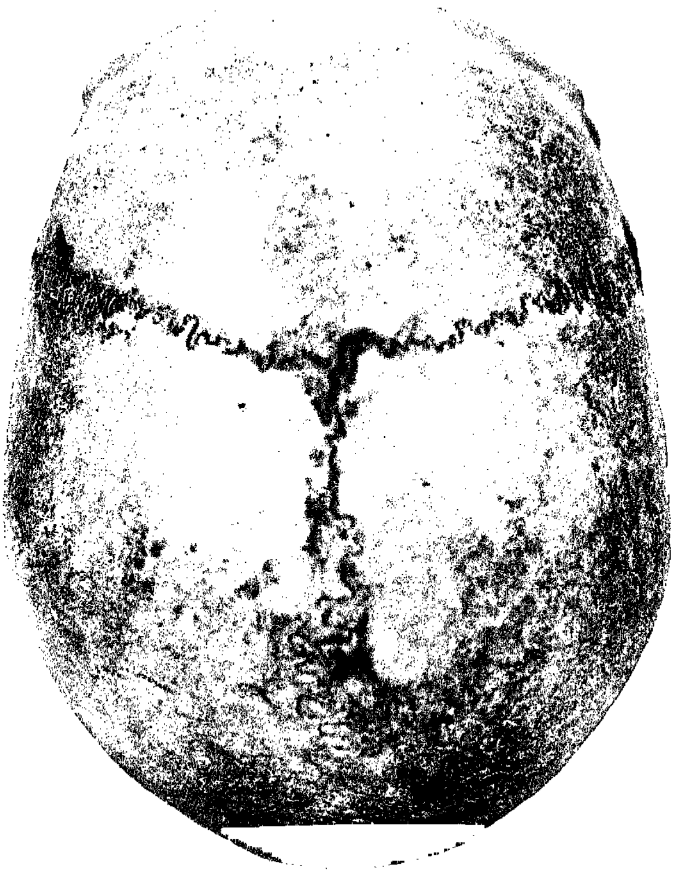
As there were approximately 40 more or less complete mandibles associated with the relatively homogeneous series of male skulls in the Dunstable collection, it seemed desirable that the mean measurements of the principal mandibular characters

* *Biometrika*, Vol. xxiv. 1932, pp. 135—202.
† *Ibid.* Vol. xiv. 1923, pp. 253—260.
‡ *Ibid.* Vol. xviii. 1926, pp. 55—98 (Table xviii).
§ *Ibid.* Vol. xviii. 1926, pp. 1—55 (Appendix C).
‖ *Ibid.* Vol. xix. 1927, p. 149.
¶ *Ibid.* Vol. xvi. 1924, pp. 103—104.
** *Ibid.* Vol. xx^B. 1928. pp. 279—293.

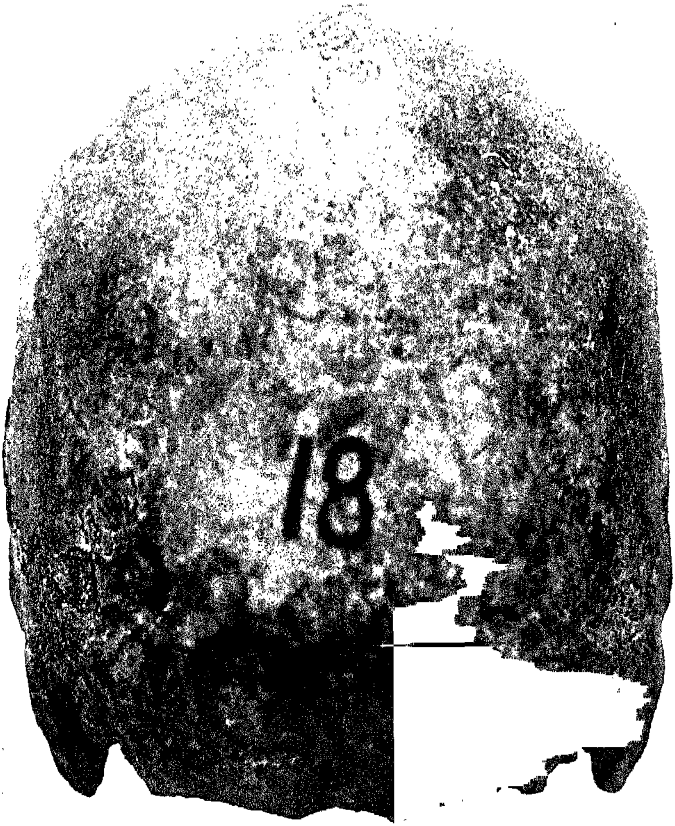
			Indices							Angles in degrees					
ml	100 × $c_r h/ml$	*l*	100 × $g_o g_o/c_x l$	100 × rb'/rl	100 × $c_y b/c_y l$	100 × $g_o g_o/c_r c_r$	100 × $c_y h/c_r h$	100 × $ih'/c_y c_r$	100 × $d_i h/c_r h$	$M\angle$	$R\angle$	$G\angle$	$O\angle$	$O'\angle$	$F\angle$
105·0	60·0		150·0	49·2	—	110·1	93·7	45·7	56·3	120·0	62·0	80·0	56·0	57·0	86·0
112·0	66·1		135·4	48·5	42·9	102·8	86·5	50·0	50·0	118·0	81·0	72·0	70·0	71·0	90·0
—	—		—	—	—	—	—	—	—	—	—	—	—	—	—
109·0	—		136·5	50·0	39·5	—	—	—	—	125·0	—	73·0	64·0	66·0	90·5
108·0	70·4		125·0	42·7	43·8	93·2	93·4	34·7	43·4	115·0	74·0	67·0	60·0	—	—
103·0	—		133·8	42·8	35·7	—	—	—	—	121·0	—	70·0	61·0	65·0	—
99·0	64·9		115·9	56·3	41·3	—	98·0	42·9	54·4	110·0	72·0	67·0	77·5	78·5	—
115·0	55·7		129·5	55·7	41·7	95·3	81·3	32·4	60·9	128·0	69·0	81·0	66·0	66·0	87·0
116·0	59·5		120·0	47·8	44·2	106·3	84·1	40·0	59·4	121·0	72·0	68·0	61·5	63·0	85·0
96·0	68·8		144·4	55·4	35·0	108·3	87·9	45·8	50·0	115·0	78·0	76·0	73·0	73·0	89·0
112·5	60·4		135·6	50·0	46·5	94·3	85·3	35·7	57·4	128·0	65·5	68·0	70·0	71·0	93·0
105·0	61·0		125·0	48·8	36·8	—	81·3	31·6	56·3	127·0	70·0	68·0	—	—	90·0
114·0	55·3		121·6	47·7	—	97·5	82·5	31·7	55·6	131·0	68·0	67·0	65·0	65·0	89·5
108·5	69·1		137·5	36·6	42·3	—	92·0	36·6	46·7	120·0	72·0*	75·0	59·0	60·5	83·5
—	—		—	—	—	—	86·4	—	54·2	—	66·0*	75·0	68·0	68·5	87·0
103·0	—		127·0	45·3	39·1	—	—	—	—	114·0	63·0*	69·0	—	—	—
105·0	52·4		123·3	42·7	40·0	—	96·5	39·5	65·5	124·5	59·0	69·0	68·0	69·0	—
—	—		126·7	49·2	—	102·2	92·6	35·5	56·2	119·0	67·5	70·0	64·5	65·5	—
105·0	60·0		124·7	57·4	41·9	94·8	87·3	45·3	52·9	123·0	70·0	69·0	72·0	72·0	—
07·0	58·9		130·0	49·2	40·0	87·5	90·5	35·4	57·1	126·0	62·0	70·0	83·0	83·0	89·9
—	—		—	—	39·1	—	93·8	—	52·3	119·5	—	72·0	69·0	70·0	88·0
—	—		132·5	50·7	—	109·7	93·8	33·9	59·4	122·0	62·0*	70·0	65·0	67·0	—
07·0	51·4		136·5	53·5	40·0	104·1	94·5	34·2	61·8	127·0	59·0	73·0	73·5	74·5	—
06·0	67·5		103·6	46·5	36·9	81·3	85·3	37·2	52·4	114·0	76·0	61·0	73·0	72·0	85·0
09·0	57·8		162·9	52·0	50·0	116·0	76·2	38·2	59·2	130·0	69·0	82·5	66·0	65·0	96·0
02·5	65·4		108·2	51·1	43·2	82·0	95·5	43·3	53·7	112·0	74·0	60·0	69·0	69·0	82·5
04·5	56·0		125·8	58·5	40·5	109·9	106·0	28·9	56·5	116·0	72·0	70·0	72·0	72·0	88·5
08·8	57·9		131·3	48·9	41·7	98·2	100·0	35·1	54·0	121·0	58·5	70·0	69·0	68·0	81·0
02·2	56·9		138·6	55·0	43·2	108·4	94·5	36·4	61·9	122·0	64·0	73·0	69·0	72·5	102·5
01·2	67·2		133·8	48·0	47·8	110·6	97·1	34·3	45·6	113·5	70·0	72·0	60·0	60·0	89·3
12·0	63·4		110·6	58·3	52·2	91·9	91·5	38·6	56·3	114·0	74·0	63·0	68·0	66·0	84·9
09·5	57·5		153·6	49·4	27·3	106·0	83·7	36·1	60·6	132·0	58·5	78·0	69·0	70·0	—
01·0	63·5		106·3	52·4	39·1	92·2	92·0	34·3	55·9	113·0	75·0	65·0	66·0	69·0	—
02·5	54·6		143·8	59·3	42·5	115·4	83·9	34·9	62·5	127·0	68·5	78·0	59·0	60·0	—
11·0	54·1		138·8	71·9	56·3	—	86·7	33·3	60·0	126·0	64·0	77·0	67·0	68·0	92·5
—	—		—	—	42·9	—	93·0	38·5	54·9	—	—	—	74·0	77·0	83·5
—	—		118·1	53·0	45·0	—	86·8	—	—	121·0	72·0	65·0	—	—	83·5
—	—		131·5	52·5	38·1	—	93·4	—	55·7	120·0	67·0	71·0	80·0	77·5	91·5
13·0	59·3		117·3	48·4	—	—	77·6	33·7	—	123·0	73·0	69·0	—	—	—
04·0	63·5	2	145·6	46·3	37·5	101·0	90·2	41·8	54·5	126·0	73·0	76·0	76·0	77·0	86·0
05·0	62·9	1	137·5	54·5	49·3	97·1	89·4	42·8	54·5	122·0	67·0	74·0	62·0	63·5	89·0
07·5	71·6	b	109·1	53·9	58·2	84·2	94·8	41·7	44·2	108·0	80·0	61·0	75·0	73·0	85·0



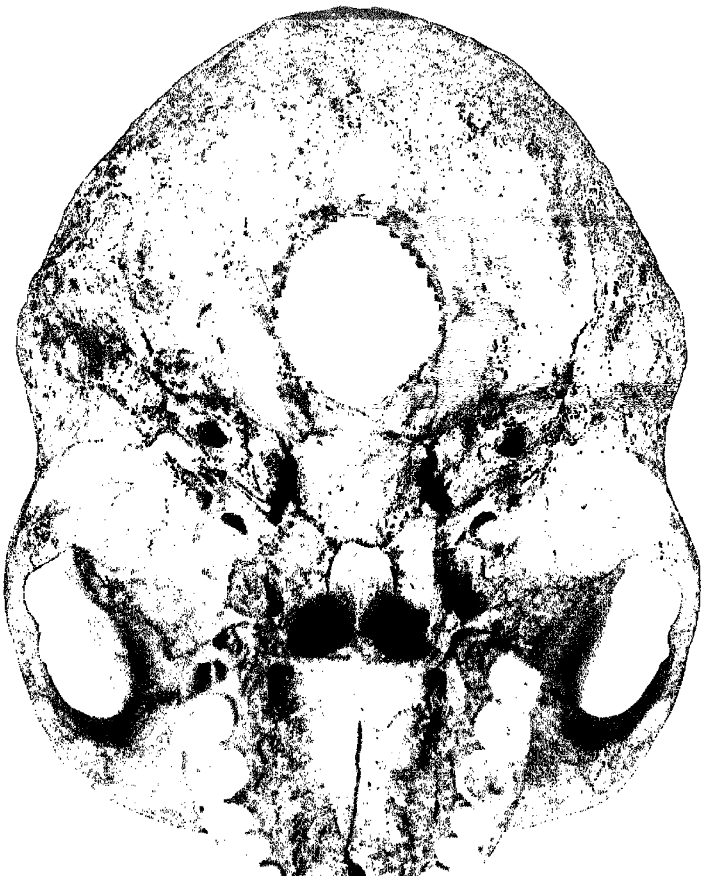
Typical Male Skull, No. 18. *Norma lateralis.* × *circa* $\frac{1}{3}$.



Typical Male Skull, No. 18. *Norma verticalis.* Life size.



Typical Male Skull, No. 18. *Norma occipitalis*. Life size.



Typical Male Skull, No. 18. *Norma basalis.* Life size.

(*a*)

(*b*)

Male Mandible, Skull, No. 18. (*a*) *Norma verticalis*, (*b*) *Norma lateralis*. Life size.

should be placed on record. Only male mandibles have been dealt with. The measurements taken were those listed by Morant in his memoir in *Biometrika*, Vol. XIV. 1923, and the technique of measurement which he describes therein was carefully followed. The symbols which are used to represent the various characters are those introduced by Morant and used in all the memoirs to which reference has been made. Although the number of mandibles was only 42, defects in some of which occasionally precluded the measurement of particular characters, so that the number is not quite so large as the number of Anglo-Saxon male mandibles measured by Morant, it was deemed advisable to calculate the variabilities of the various characters in order to provide provisional estimates of these. The estimate of the variability of a mandibular character, which is based on a number of observations that may possibly be no more than adequate to provide a reliable criterion, is better than no estimate at all of the variability. So far, no attempt has apparently been made to furnish, or at least to publish, even approximate measures of the variabilities of the several mandibular characters.

The means and variabilities of the mandibular characters are shown in Table V in apposition to the means of the corresponding characters in series of mandibles belonging to Anglo-Saxon, Badari Egyptian, Tibetan A and Fukien male skulls. The last-mentioned Asiatic group was selected for comparison in preference to the Tamil or Hylam mandibles, for which mean measurements have also been published by Harrower, because it possesses, according to this author, a heavier and more solid type of mandible than either of these two racial series.

From a comparison of the data in Table V, it is evident that there is a very close resemblance between the Dunstable and Anglo-Saxon mandibles. The mean values of many of the corresponding characters in these two types are almost identical, and few of the remaining characters diverge in average value to such a degree that the differences can be considered real or significant on such numbers of observations as are available. It may readily be seen that the mandibles of the other skull types in the table are, as might be expected, less like the Dunstable type.

Unfortunately, there is no other male series of British mandibles with which the Dunstable type can be compared. Miss Hooke* has published, however, the individual measurements of the characters in a series of about 60 unsexed mandibles belonging to the Farringdon Street 17th Century Londoners. For the indices and angles—the characters which describe the relative proportions and general form of the mandible—in this unsexed series, the means and variabilities have been calculated. These are shown in Table VI in comparison with the corresponding constants for the Dunstable male series and with the means computed for Morant's Anglo-Saxon series by taking males and females together. Morant has stated that in his series the male and female indices and angles are so similar that, for the small numbers dealt with, the differences would almost certainly not be significant. From a brief scrutiny of Table VI it is evident that

* *Biometrika*, Vol. XVIII. 1926, pp. 1—55 (Appendix C).

TABLE V.

Showing a Comparison of the Mean Values of the Mandibular Characters in the Dunstable and other series of Male Skulls.

Characters	Dunstable			Anglo-Saxon		Tibetan A		Badari Egyptian		Fukien Chinese	
	Means	S. D.	No.	Means	No.	Means	No.	Means	No.	Means	No.
w_1	120·99 ± 1·02*	5·47 ± 0·72*	29	123·7	25	117·0	25	109·5	30	121·9	38
w_2	102·93 ± 1·05	6·63 ± 0·74	40	103·2	45	96·2	25	88·8	32	101·0	38
h_1	32·51 ± 0·39	2·46 ± 0·28	39	33·1	40	30·6	25	32·6	34	35·2	38
zz	45·43 ± 0·37	2·38 ± 0·26	42	45·3	57	45·7	25	43·4	30	46·8	38
$c_r c_r$	99·34 ± 1·14	5·92 ± 0·81	27	100·3	27	93·8	25	80·8	29	97·2	38
rb	36·49 ± 0·51	3·29 ± 0·36	42	36·4	58	37·2	25	36·7	34	39·4	38
rb'	32·75 ± 0·45	2·93 ± 0·32	42	33·2	61	32·1	25	33·6	39	34·4	38
G_2'	47·58 ± 0·36	2·30 ± 0·25	41	48·7	43	46·7	25	41·1	29	43·9	38
$c_y c_r$	35·70 ± 0·63	3·86 ± 0·45	37	33·9	40	34·2	25	33·8	37	35·4	38
$g_o g_o$	98·40 ± 1·10	6·95 ± 0·78	40	100·4	33	92·8	24	83·9	31	95·4	38
$g_n g_o$, L	84·96 ± 0·60	3·82 ± 0·43	40	87·9	38	83·8	24	82·0	31	83·1	38
$g_n g_o$, R	84·92 ± 0·67	4·26 ± 0·48	40	80·9	41	83·8	24	82·4	31	82·3	38
$c_y l$	21·31 ± 0·27	1·59 ± 0·19	35	21·7	38	18·8	25	20·3	36	20·1	38
$c_y b$	9·01 ± 0·26	1·57 ± 0·19	36	9·5	42	8·1	25	9·8	37	8·9	38
$m_2 p_1$	27·99 ± 0·21	1·34 ± 0·15	39	28·1	59	29·2	24	27·3	33	30·2	33
$p_a d_1$	28·28 ± 0·42	2·61 ± 0·30	39	28·2	38	25·4	25	26·7	32	30·0	38
$p_a g_n$	7·25 ± 0·27	1·77 ± 0·19	42	7·1	59	8·2	25	8·5	30	7·5	38
$p_a d_f$	25·02 ± 0·50	2·94 ± 0·35	35	25·3	41	23·0	24	23·4	31	27·5	38
$g_n d_f$	29·57 ± 0·41	2·45 ± 0·29	35	30·0	41	28·3	25	29·6	31	32·9	38
$p_a p_b$	4·55 ± 0·41	2·16 ± 0·29	28	3·1	6	3·6	25	—	—	5·3	38
$g_o p_a g_o$	193·40 ± 1·45	9·14 ± 1·02	40	198·8	30	190·4	24	189·5	31	189·8	38
ih	48·41 ± 0·83	5·07 ± 0·59	37	47·9	51	41·6	25	44·3	35	47·2	38
ih'	13·39 ± 0·27	1·60 ± 0·19	34	13·6	35	15·0	25	12·2	33	13·6	38
$c_r h$	65·00 ± 0·88	5·41 ± 0·62	38	65·7	48	60·8	25	61·8	33	65·9	38
$c_y h$	58·55 ± 0·92	5·93 ± 0·65	42	59·4	44	50·8	25	53·8	35	56·8	38
$d_1 h$	35·50 ± 0·37	2·28 ± 0·26	39	36·3	40	35·0	24	36·1	31	39·7	38
$m_2 h$	26·60 ± 0·35	2·25 ± 0·25	41	27·2	51	24·0	25	24·5	31	28·6	34
$p_1 h$	30·66 ± 0·37	2·37 ± 0·26	40	30·9	54	28·9	25	30·4	30	32·8	38
$c_p l$	76·10 ± 0·79	4·90 ± 0·56	38	77·7	42	74·6	25	70·2	33	73·8	38
rl	64·69 ± 0·70	4·29 ± 0·49	38	64·0	45	58·4	25	57·6	33	61·5	38
ml	106·76 ± 0·60	4·04 ± 0·56	34	107·2	31	105·2	25	101·2	33	102·8	38
100 $c_r h/ml$	59·27 ± 1·01	5·65 ± 0·72	31	60·9	27	57·8	25	61·0	32	64·0	38
100 $c_r c_r/ml$	93·48 ± 1·11	5·57 ± 0·79	25	94·4	15	89·2	25	80·2	29	94·7	38
100 $g_o g_o/c_p l$	129·67 ± 2·12	13·05 ± 1·50	38	129·0	32	124·9	24	110·4	30	129·9	38
100 rb'/rl	49·55 ± 0·84	5·16 ± 0·59	38	51·5	45	55·3	25	58·6	33	56·0	38
100 $c_y b/c_y l$	42·33 ± 0·99	5·86 ± 0·70	35	44·0	38	43·3	25	48·4	36	44·5	38
100 $g_o g_o/c_r c_r$	100·01 ± 1·85	9·62 ± 1·31	27	90·3	19	98·9	24	97·4	28	98·1	38
100 $c_y h/c_r h$	89·97 ± 1·01	6·21 ± 0·71	38	89·9	40	83·8	25	86·9	32	86·0	38
100 $ih'/c_y c_r$	37·65 ± 0·82	4·77 ± 0·58	34	40·4	35	44·0	25	36·9	32	41·3	38
100 $d_1 h/c_r h$	55·34 ± 0·84	5·02 ± 0·59	36	55·2	30	57·9	24	58·6	29	60·4	38
$M\angle$	120°·88 ± 0·96	5°·94 ± 0·67	39	120°·3	47	125°·3	25	120°·0	34	121°·0	38
$R\angle$	68°·70 ± 0·98	5°·95 ± 0·69	37	72°·0	36	70°·5	25	73°·7	31	74°·7	38
$G\angle$	70°·86 ± 0·83	5°·26 ± 0.59	40	68°·6	31	67°·1	24	60°·8	31	70°·5	38
$C\angle$	68°·11 ± 0·99	6°·05 ± 0·70	37	68°·2	32	64°·8	25	71°·7	28	—	—
$C'\angle$	69°·04 ± 0·93	5°·60 ± 0·66	36	68°·3	35	62°·9	24	73°·2	28	77°·1	38
$F\angle$	88°·40 ± 0·81	4°·31 ± 0·58	28	87°·0	6	90°·6	25	—	—	93°·0	38

* Standard errors.

TABLE VI.

Showing a Comparison of the Mandibular Indices and Angles in the Dunstable Skulls with those in Anglo-Saxons and 17th Century Londoners.

Characters	Dunstable (M.)			Anglo-Saxon (M. and F.)		Farringdon Street (M. and F.)		
	Means	σ	No.	Means	No.	Means	σ	No.
$100\, c_r h/ml$	59·27 ± 1·01*	5·65 ± 0·72*	31	59·4	65	58·59 ± 0·78*	6·06 ± 0·55*	61
$100\, c_r c_r/ml$	93·48 ± 1·11	5·57 ± 0·79	25	92·7	41	90·43 ± 0·94	6·60 ± 0·66	49
$100\, g_o g_o/c_p l$	129·67 ± 2·12	13·05 ± 1·50	38	127·5	67	126·35 ± 1·56	12·21 ± 1·10	61
$100\, rb'/rl$	49·55 ± 0·84	5·16 ± 0·59	38	52·2	88	51·57 ± 0·78	6·21 ± 0·55	63
$100\, c_y b/c_y l$	42·33 ± 0·99	5·86 ± 0·70	35	43·9	74	48·64 ± 0·94	7·35 ± 0·66	61
$100\, g_a g_a/c_r c_r$	100·01 ± 1·85	9·62 ± 1·31	27	99·3	42	99·87 ± 1·25	8·46 ± 0·88	46
$100\, c_y h/c_r h$	89·97 ± 1·01	6·21 ± 0·71	38	89·8	80	87·71 ± 0·99	7·80 ± 0·70	62
$100\, ih'/c_y c_r$	37·65 ± 0·82	4·77 ± 0·58	34	39·7	70	40·64 ± 0·80	6·14 ± 0·57	59
$100\, d_1 h/c_r h$	55·34 ± 0·84	5·02 ± 0·59	36	55·7	66	54·10 ± 0·86	6·75 ± 0·61	61
$M\angle$	120°·88 ± 0·95	5°·94 ± 0·67	39	121°·4	96	124°·11 ± 0·88	6°·09 ± 0·62	63
$R\angle$	68°·70 ± 0·98	5°·95 ± 0·69	37	70°·1	72	70°·22 ± 1·00	7°·65 ± 0·71	59
$O\angle$	70°·86 ± 0·83	5°·26 ± 0·59	40	68°·5	65	67°·87 ± 0·73	5°·70 ± 0·52	61
$C\angle$	68°·11 ± 0·99	6°·05 ± 0·70	37	69°·0	57	64°·42 ± 0·90	7°·05 ± 0·64	62
$C'\angle$	69°·04 ± 0·93	5°·80 ± 0·66	36	69°·4	67	66°·49 ± 0·85	6°·84 ± 0·60	65

* Standard errors.

the Dunstable mandible in its general form resembles the Anglo-Saxon mandible more closely than it does that of the 17th Century Londoner. In seven of the fourteen characters compared, even though the number of specimens available is so relatively small, the Dunstable mandible probably differs significantly from the Londoner, whereas the mean values of the characters in the Anglo-Saxon mandible are so similar to those in the Dunstable mandible that probably only one or two of the differences observed can be considered real. The fact that the Dunstable mandible resembles the Anglo-Saxon mandible more closely than it does that of the 17th Century Londoner from Farringdon Street in no way controverts the conclusion that the Dunstable skull is more closely related to the Bronze Age type than to the Anglo-Saxon, as it has been shown that the Dunstable cranium is more closely related to the Anglo-Saxon cranium than to that of the 17th Century Londoner from Whitechapel, which is very similar to the contemporary series from Farringdon Street.

The number of fairly complete mandibles available in the Dunstable series is obviously inadequate to justify any attempt to calculate the correlation between the various mandibular characters.

ON THE APPLICATIONS OF THE DOUBLE BESSEL FUNCTION $\mathfrak{K}_{\tau_1, \tau_2}(x)$ TO STATISTICAL PROBLEMS.

By KARL PEARSON.

PART I. THEORETICAL.

(i) I left over from the discussion in *Biometrika*, Vol. XXIV. pp. 293—343 the case in which two variables, u and v, follow Type III curves of the form

$$y_u = \frac{M\gamma_1^{\tau_1+1}}{\Gamma(\tau_1+1)} u^{\tau_1} e^{-\gamma_1 u} \text{ and } y_v = \frac{M\gamma_2^{\tau_2+1}}{\Gamma(\tau_2+1)} v^{\tau_2} e^{-\gamma_2 v} \quad \ldots\ldots\ldots\ldots\ldots(\text{i}),$$

leading to the surface

$$w = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_1+1)\,\Gamma(\tau_2+1)} e^{-(\gamma_1 u+\gamma_2 v)} u^{\tau_1} v^{\tau_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii}),$$

where γ_1 and γ_2 and τ_1 and τ_2 are not equal each to each. We require to discuss the distribution surface of $Y = v - u$.

Let $X = v + u$, and we have

$$w\,[du, dv] = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1} e^{-\frac{1}{2}(\gamma_2-\gamma_1)Y}}{\Gamma(\tau_1+1)\,\Gamma(\tau_2+1)\,2^{\tau_1+\tau_2+1}} e^{-\frac{1}{2}(\gamma_1+\gamma_2)X}(X-Y)^{\tau_1}(X+Y)^{\tau_2}[dX, dY] \quad \ldots\ldots\ldots(\text{iii}).$$

Let us write
$$z_0 = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_1+1)\,\Gamma(\tau_2+1)\,2^{\tau_1+\tau_2+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{iv}).$$

Now we want to integrate w from $X = Y$ to $X = \infty$; let us put $X = Yt$ and integrate t from 1 to ∞.

Hence the distribution curve for Y is

$$z = z_0 e^{-\frac{1}{2}(\gamma_2-\gamma_1)Y} Y^{\tau_2+\tau_1+1} \int_1^\infty e^{-\frac{1}{2}(\gamma_1+\gamma_2)Yt}(t-1)^{\tau_1}(t+1)^{\tau_2} dt \quad \ldots\ldots\ldots(\text{v}).$$

This curve is closely allied to the Bessel Functions of the second kind with imaginary argument.

Let us consider the expression

$$\phi(Y) = Y^{\tau_2+\tau_1+1} \int_1^\infty e^{-\frac{1}{2}(\gamma_1+\gamma_2)Yt}(t-1)^{\tau_1}(t+1)^{\tau_2} dt \quad \ldots\ldots\ldots\ldots(\text{vi}),$$

then if we had started below the OX line (see Diagram Fig. 1, p. 295 of the paper referred to above), we should have had to integrate X from $-Y$ to ∞, or t from -1 to ∞; then by changing t to $-t$, we obtain precisely the expression (vi) with τ_1 and τ_2 interchanged. Thus the only thing which remains to be changed when we put Y negative is the term $e^{-\frac{1}{2}(\gamma_2-\gamma_1)Y}$, which we accomplish by interchanging γ_1 and γ_2.

Thus the equation to our curve is

$$z = z_0 e^{-\frac{1}{2}(\gamma_2-\gamma_1)Y}\phi(|Y|) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vii}),$$

where Y is to be taken from $-\infty$ to $+\infty$, but τ_1 and τ_2 are to be interchanged when Y is negative in ϕ.

(ii) I start with the integral

$$W = \int_1^\infty e^{-\{c_1 Y(t-1)+c_2 Y(t+1)\}}\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{viii}).$$

Then

$$\frac{d^{\tau_1+\tau_2}W}{dc_1^{\tau_1}dc_2^{\tau_2}} = (-1)^{\tau_1+\tau_2}\int_1^\infty e^{-\{c_1 Y(t-1)+c_2 Y(t+1)\}}\,Y^{\tau_1+\tau_2}(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$

$$= (-1)^{\tau_1+\tau_2} e^{-(c_2-c_1)Y}\,Y^{\tau_1+\tau_2}\int_1^\infty e^{-(c_1+c_2)Yt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt \quad \ldots(\text{ix}).$$

Returning to (viii),

$$W = e^{-(c_2-c_1)Y}\int_1^\infty e^{-(c_1Y+c_2Y)t}\,dt$$

$$= e^{-(c_2-c_1)Y}\left[-\frac{e^{-(c_1+c_2)Yt}}{(c_1+c_2)Y}\right]_1^\infty,$$

or, c_1 and c_2 being positive,

$$W = \frac{e^{-2c_2Y}}{(c_1+c_2)Y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{x})$$

Accordingly $$\frac{d^{\tau_1}W}{dc_1^{\tau_1}} = (-1)^{\tau_1}\frac{e^{-2c_2Y}}{Y(c_1+c_2)^{\tau_1+1}}\Gamma(\tau_1+1)$$

and $$\frac{d^{\tau_1+\tau_2}W}{dc_1^{\tau_1}dc_2^{\tau_2}} = \frac{(-1)^{\tau_1}\Gamma(\tau_1+1)}{Y}\,\frac{d^{\tau_2}}{dc_2^{\tau_2}}\left(\frac{e^{-2c_2Y}}{(c_1+c_2)^{\tau_1+1}}\right).$$

Applying Leibnitz's Theorem

$$\frac{d^{\tau_1+\tau_2}W}{dc_1^{\tau_1}dc_2^{\tau_2}} = \frac{(-1)^{\tau_1}\Gamma(\tau_1+1)}{Y}\left[\frac{(-2Y)^{\tau_2}}{(c_1+c_2)^{\tau_1+1}} + \tau_2\frac{(-2Y)^{\tau_2-1}\{-(\tau_1+1)\}}{(c_1+c_2)^{\tau_1+2}}\right.$$

$$\left. + \frac{\tau_2(\tau_2-1)}{1.2}\frac{(-2Y)^{\tau_2-2}}{(c_1+c_2)^{\tau_1+3}}(-(\tau_1+1)\times-(\tau_1+2)) + \ldots\right]e^{-2c_2Y}$$

$$= \frac{(-1)^{\tau_1+\tau_2}2^{\tau_2}Y^{\tau_2-1}e^{-2c_2Y}\,\Gamma(\tau_1+1)}{(c_1+c_2)^{\tau_1+1}}$$

$$\times\left[1 + \frac{\tau_2(\tau_1+1)}{(c_1+c_2)\,2Y} + \frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{1.2\,(c_1+c_2)^2(2Y)^2} + \ldots\right] \ldots(\text{xi}).$$

Now put $c_1 = \frac{1}{2}\gamma_1$, $c_2 = \frac{1}{2}\gamma_2$, and we have from (vi) and (ix)

$$\phi(|Y|) = Y^{\tau_1+\tau_2+1}\int_1^\infty e^{-\frac{1}{2}(\gamma_1+\gamma_2)Yt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$

$$= (-1)^{\tau_1+\tau_2}e^{(c_2-c_1)Y}\,Y\frac{d^{\tau_1+\tau_2}W}{dc_1^{\tau_1}dc_2^{\tau_2}}$$

$$= \frac{2^{\tau_2}Y^{\tau_2}e^{-\frac{1}{2}(\gamma_1+\gamma_2)Y}\,\Gamma(\tau_1+1)}{(\frac{1}{2}(\gamma_1+\gamma_2))^{\tau_1+1}}\left[1 + \frac{\tau_2(\tau_1+1)}{(\gamma_1+\gamma_2)Y} + \frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{1.2\,(\gamma_1+\gamma_2)^2Y^2} + \ldots\right].$$

Hence (vii) and (iv) for $Y_1 = v - u$ positive,

$$z = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_2+1)(\gamma_1+\gamma_2)^{\tau_1+1}} e^{-\gamma_2 Y} Y^{\tau_2}$$
$$\times\left[1 + \frac{\tau_2(\tau_1+1)}{1!(\gamma_1+\gamma_2)Y} + \frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{2!(\gamma_1+\gamma_2)^2 Y^2} + \dots + \frac{\tau_2!(\tau_1+\tau_2)!}{\tau_2!\tau_1!(\gamma_1+\gamma_2)^{\tau_2}Y^{\tau_2}}\right] \quad \text{.........(xii)}.$$

The last term if the series be finite, i.e. τ_2 an integer, will be as above.

It follows that for $Y = 0$ we have

$$z_{Y=0} = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}} \cdot \frac{\Gamma(\tau_1+\tau_2+1)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)} \quad \text{..............(xiii)}.$$

This expression for $z_{Y=0}$ is symmetrical in τ_1, τ_2, γ_1 and γ_2. We can take for the $u - v$ side of the curve

$$z = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_1+1)(\gamma_1+\gamma_2)^{\tau_2+1}} e^{-\gamma_1 Y} Y^{\tau_1}$$
$$\times\left[1 + \frac{\tau_1(\tau_2+1)}{1!(\gamma_1+\gamma_2)Y} + \frac{\tau_1(\tau_1-1)(\tau_2+1)(\tau_2+2)}{2!(\gamma_1+\gamma_2)^2 Y^2} + \dots + \frac{\tau_1!(\tau_1+\tau_2)!}{\tau_1!\tau_2!(\gamma_1+\gamma_2)^{\tau_1}Y^{\tau_1}}\right] \quad \text{.........(xiv)},$$

leading to the same value of $z_{Y=0}$.

(iii) In order to discuss the moments of the curve in (v) we require the sum of the first p terms of the negative binomial $(1-x)^{-m}$. The following elegant proof was provided for me by Mr E. C. Fieller.

The remainder after p terms of the Maclaurin series in the integral form for a function $f(x)$ is

$$R = \frac{1}{\Gamma(p)}\int_0^x f^{(p)}(x-t)\,t^{p-1}dt.$$

But for the binomial $f(x) = (1-x)^{-m}$, and accordingly

$$f^{(p)}(x) = (-1)^p\{-m(-m-1)(-m-2)\dots(-m-p+1)\}(1-x)^{-m-p}$$
$$= \frac{\Gamma(m+p)}{\Gamma(m)}(1-x)^{-m-p},$$

therefore
$$R = \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(p)}\int_0^x (1-x+t)^{-m-p}t^{p-1}dt.$$

Take
$$1-u = (1-x)/(1-x+t),$$

then $t=0$, $u=0$, and $t=x$, $u=x$, $dt = \dfrac{1-x}{(1-u)^2}du$, and we have

$$R = \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(p)}\int_0^x (1-u)^{m-1}u^{p-1}du \times (1-x)^{-m}$$
$$= \frac{\Gamma(m+p)}{\Gamma(m)\Gamma(p)} \mathrm{B}_x(p,m) \times (1-x)^{-m}$$
$$= (1-x)^{-m} \times I_x(p,m),$$

where $I_x(p, m)$ is the ratio of the incomplete to the complete beta-function. Accordingly the sum of the first p terms of $(1-x)^{-m}$

$$\begin{aligned}&=(1-x)^{-m}-R\\&=(1-x)^{-m}(1-I_x(p,m))=(1-x)^{-m}I_{1-x}(m,p)\quad\ldots\ldots\ldots\ldots(\text{xv}),\end{aligned}$$

which is the result we require.

Returning to Equation (xii) we have

$$\begin{aligned}\int_0^\infty zY^s dY=&\frac{M\gamma_1^{\tau_1+1}}{\Gamma(\tau_2+1)(\gamma_1+\gamma_2)^{\tau_1+1}\gamma_2^s}\int_0^\infty e^{-y}y^{\tau_2+s}\\&\times\left(1+\frac{\tau_2(\tau_1+1)}{1!}\frac{\psi}{y}+\frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{2!}\frac{\psi^2}{y^2}+\ldots\right.\\&\left.+\frac{\tau_2!(\tau_1+\tau_2)!}{\tau_1!\,\tau_2!}\frac{\psi^{\tau_2}}{y^{\tau_2}}\right)dy\quad\ldots\ldots\ldots\ldots(\text{xvi}),\end{aligned}$$

where $y=\gamma_2 Y$ and $\psi=\gamma_2/(\gamma_1+\gamma_2)$.

Integrating out with regard to y we have

$$\begin{aligned}\int_0^\infty zY^s dY=&\frac{M(1-\psi)^{\tau_1+1}}{\Gamma(\tau_2+1)\gamma_2^s}\left(\Gamma(\tau_2+s+1)+\frac{(\tau_1+1)}{1!}\tau_2\Gamma(\tau_2+s)\psi\right.\\&\left.+\frac{(\tau_1+1)(\tau_1+2)}{2!}\tau_2(\tau_2-1)\Gamma(\tau_2+s-1)\psi^2+\ldots+\frac{(\tau_1+\tau_2)!}{\tau_1!\,\tau_2!}\tau_2!\,\Gamma(s)\psi^{\tau_2}\right)\\&\ldots\ldots\ldots(\text{xvii}).\end{aligned}$$

First take $s=0$,

$$\begin{aligned}\int_0^\infty zY^s dY=&\frac{M(1-\psi)^{\tau_1+1}}{\Gamma(\tau_2+1)}\Gamma(\tau_2+1)\left(1+\frac{\tau_1+1}{1!}\psi+\frac{(\tau_1+1)(\tau_1+2)}{2!}\psi^2+\ldots\right.\\&\left.+\frac{(\tau_1+\tau_2)!}{\tau_1!\,\tau_2!}\psi^{\tau_2}\right)\\=&M(1-\psi)^{\tau_1+1}\times\text{sum of first }\tau_2\text{ terms of the negative binomial }(1-\psi)^{-(\tau_1+1)}\\=&MI_{1-\psi}(\tau_1+1,\tau_2+1)=MI_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1),\end{aligned}$$

by the Lemma just demonstrated.

Hence the total area

$$\begin{aligned}&=MI_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1)+MI_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+1,\tau_1+1)\\&=\frac{M}{B(\tau_1,\tau_2)}\left(\int_0^{\frac{\gamma_1}{\gamma_1+\gamma_2}}x^{\tau_1}(1-x)^{\tau_2}dx+\int_0^{\frac{\gamma_2}{\gamma_1+\gamma_2}}x^{\tau_2}(1-x)^{\tau_1}dx\right).\end{aligned}$$

In the last integral write $1-x$ for x, and we have for the total area

$$\frac{M}{B(\tau_1,\tau_2)}\left\{\int_0^{\frac{\gamma_1}{\gamma_1+\gamma_2}}x^{\tau_1}(1-x)^{\tau_2}dx+\int_{\frac{\gamma_1}{\gamma_1+\gamma_2}}^1 x^{\tau_1}(1-x)^{\tau_2}dx\right\},$$

but the sum of the two integrals in the curled brackets makes up $B(\tau_1,\tau_2)$. Hence the total area equals M as it should do.

Next take $s = 1$.

We have for moment about $Y = 0$,

$$M\mu_1' = \frac{M(1-\psi)^{\tau_1+1}}{\Gamma(\tau_2+1)\gamma_2}\Big\{\Gamma(\tau_2+2) + \frac{\tau_1+1}{1!}\tau_2\Gamma(\tau_2+1)\psi + \frac{(\tau_1+1)(\tau_1+2)}{2!}\tau_2(\tau_2-1)\Gamma(\tau_2)\psi^2 + \ldots + \frac{\tau_2!(\tau_1+\tau_2)!}{\tau_1!\tau_2!}\psi^{\tau_2}\Big\}$$

– (a similar expression w τ_1 and τ_2 interchanged and γ_1 and γ_2 also).

The expression in curled brackets, call it Q, may be read as

$$Q = \Gamma(\tau_2+1)\Big(\tau_2+1+\frac{\tau_1+1}{1!}(\tau_2+1-1)\psi + \frac{(\tau_1+1)(\tau_1+2)}{2!}(\tau_2+1-2)\psi^2 + \frac{(\tau_1+1)(\tau_1+2)(\tau_1+3)}{3!}(\tau_2+1-3)\psi^3+\ldots\Big)$$

$$= \Gamma(\tau_2+1)\left((\tau_2+1)(1-\psi)^{-(\tau_1+1)}I_{1-\psi}(\tau_1+1,\tau_2+1) - (\tau_1+1)\psi(1-\psi)^{-(\tau_1+2)}I_{1-\psi}(\tau_1+2,\tau_2)\right).$$

Thus

$$\mu_1' = \frac{\tau_2+1}{\gamma_2}I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1) - \frac{\tau_1+1}{\gamma_1}I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2) - \frac{\tau_1+1}{\gamma_1}I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+1,\tau_1+1) + \frac{\tau_2+1}{\gamma_2}I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+2,\tau_1),$$

$$= \frac{\tau_2+1}{\gamma_2}\left(I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1) + I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+2,\tau_1)\right) - \frac{\tau_1+1}{\gamma_1}\left(I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+1,\tau_1+1) + I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2)\right) \quad \ldots\ldots\text{(xviii)}.$$

But $$I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1) + I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+2,\tau_1)$$

$$= \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\left(\int_0^{\frac{\gamma_1}{\gamma_1+\gamma_2}} x^{\tau_1}(1-x)^{\tau_2}dx + \frac{\tau_1}{\tau_2+1}\int_0^{\frac{\gamma_2}{\gamma_1+\gamma_2}} x^{\tau_2+1}(1-x)^{\tau_1-1}dx\right),$$

or, integrating the second integral by parts,

$$= \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\left(\int_0^{\frac{\gamma_1}{\gamma_1+\gamma_2}} x^{\tau_1}(1-x)^{\tau_2}dx + \int_0^{\frac{\gamma_2}{\gamma_1+\gamma_2}} x^{\tau_2}(1-x)^{\tau_1}dx - \frac{\gamma_2}{\tau_2+1}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\right).$$

Since the sum of the two integrals is as before the complete Γ-function, we find from (xviii)

$$\mu_1' = \frac{\tau_2+1}{\gamma_2} - \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}} - \frac{\tau_1+1}{\gamma_1} + \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}},$$

or, since the second term in both lines is symmetrical in τ_1, γ_1 and τ_2, γ_2,

$$\mu_1' = \frac{\tau_2+1}{\gamma_2} - \frac{\tau_1+1}{\gamma_1} \quad \text{.........................(xix).}$$

We have next to take $s = 2$ in (xvii), and have

$$\mu_2' = \frac{(1-\psi)^{\tau_1+1}}{\Gamma(\tau_2+1)\gamma_2^2}\Big(\Gamma(\tau_2+3) + \frac{\tau_1+1}{1!}\tau_2\Gamma(\tau_2+2)\psi$$
$$+ \frac{(\tau_1+1)(\tau_1+2)}{2!}\tau_2(\tau_2-1)\Gamma(\tau_2+1)\psi^2$$
$$+ \frac{(\tau_1+1)(\tau_1+2)(\tau_1+3)}{3!}\tau_2(\tau_2-1)(\tau_2-2)\Gamma(\tau_2)\psi^3 + \dots \text{ up to } \psi^{\tau_2}\Big)$$

+ a similar series with interchanges of τ_1, τ_2 and γ_1, γ_2

$$= \frac{(1-\psi)^{\tau_1+1}}{\gamma_2^2}\Big((\tau_2+2)(\tau_2+1) + \frac{\tau_1+1}{1!}(\tau_2+1)\tau_2\psi + \frac{(\tau_1+1)(\tau_2+2)}{2!}\tau_2(\tau_2-1)\psi^2$$
$$+ \frac{(\tau_1+1)(\tau_1+2)(\tau_1+3)}{3!}(\tau_2-1)(\tau_2-2)\psi^2 + \dots \text{ up to } \psi^{\tau_2}\Big)$$

+ a similar series with proper interchanges.

Replacing $(\tau_2+1)\tau_2$ by $(\tau_2+2-1)(\tau_2+1-1)$, $\tau_2(\tau_2-1)$ by $(\tau_2+2-2)(\tau_1+1-2)$, $(\tau_2-1)(\tau_2-2)$ by $(\tau_2+2-3)(\tau_2+1-3)$ etc., we can rewrite our μ_2' as

$$\mu_2' = \frac{(1-\psi)^{\tau_1+1}}{\gamma_2^2}\Big[(\tau_2+2)(\tau_2+1)\Big\{1 + \frac{\tau_1+1}{1!}\psi + \frac{(\tau_1+1)(\tau_1+2)}{2!}\psi^2 + \dots \text{ to } \psi^{\tau_2}\Big\}$$
$$-(2\tau_2+3)(\tau_1+1)\psi\Big\{1 + \frac{\tau_1+2}{1!}\psi + \frac{(\tau_1+2)(\tau_1+3)}{2!}\psi^2 + \dots \text{ to } \psi^{\tau_2-1}\Big\}$$
$$+(\tau_1+1)\psi\Big\{1 + 2\frac{(\tau_1+2)\psi}{1!} + 3\frac{(\tau_1+2)(\tau_2+3)}{2!}\psi^2 + \dots \text{ to } \psi^{\tau_2-1}\Big\}\Big]$$

+ a similar series with proper interchanges.

The first series in curled brackets is $(1-\psi)^{-(\tau_1+1)}I_{1-\psi}(\tau_1+1, \tau_2+1)$; the second series in curled brackets is $(1-\psi)^{-(\tau_1+2)}I_{1-\psi}(\tau_1+2, \tau_2)$.

To obtain the third series we note that, if

$$S = 1 + \frac{\tau_1+2}{1!}\psi + \frac{(\tau_1+2)(\tau_2+3)}{2!}\psi^2 + \dots \text{ up to } \psi^{\tau_2-1}$$
$$= (1-\psi)^{-(\tau_1+2)}I_{1-\psi}(\tau_1+2, \tau_2),$$

then the third series $= \dfrac{d(S\psi)}{d\psi}$

$$= (1-\psi)^{-(\tau_1+2)}I_{1-\psi}(\tau_1+2, \tau_2) + (\tau_1+2)\psi(1-\psi)^{-(\tau_1+3)}I_{1-\psi}(\tau_1+2, \tau_2)$$
$$-\psi(1-\psi)^{-(\tau_1+2)}\frac{(1-\psi)^{\tau_1+1}\psi^{\tau_2-1}}{B(\tau_1+2, \tau_2)}.$$

Substituting

$$\mu_2' = \frac{1}{\gamma_2^2}\Big[(\tau_2+2)(\tau_2+1)I_{1-\psi}(\tau_1+1,\tau_2+1) - (2\tau_2+3)(\tau_1+1)\frac{\psi}{1-\psi}I_{1-\psi}(\tau_1+2,\tau_2)$$
$$+(\tau_1+1)\frac{\psi}{1-\psi}I_{1-\psi}(\tau_1+2,\tau_2) + (\tau_1+1)(\tau_1+2)\frac{\psi^2}{(1-\psi)^2}I_{1-\psi}(\tau_1+2,\tau_2)$$
$$-\frac{\psi^{\tau_2}(1-\psi)^{\tau_1}\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\tau_2\psi\Big]$$

+ a similar series with proper interchanges,

$$= \frac{(\tau_2+1)(\tau_2+2)}{\gamma_2^2}I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1) - \frac{2(\tau_2+1)(\tau_1+1)}{\gamma_1\gamma_2}I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2)$$
$$+\frac{(\tau_1+1)(\tau_1+2)}{\gamma_1^2}I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2) - \frac{\gamma_2^{\tau_2}\gamma_1^{\tau_1}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\frac{\tau_2}{\gamma_2}$$

+ a similar series with proper interchanges.

We can now combine the two series and find

$$\mu_2' = \frac{(\tau_2+1)(\tau_2+2)}{\gamma_2^2}\left\{I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1) + I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+2,\tau_1)\right\}$$
$$-\frac{2(\tau_2+1)(\tau_1+1)}{\gamma_1\gamma_2}\left\{I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2) + I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+2,\tau_1)\right\}$$
$$+\frac{(\tau_1+1)(\tau_1+2)}{\gamma_1^2}\left\{I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+1,\tau_1+1) + I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2)\right\}$$
$$-\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\left(\frac{\tau_1}{\gamma_1}+\frac{\tau_2}{\gamma_2}\right) \quad \ldots\ldots\ldots\ldots\ldots(\text{XX}).$$

But we have already seen that

$$I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+1,\tau_2+1) + I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+2,\tau_1) = 1 - \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\frac{\gamma_2}{\tau_2+1}.$$

Similarly

$$I_{\frac{\gamma_2}{\gamma_1+\gamma_2}}(\tau_2+1,\tau_1+1) + I_{\frac{\gamma_1}{\gamma_1+\gamma_2}}(\tau_1+2,\tau_2) = 1 - \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\frac{\gamma_1}{\tau_1+1}.$$

Again integrating by parts one finds

$$I_{1-\psi}(\tau_1+2,\tau_2) = -\frac{\gamma_1}{\tau_1+1}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)} + I_{1-\psi}(\tau_1+1,\tau_2+1),$$

and similarly

$$I_{\psi}(\tau_2+2,\tau_1) = -\frac{\gamma_2}{\tau_2+1}\frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}}\frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)} + I_{\psi}(\tau_2+1,\tau_1+1).$$

Substituting in (xx) and remembering that

$$I_{1-\psi}(\tau_1+1,\,\tau_2+1)+I_\psi(\tau_2+1,\,\tau_1+1)=1,$$

we find

$$\mu_2' = \frac{(\tau_2+1)(\tau_2+2)}{\gamma_2^2} - \frac{2(\tau_2+1)(\tau_1+1)}{\gamma_1\gamma_2} + \frac{(\tau_1+1)(\tau_1+2)}{\gamma_1^2}$$

$$- \frac{\gamma_1^{\tau_1}\gamma_2^{\tau_2}}{(\gamma_1+\gamma_2)^{\tau_1+\tau_2+1}} \frac{\Gamma(\tau_1+\tau_2+2)}{\Gamma(\tau_1+1)\,\Gamma(\tau_2+1)}$$

$$\times \left\{\frac{\tau_2+2}{\gamma_2} + \frac{\tau_1+2}{\gamma_1} - \frac{2(\tau_2+1)}{\gamma_2} - \frac{2(\tau_1+1)}{\gamma_1} + \frac{\tau_2}{\gamma_2} + \frac{\tau_1}{\gamma_1}\right\},$$

and since the last term in curled brackets vanishes,

$$\mu_2' = \frac{(\tau_2+1)(\tau_2+2)}{\gamma_2^2} - \frac{2(\tau_2+1)(\tau_1+1)}{\gamma_1\gamma_2} + \frac{(\tau_1+1)(\tau_1+2)}{\gamma_1^2} \quad \text{......(xxi).}$$

Subtract $\mu_1'^2$ and we have $\qquad \mu_2 = \dfrac{\tau_2+1}{\gamma_2^2} + \dfrac{\tau_1+1}{\gamma_1^2}$.

Putting $s=3$ in (xvii) and proceeding in the same manner, we find after considerable algebra

$$\mu_3' = \frac{(\tau_2+3)(\tau_2+2)(\tau_1+1)}{\gamma_2^3} - \frac{3(\tau_2+2)(\tau_2+1)(\tau_1+1)}{\gamma_2^2\gamma_1}$$

$$+ \frac{3(\tau_2+1)(\tau_1+1)(\tau_1+2)}{\gamma_2\gamma_1^2} - \frac{(\tau_1+3)(\tau_1+2)(\tau_2+1)}{\gamma_1^3} \quad \text{......(xxii).}$$

Hence transferring to the mean we find

$$\mu_3 = 2\left(\frac{\tau_2+1}{\gamma_2^3} - \frac{\tau_1+1}{\gamma_1^3}\right) \quad \text{........................(xxiii).}$$

The algebraic labour for μ_4' and its transference to the mean is so considerable that it is desirable to use the characteristic function of the distribution.

We have first to find $\qquad \displaystyle\int z e^{\omega Y}\, dY,$

and this by (xii)

$$= \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_2+1)(\gamma_1+\gamma_2)^{\tau_1+1}} \int_0^\infty e^{-(\gamma_2-\omega)Y}\, Y^{\tau_2} \left(1 + \frac{\tau_2(\tau_1+1)}{1!\,(\gamma_1+\gamma_2)\,Y}\right.$$

$$\left. + \frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{2!\,(\gamma_1+\gamma_2)^2\,Y^2} + \ldots + \frac{\tau_2!\,(\tau_1+\tau_2)!}{\tau_2!\,\tau_1!\,(\gamma_1+\gamma_2)^{\tau_2}\,Y^{\tau_2}}\right) dY$$

+ a similar series with the proper interchanges.

Write $(\gamma_2-\omega)\,Y = y$, and we have for the characteristic function $F(\omega)$,

$$F(\omega) = \frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{\Gamma(\tau_2+1)(\gamma_1+\gamma_2)^{\tau_1+1}} \frac{1}{(\gamma_2-\omega)^{\tau_2+1}} \int_0^\infty e^{-y}\, y^{\tau_2}$$

$$\times \left\{1 + \frac{\tau_2(\tau_1+1)}{1!\,(\gamma_1+\gamma_2)}\left(\frac{\gamma_2-\omega}{y}\right) + \frac{\tau_2(\tau_2-1)(\tau_1+1)(\tau_1+2)}{2!\,(\gamma_1+\gamma_2)^2}\left(\frac{\gamma_2-\omega}{y}\right)^2\right.$$

$$\left. + \ldots + \frac{\tau_2!\,(\tau_1+\tau_2)!}{\tau_2!\,\tau_1!\,(\gamma_1+\gamma_2)^{\tau_2}}\left(\frac{\gamma_2-\omega}{y}\right)^{\tau_2}\right\} dy + \text{a similar series,}$$

or, $$F(\omega)=\frac{M\gamma_1^{\tau_1+1}\gamma_2^{\tau_2+1}}{(\gamma_1+\gamma_2)^{\tau_1+1}}\frac{1}{(\gamma_2-\omega)^{\tau_2+1}}\left(1-\frac{\gamma_2-\omega}{\gamma_1+\gamma_2}\right)^{-(\tau_2+1)}I_{1-\frac{\gamma_2-\omega}{\gamma_1+\gamma_2}}(\tau_1+1,\ \tau_2+1)$$

+ a similar series with ω changed to $-\omega$, since V becomes $-V$, and τ_1, γ_1 changed to τ_2, γ_2, etc.

Hence
$$F(\omega)=\frac{M}{\left(1-\frac{\omega}{\gamma_2}\right)^{\tau_2+1}\left(1+\frac{\omega}{\gamma_1}\right)^{\tau_1+1}}I_{\frac{\gamma_1+\omega}{\gamma_1+\gamma_2}}(\tau_1+1,\ \tau_2+1)$$
$$+\frac{M}{\left(1-\frac{\omega}{\gamma_2}\right)^{\tau_2+1}\left(1+\frac{\omega}{\gamma_1}\right)^{\tau_1+1}}I_{\frac{\gamma_2-\omega}{\gamma_1+\gamma_2}}(\tau_2+1,\ \tau_1+1).$$

But the sum of the two incomplete B-function ratios is unity. Accordingly the characteristic function*

$$F(\omega)=\frac{1}{\left(1-\frac{\omega}{\gamma_2}\right)^{\tau_2+1}\left(1+\frac{\omega}{\gamma_1}\right)^{\tau_1+1}}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{xxiv}).$$

Now the logarithm of the characteristic function is given by

$$\log F(\omega)=\frac{\lambda_1\omega}{1!}+\frac{\lambda_2\omega^2}{2!}+\ldots+\frac{\lambda_s\omega^s}{s!}+\ldots\ \ \ldots\ldots\ldots\ldots\ldots(\text{xxv}),$$

where $\lambda_1, \lambda_2, \ldots \lambda_s \ldots$ are the semi-invariants. Thus

$$\begin{aligned}\frac{\lambda_1\omega}{1!}&+\frac{\lambda_2\omega^2}{2!}+\ldots+\frac{\lambda_s\omega^s}{s!}+\ldots\\
&=-(\tau_2+1)\log\left(1-\frac{\omega}{\gamma_2}\right)-(\tau_1+1)\log\left(1+\frac{\omega}{\gamma_1}\right)\\
&=(\tau_2+1)\left(\frac{\omega}{\gamma_2}+\frac{1}{2}\frac{\omega^2}{\gamma_2^2}+\frac{1}{3}\frac{\omega^3}{\gamma_2^3}+\ldots+\frac{1}{s}\frac{\omega^s}{\gamma_2^s}+\ldots\right)\\
&\qquad-(\tau_1+1)\left(\frac{\omega}{\gamma_1}-\frac{1}{2}\frac{\omega^2}{\gamma_1^2}+\frac{1}{3}\frac{\omega^3}{\gamma_1^3}+\ldots+\frac{(-1)^{s-1}}{s}\frac{\omega^s}{\gamma_1^s}+\ldots\right).\end{aligned}$$

Equating powers of ω we have

$$\left.\begin{aligned}\lambda_1&=\mu_1'=\frac{\tau_2+1}{\gamma_2}-\frac{\tau_1+1}{\gamma_1}\\
\lambda_2&=\mu_2=1\left(\frac{\tau_2+1}{\gamma_2^2}+\frac{\tau_1+1}{\gamma_1^2}\right)\\
\lambda_3&=\mu_3=1\,.\,2\left(\frac{\tau_2+1}{\gamma_2^3}-\frac{\tau_1+1}{\gamma_1^3}\right)\\
\lambda_4&=\mu_4-3\mu_2^2=1\,.\,2\,.\,3\left(\frac{\tau_2+1}{\gamma_2^4}+\frac{\tau_1+1}{\gamma_1^4}\right)\\
\lambda_5&=\mu_5-10\mu_2\mu_3=1\,.\,2\,.\,3\,.\,4\left(\frac{\tau_2+1}{\gamma_2^5}-\frac{\tau_1+1}{\gamma_1^5}\right)\end{aligned}\right\}\ \ldots\ldots\ldots(\text{xxvi}),$$

and so on.

* The area M of the curve must be made unity to obtain the characteristic function.

The first three semi-invariants check with the values found directly for the first three moments.

The above equations suffice to give the mean, standard deviation and β_1, β_2 for the distribution of the difference of any two statistical coefficients, satisfying equations of Type III, when these coefficients are measured from the start of their distributions.

(iv) The previous investigation will not only have made it clear that we are dealing with a function closely allied to the $K_m(x)$ Bessel Function in a generalised form, but also that it is desirable to work out some of its properties. We may write our curve in the form

$$z = z_0' e^{-\frac{1}{2}(\gamma_2-\gamma_1)Y} Y^{\frac{1}{2}(\tau_1+\tau_2+1)} \mathcal{K}_{\tau_1,\tau_2}(\tfrac{1}{2}(\gamma_1+\gamma_2)Y) \ldots\ldots\ldots\ldots(\text{xxvii}),$$

where, if $\nu = \frac{1}{2}(\tau_1+\tau_2+1)$,

$$\mathcal{K}_{\tau_1,\tau_2}(\tfrac{1}{2}(\gamma_1+\gamma_2)Y) = c_0(\tfrac{1}{2}(\gamma_1+\gamma_2)Y)^\nu \int_1^\infty e^{-\frac{1}{2}(\gamma_1+\gamma_2)Yt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$

(see Equation (v)), c_0 being a constant.

Thus
$$\mathcal{K}_{\tau_1,\tau_2}(x) = c_0 x^\nu \int_1^\infty e^{xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt.$$

If we choose c_0 so that when $\tau_1=\tau_2$ we have $\mathcal{K}_{\tau_1,\tau_2}(x) = K_\nu(x)$, where ν is now $\tau_1+\frac{1}{2}$, we see that our $\mathcal{K}_{\tau_1,\tau_2}(x)$ will pass over into the Bessel Function $K_\nu(x)$; this requires us to take $c_0 = \sqrt{\pi}/(2^\nu\Gamma(\nu+\frac{1}{2}))$. Thus our curve becomes

$$z = \frac{1}{2}\frac{M(\gamma_1+\gamma_2)}{\Gamma(\tau_1+1)\Gamma(\tau_2+1)}\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)^{\tau_1+1}\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)^{\tau_2+1}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})}e^{-\rho\frac{1}{2}(\gamma_1+\gamma_2)Y}$$
$$\times(\tfrac{1}{2}(\gamma_1+\gamma_2)Y)^\nu \mathcal{K}_{\tau_1,\tau_2}(\tfrac{1}{2}(\gamma_1+\gamma_2)Y)\ldots\ldots\ldots(\text{xxviii}),$$

where $\rho = (\gamma_2-\gamma_1)/(\gamma_1+\gamma_2)$.

I propose to call

$$\mathcal{K}_{\tau_1,\tau_2}(x) = \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}x^\nu\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt \quad (\nu=\tfrac{1}{2}(\tau_1+\tau_2+1)) \ldots\ldots\ldots(\text{xxix})$$

the Double Bessel Function*, and to study some of its properties. We may first find the differential equation which it satisfies.

Write for brevity $u = \mathcal{K}_{\tau_1,\tau_2}(x)$, and differentiate twice,

$$\frac{du}{dx} = \frac{\nu u}{x} - x^\nu\int_1^\infty e^{-xt}t(t-1)^{\tau_1}(t+1)^{\tau_2}dt,$$

$$\frac{d^2u}{dx^2} = \frac{\nu(\nu-1)u}{x^2} - 2\nu x^{\nu-1}\int_1^\infty e^{-xt}t(t-1)^{\tau_1}(t+1)^{\tau_2}dt + x^\nu\int_1^\infty e^{-xt}t^2(t-1)^{\tau_1}(t+1)^{\tau_2}dt.$$

Accordingly

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} = \nu^2 u + x^{\nu+1}\int_1^\infty e^{-xt}\{x(t^2-1)-(2\nu+1)t\}(t-1)^{\tau_1}(t+1)^{\tau_2}dt + x^2 u.$$

* Of course only a Double Bessel Function of the second order and imaginary argument.

Now

$$\int_1^\infty e^{-xt}x(t^2-1)(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$

$$=\int_1^\infty \frac{d(-e^{-xt})}{dt}(t-1)^{\tau_1+1}(t+1)^{\tau_2+1}dt$$

$$=\int_1^\infty e^{-xt}\{(\tau_1+1)(t+1)+(\tau_2+1)(t-1)\}(t-1)^{\tau_1}(t+1)^{\tau_2}dt,$$

since the term between limits vanishes,

$$=\int_1^\infty e^{-xt}t(2\nu+1)(t-1)^{\tau_1}(t+1)^{\tau_2}dt+(\tau_1-\tau_2)\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt.$$

Substituting we have*

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}-u(\nu^2+(\tau_1-\tau_2)x+x^2)=0\qquad\text{(xxx)}.$$

This is the differential equation satisfied by $\mathcal{K}_{\tau_1,\tau_2}(x)$. It may also be written

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}-u\{(\tau_1+\tfrac{1}{2})(\tau_2+\tfrac{1}{2})+(x+\tfrac{1}{2}(\tau_1-\tau_2))^2\}=0\quad\text{...(xxx }bis\text{)},$$

which shows its relation to $K_{\tau+\frac{1}{2}}(x)$ when $\tau_1=\tau_2$, for $K_{\tau+\frac{1}{2}}$ satisfies the equation

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}-u\{(\tau+\tfrac{1}{2})^2+x^2\}=0.$$

(v) We next turn to the recurrence formulae. Consider the expression

$$I=\int_1^\infty e^{-xt}\frac{d}{dt}\{(t-1)^{\tau_1}(t+1)^{\tau_2}\}dt.$$

Integrating by parts, since the part between limits vanishes, we have

$$I=x\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt.$$

Differentiating out,

$$I=\int_1^\infty e^{-xt}\{\tau_1(t+1)+\tau_2(t-1)\}(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt$$

$$=(2\nu-1)\int_1^\infty e^{-xt}t(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt+(\tau_1-\tau_2)\int_1^\infty e^{-xt}(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt.$$

Accordingly

$$\int_1^\infty e^{-xt}t(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt$$

$$=\frac{x}{2\nu-1}\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt-\frac{\tau_1-\tau_2}{2\nu-1}\int_1^\infty e^{-xt}(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt\qquad\text{(xxxi)}.$$

Multiply both sides by $\sqrt{\pi}x^{\nu-1}/(2^{\nu-1}\Gamma(\nu-\frac{1}{2}))$ and express the right-hand side in terms of the $\mathcal{K}_{\tau_1,\tau_2}$ functions. We have

$$\frac{\sqrt{\pi}x^{\nu-1}}{2^{\nu-1}\Gamma(\nu-\frac{1}{2})}\int_1^\infty e^{-xt}t(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt$$

$$=\mathcal{K}_{\tau_1,\tau_2}(x)-\frac{\tau_1-\tau_2}{2\nu-1}\mathcal{K}_{\tau_1-1,\tau_2-1}\qquad\text{(xxxii)}.$$

* The other solution of this equation is the Double Bessel Function $I_{\tau_1,\tau_2}(x)$.

The left-hand side may also be expressed in terms of $\mathcal{K}_{\tau_1-1,\tau_2}(x)$ and $\mathcal{K}_{\tau_1,\tau_2-1}(x)$ as

$$\frac{\Gamma(\nu)}{\Gamma(\nu-\frac{1}{2})}\frac{1}{\sqrt{2x}}(\mathcal{K}_{\tau_1-1,\tau_2}(x)+\mathcal{K}_{\tau_1,\tau_2-1}(x)),$$

so that we have

$$\mathcal{K}_{\tau_1,\tau_2}(x)=\frac{\Gamma(\frac{1}{2}(\tau_1+\tau_2+1))}{\Gamma(\frac{1}{2}(\tau_1+\tau_2))}\frac{1}{\sqrt{2x}}(\mathcal{K}_{\tau_1-1,\tau_2}(x)+\mathcal{K}_{\tau_1,\tau_2-1}(x))$$
$$+\frac{\tau_1-\tau_2}{\tau_1+\tau_2}\mathcal{K}_{\tau_1-1,\tau_2-1}(x) \quad \text{.........(xxxiii)},$$

from which a table of $\mathcal{K}_{\tau_1,\tau_2}(x)$ could be fairly readily computed.

Next consider $\mathcal{K}_{\tau_1+1,\tau_2+1}(x)$; we have

$$\mathcal{K}_{\tau_1+1,\tau_2+1}(x)=\frac{\sqrt{\pi}\,x^{\nu+1}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_1^\infty e^{-xt}(t-1)^{\tau_1+1}(t+1)^{\tau_2+1}dt$$
$$=\frac{\sqrt{\pi}\,x^{\nu}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_1^\infty \frac{d}{dt}(-e^{-xt})(t-1)^{\tau_1+1}(t+1)^{\tau_2+1}dt,$$

or integrating by parts,

$$=\frac{\sqrt{\pi}\,x^{\nu}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\int_1^\infty e^{-xt}\{(\tau_1+1)(t+1)+(\tau_2+1)(t-1)\}(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$
$$=\frac{\sqrt{\pi}\,x^{\nu}}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}\left[(2\nu+1)\int_1^\infty e^{-xt}t(t-1)^{\tau_1}(t+1)^{\tau_2}dt\right.$$
$$\left.+(\tau_1-\tau_2)\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt\right]$$
$$=\frac{\sqrt{\pi}\,x^{\nu}}{2^{\nu}\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-xt}t(t-1)^{\tau_1}(t+1)^{\tau_2}dt+\frac{\tau_1-\tau_2}{2\nu+1}\mathcal{K}_{\tau_1,\tau_2}(x) \quad \text{.........(xxxiv)}.$$

Again

$$x\int_1^\infty e^{-xt}t(t-1)^{\tau_1}(t+1)^{\tau_2}dt=-\int_1^\infty \frac{d(e^{-xt})}{dt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$
$$=\int_1^\infty e^{-xt}[(t-1)^{\tau_1}(t+1)^{\tau_2}+\{\tau_1 t(t+1)+\tau_2 t(t-1)\}(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}]dt$$
$$=\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt+(2\nu-1)\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt$$
$$+(\tau_1-\tau_2)\int_1^\infty e^{-xt}t(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt$$
$$+(2\nu-1)\int_1^\infty e^{-xt}(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt.$$

Multiply both sides of this equation by $\dfrac{\sqrt{\pi}\,x^{\nu-1}}{2^\nu\Gamma(\nu+\frac{1}{2})}$ and we have

$$\frac{\sqrt{\pi}\,x^{\nu}}{2^{\nu}\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-xt}t(t-1)^{\tau_1}(t+1)^{\tau_2}dt=\frac{2\nu}{x}\mathcal{K}_{\tau_1,\tau_2}(x)$$
$$+\frac{(\tau_1-\tau_2)}{2\nu-1}\frac{\sqrt{\pi}\,x^{\nu-1}}{2^{\nu-1}\Gamma(\nu-\frac{1}{2})}\int_1^\infty e^{-xt}t(t-1)^{\tau_1-1}(t+1)^{\tau_2-1}dt+\mathcal{K}_{\tau_1-1,\tau_2-1}(x),$$

and thus by (xxxii),

$$= \frac{2\nu}{x} \mathfrak{K}_{\tau_1, \tau_2}(x) + \frac{\tau_1 - \tau_2}{2\nu - 1} \mathfrak{K}_{\tau_1, \tau_2}(x) - \frac{(\tau_1 - \tau_2)^2}{(2\nu - 1)^2} \mathfrak{K}_{\tau_1 - 1, \tau_2 - 1}(x) + \mathfrak{K}_{\tau_1 - 1, \tau_2 - 1}(x) \quad \text{.........(xxxv).}$$

Hence substituting (xxxv) in (xxxiv),

$$\mathfrak{K}_{\tau_1+1, \tau_2+1}(x) = \mathfrak{K}_{\tau_1-1, \tau_2-1}(x) + \left(\frac{2\nu}{x} + \frac{\tau_1 - \tau_2}{2\nu - 1}\right) \mathfrak{K}_{\tau_1, \tau_2}(x) - \left(\frac{\tau_1 - \tau_2}{2\nu - 1}\right)^2 \mathfrak{K}_{\tau_1-1, \tau_2-1}(x) + \frac{\tau_1 - \tau_2}{2\nu + 1} \mathfrak{K}_{\tau_1, \tau_2}(x).$$

Or finally,

$$\mathfrak{K}_{\tau_1+1, \tau_2+1}(x) - \mathfrak{K}_{\tau_1-1, \tau_2-1}(x) = \frac{2\nu}{x}\left(1 + \frac{2x(\tau_1 - \tau_2)}{4\nu^2 - 1}\right) \mathfrak{K}_{\tau_1, \tau_2}(x) - \left(\frac{\tau_1 - \tau_2}{2\nu - 1}\right)^2 \mathfrak{K}_{\tau_1-1, \tau_2-1}(x) \quad \text{.........(xxxvi).}$$

This formula reduces, when $\tau_1 = \tau_2$, to the familiar and much simpler one

$$K_{\nu+1}(x) - K_{\nu-1}(x) = \frac{2\nu}{x} K_\nu(x),$$

for the Bessel Function of second order and imaginary argument*. By means of (xxxvi) we express $\mathfrak{K}_{\tau_1+1, \tau_2+1}(x)$ in terms of $\mathfrak{K}_{\tau_1, \tau_2}(x)$ and $\mathfrak{K}_{\tau_1-1, \tau_2-1}(x)$. We shall find it convenient to use the symbol $T_{\tau_1, \tau_2}(x)$ in the following manner:

$$T_{\tau_1, \tau_2}(x) = \frac{1}{\sqrt{\pi}} \frac{1}{2^\nu} \frac{1}{\Gamma(\nu + \frac{1}{2})} x^\nu \mathfrak{K}_{\tau_1, \tau_2}(x) \quad \text{..............(xxxvii).}$$

Then by (xxvii) if $Y' = \frac{1}{2}(\gamma_1 + \gamma_2) Y$ and $\rho = (\gamma_2 - \gamma_1)/(\gamma_1 + \gamma_2)$,

$$z = z_0 e^{-\rho Y'} \left(\frac{Y'}{\frac{1}{2}(\gamma_1 + \gamma_2)}\right)^{2\nu} \int_1^\infty e^{-Y't}(t-1)^{\tau_1}(t+1)^{\tau_2}\,dt,$$

and by (iv),

$$z = \frac{1}{2} M (\gamma_1 + \gamma_2) \frac{\left(\frac{\gamma_1}{\gamma_1 + \gamma_2}\right)^{\tau_1+1} \left(\frac{\gamma_2}{\gamma_1 + \gamma_2}\right)^{\tau_2+1}}{\Gamma(\tau_1 + 1)\Gamma(\tau_2 + 1)} e^{-\rho Y'} Y'^\nu \left(Y'^\nu \int_1^\infty e^{-Y't}(t-1)^{\tau_1}(t+1)^{\tau_2}\,dt\right)$$

$$= \frac{1}{2} M (\gamma_1 + \gamma_2) \left(\frac{\gamma_1}{\gamma_1 + \gamma_2}\right)^{\tau_1+1} \left(\frac{\gamma_2}{\gamma_1 + \gamma_2}\right)^{\tau_2+1} \frac{e^{-\rho Y'}\Gamma(\nu + \frac{1}{2}) 2^\nu}{\Gamma(\tau_1 + 1)\Gamma(\tau_2 + 1)\sqrt{\pi}} Y'^\nu \mathfrak{K}_{\tau_1, \tau_2}(Y')$$

$$= \frac{1}{2} M (\gamma_1 + \gamma_2) \frac{(1+\rho)^{\tau_1+1}(1-\rho)^{\tau_2+1}}{\sqrt{\pi}\,\Gamma(\tau_1 + 1)\Gamma(\tau_2 + 1)} \frac{\Gamma(\nu + \frac{1}{2}) e^{-\rho Y'}}{2^{\nu+1}} Y'^\nu \mathfrak{K}_{\tau_1, \tau_2}(Y') \quad \text{...(xxxviii),}$$

or by (xxxvii),

$$z = M \tfrac{1}{2}(\gamma_1 + \gamma_2) \frac{(1+\rho)^{\tau_1+1}(1-\rho)^{\tau_2+1}}{\Gamma(\tau_1 + 1)\Gamma(\tau_2 + 1)} \Gamma^2(\nu + \tfrac{1}{2}) e^{-\rho Y'} T_{\tau_1, \tau_2}(Y') \quad \text{(xxxix).}$$

The element of frequency is

$$z\,dY = z\,dY'/\tfrac{1}{2}(\gamma_1 + \gamma_2).$$

Hence

$$z\,dY = M(1+\rho)^{\tau_1+1}(1-\rho)^{\tau_2+1} \frac{\Gamma^2(\nu + \frac{1}{2})}{\Gamma(\tau_1 + 1)\Gamma(\tau_2 + 1)} e^{-\rho Y'} T_{\tau_1, \tau_2}(Y')\,dY' \quad \text{...(xl).}$$

Putting $\tau_1 = \tau_2 = \nu - \frac{1}{2}$ we have

$$z\,dY = M(1 - \rho^2)^{\nu+\frac{1}{2}} e^{-\rho Y'} T_\nu(Y')\,dY' \quad \text{.................(xl \textit{bis}),}$$

* Watson, *Theory of Bessel Functions*, p. 79.

which agrees with the value given in *Biometrika*, Vol. XXI, p. 183, Equation (xli), if we remember that we are here dealing with one-half the full curve, and that ρ may be either positive or negative. Sometimes one and sometimes the other of the two expressions $\mathcal{K}_{\tau_1,\tau_2}(Y')$ and $T_{\tau_1,\tau_2}(Y')$ may be the more serviceable.

Writing (xxxvi) in the form

$$\mathcal{K}_{\tau_1+1,\tau_2+1}(x) = \frac{2\nu}{x}\left(1 + \frac{2x(\tau_1-\tau_2)}{4\nu^2-1}\right)\mathcal{K}_{\tau_1,\tau_2}(x) + \left(1 - \left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)\mathcal{K}_{\tau_1-1,\tau_2-1}(x),$$

we may replace the $\mathcal{K}_{\tau_1,\tau_2}$ by the T_{τ_1,τ_2} functions and find

$$T_{\tau_1+1,\tau_2+1}(x) = \frac{2\nu}{2\nu+1}\left(1 + \frac{2x(\tau_1-\tau_2)}{4\nu^2-1}\right)T_{\tau_1,\tau_2}(x) + \frac{x^2}{4\nu^2-1}\left(1 - \left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)T_{\tau_1-1,\tau_2-1}(x) \quad \text{.........(xli).}$$

When we make $\tau_1 = \tau_2 = \nu - \frac{1}{2}$, this becomes

$$T_{\nu+1}(x) = \frac{2\nu}{2\nu+1}T_\nu(x) + \frac{x^2}{4\nu^2-1}T_{\nu-1}(x),$$

and is identical with the formula (xliv) on p. 184 of *Biometrika*, Vol. XXI.

(vi) In the next place we may consider the differential coefficients of our functions

$$\frac{d}{dx}(x^\nu\mathcal{K}_{\tau_1,\tau_2}(x)) = 2\nu x^{\nu-1}\mathcal{K}_{\tau_1,\tau_2}(x) - \frac{\sqrt{\pi}\,x^{2\nu}}{2^\nu\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-xt}\,t\,(t-1)^{\tau_1}(t+1)^{\tau_2}\,dt,$$

and thus by (xxxv),

$$= 2\nu x^{\nu-1}\mathcal{K}_{\tau_1,\tau_2}(x) - x^\nu\left(\frac{2\nu}{x}\mathcal{K}_{\tau_1,\tau_2}(x) + \mathcal{K}_{\tau_1-1,\tau_2-1}(x) + \frac{\tau_1-\tau_2}{2\nu-1}\mathcal{K}_{\tau_1,\tau_2}(x) - \left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\mathcal{K}_{\tau_1-1,\tau_2-1}(x)\right)$$

$$= -x\,.\,x^{\nu-1}\mathcal{K}_{\tau_1-1,\tau_2-1}(x)\left(1 - \left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right) - \frac{\tau_1-\tau_2}{2\nu-1}x^\nu\mathcal{K}_{\tau_1,\tau_2}(x).$$

Thus

$$\frac{1}{x}\frac{d}{dx}(x^\nu\mathcal{K}_{\tau_1,\tau_2}(x)) = -\left(1 - \left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)x^{\nu-1}\mathcal{K}_{\tau_1-1,\tau_2-1}(x) - \frac{\tau_1-\tau_2}{(2\nu-1)x}x^\nu\mathcal{K}_{\tau_1,\tau_2}(x) \quad \text{.........(xlii).}$$

This reduces to the familiar relation

$$\frac{1}{x}\frac{d}{dx}(x^\nu K_\nu(x)) = -x^{\nu-1}K_{\nu-1}(x)$$

of the Bessel K-function, when $\tau_1 = \tau_2 = \nu - \frac{1}{2}$*. The corresponding result in terms of $T_{\tau_1,\tau_2}(x)$ is

$$\frac{d}{dx}(T_{\tau_1,\tau_2}(x)) = -\frac{x}{2\nu-1}\left(1 - \left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)T_{\tau_1-1,\tau_2-1}(x) - \frac{\tau_1-\tau_2}{2\nu-1}T_{\tau_1,\tau_2}(x) \quad \text{.........(xliii).}$$

If $\tau_1 = \tau_2$, then

$$\frac{d}{dx}(T_\nu(x)) = -\frac{x}{2\nu-1}T_{\nu-1}(x).$$

* Watson, *Theory of Bessel Functions*, p. 79, Equation (5), with $m=1$.

We may put (xliii) by aid of (xli) in the form

$$T_{\tau_1+1,\tau_2+1}(x)=\left(\frac{2\nu}{2\nu+1}+\frac{\tau_1-\tau_2}{(2\nu+1)^2}x\right)T_{\tau_1,\tau_2}(x)-\frac{x}{2\nu+1}\frac{dT_{\tau_1,\tau_2}(x)}{dx}$$
........(xliii *bis*),

whence if $\tau_1=\tau_2$ the result

$$T_{\nu+1}(x)=\frac{2\nu}{2\nu+1}T_\nu(x)-\frac{x}{2\nu+1}\frac{dT_{\tau_1,\tau_2}(x)}{dx}$$

of *Biometrika*, Vol. XXIV. p. 309, Equation (1) directly follows.

(vii) These results enable us to determine the equation for the mode of the curve

$$z=z_0'e^{-\rho Y'}T_{\tau_1,\tau_2}(Y'),$$

where
$$Y'=\tfrac{1}{2}(\gamma_1+\gamma_2)Y.$$

We must put dz/dY or $dz/dY'=0$, whence we obtain, if $\tilde{Y}'$ correspond to the mode, i.e. $=\frac{1}{2}(\gamma_1+\gamma_2)\tilde{Y}$, where $\tilde{Y}$ is the variate, $v-u$, at mode,

$$-\rho+\frac{1}{T_{\tau_1,\tau_2}(\tilde{Y}')}\left\{\frac{d}{dY'}(T_{\tau_1,\tau_2}(Y'))\right\}_{Y'=\tilde{Y}'}=0,$$

or, by (xliii), since $\nu=\frac{1}{2}(\tau_1+\tau_2+1)$,

$$\rho=-\frac{\tau_1-\tau_2}{\tau_1+\tau_2}-\frac{\tilde{Y}'}{\tau_1+\tau_2}\left\{1-\left(\frac{\tau_1-\tau_2}{\tau_1+\tau_2}\right)^2\right\}\frac{T_{\tau_1-1,\tau_2-1}(\tilde{Y}')}{T_{\tau_1,\tau_2}(\tilde{Y}')}\text{ ....(xliv).}$$

If we take $\tau_1=\tau_2$, this agrees with a somewhat different notation (ρ negative), with the result

$$\rho=-\frac{\tilde{Y}'}{2\nu-1}\frac{T_{\nu-1}(\tilde{Y}')}{T_\nu(\tilde{Y}')}$$

of *Biometrika*, Vol. XXI. p. 184, Equation (xlv).

The simplest form of the result is in the $\mathcal{K}_{\tau_1,\tau_2}(x)$ functions, namely

$$\rho=-\frac{\tau_1-\tau_2}{\tau_1+\tau_2}-\frac{4\tau_1\tau_2}{(\tau_1+\tau_2)^2}\frac{\mathcal{K}_{\tau_1-1,\tau_2-1}(\tilde{Y}')}{\mathcal{K}_{\tau_1,\tau_2}(\tilde{Y}')}\text{ ....(xliv \textit{bis}).}$$

Clearly if a Table be formed of $\mathcal{K}_{\tau_1,\tau_2}(x)$ or $T_{\tau_1,\tau_2}(x)$, we can for every entry of τ_1, τ_2 compute the right-hand side of (xliv) or (xliv *bis*), and then by backward interpolation obtain ρ*. Since ρ always lies between $+1$ and -1, γ_1 and γ_2 being (see Equation (i)) positive, we may have to find $\tilde{Y}'$ either from (xliv *bis*), or from the equation

$$\rho=-\frac{\tau_2-\tau_1}{\tau_2+\tau_1}-\frac{4\tau_1\tau_2}{(\tau_1+\tau_2)^2}\frac{\mathcal{K}_{\tau_2-1,\tau_1-1}(\tilde{Y}')}{\mathcal{K}_{\tau_2,\tau_1}(\tilde{Y}')}\text{ ....(xliv \textit{ter}),}$$

as the case may be.

(viii) *Tables to be computed.* Before it is possible to illustrate the statistical value of the Double Bessel Function, it is needful for tables of it to be computed Such tables will be calculated for the arguments τ_1 and τ_2 proceeding by 0·5. In

* The auxiliary Table thus formed will correspond with the columns entitled "ρ" on pp. 195—201 of *Biometrika*, Vol. XXI.

order to construct such tables by aid of the recurrence formulae of this paper we need to compute fourteen primary values of $T_{\tau_1,\tau_2}(x)$ or $\mathfrak{K}_{\tau_1,\tau_2}(x)$. The reader may be reminded of the following fundamental relations, in which $\nu=\frac{1}{2}(\tau_1+\tau_2+1)$:

$$T_{\tau_1,\tau_2}(x)=\frac{1}{2^{2\nu}}\frac{x^{2\nu}}{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+1)}\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt,$$

$$T_{\tau_1,\tau_2}(x)=\frac{1}{\sqrt{\pi}}\frac{x^{\nu}}{2^{\nu}\Gamma(\frac{1}{2}(\tau_1+\tau_2)+1)}\mathfrak{K}_{\tau_1,\tau_2}(x), \text{ see Equation (xxxvii)},$$

$$\mathfrak{K}_{\tau_1,\tau_2}(x)=\sqrt{\pi}\frac{x^{\nu}}{2^{\nu}\Gamma(\frac{1}{2}(\tau_1+\tau_2)+1)}\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt,$$

see Equation (xxix).

When $\tau_1=\tau_2$, $\nu=\tau_1+\frac{1}{2}$ and $T_{\tau_1,\tau_2}(x)$ becomes $T_{\tau_1+\frac{1}{2}}(x)$, the single T-function, where

$$T_{\tau_1+\frac{1}{2}}(x)=\frac{1}{\sqrt{\pi}}\frac{x^{\tau_1+\frac{1}{2}}}{2^{\tau_1+\frac{1}{2}}\Gamma(\tau_1+1)}K_{\tau_1+\frac{1}{2}}(x),$$

and $K_{\tau_1+\frac{1}{2}}(x)$ is the Single Bessel K-function.

(1) $T_{-\frac{1}{2},-\frac{1}{2}}(x)$.

$$T_{-\frac{1}{2},-\frac{1}{2}}(x)=\frac{1}{\sqrt{\pi}}\frac{1}{\Gamma(\frac{1}{2})}K_0(x)=\frac{1}{\pi}K_0(x).$$

$K_0(x)$ will be found tabled to 21 figures by Aldis in the *R. Soc. Proc.* Vol. LXIV. pp. 219—221.

(2) $T_{-\frac{1}{2},0}(x)$.

$$T_{-\frac{1}{2},0}(x)=\frac{1}{\sqrt{2}}\frac{1}{\Gamma^2(\frac{3}{4})}\sqrt{x}\int_1^\infty e^{-xt}(t-1)^{-\frac{1}{2}}dt.$$

Take $z^2=t-1$, and we have

$$T_{-\frac{1}{2},0}(x)=\sqrt{2}\frac{1}{\Gamma^2(\frac{3}{4})}\sqrt{x}e^{-x}\int_0^\infty e^{-xz^2}dz$$

$$=\sqrt{\pi}\frac{1}{\Gamma^2(\frac{3}{4})}\sqrt{x}e^{-x}\sqrt{\frac{1}{2x}},$$

or,

$$T_{-\frac{1}{2},0}(x)=\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma^2(\frac{3}{4})}e^{-x}.$$

(3) $T_{0,-\frac{1}{2}}(x)$.

$$T_{0,-\frac{1}{2}}(x)=\frac{1}{\sqrt{2}}\frac{1}{\Gamma^2(\frac{3}{4})}\sqrt{x}\int_1^\infty e^{-xt}(t+1)^{-\frac{1}{2}}dt.$$

Put $x(t+1)=\frac{1}{2}u^2$, and we have

$$T_{0,-\frac{1}{2}}(x)=\frac{\sqrt{2\pi}}{\Gamma^2(\frac{3}{4})}e^{x}\int_{2\sqrt{x}}^\infty\frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}du$$

$$=\frac{\sqrt{2\pi}}{\Gamma^2(\frac{3}{4})}e^{x}\tfrac{1}{2}(1-\alpha_{2\sqrt{x}}).$$

Here $\frac{1}{2}(1-\alpha_{2\sqrt{x}})$ may be taken from the probability integral table.

(4) $T_{0,0}(x)$.

$$T_{0,0}(x) = \frac{1}{\sqrt{2\pi}} \sqrt{x}\, \mathfrak{K}_{0,0}(x) = \frac{1}{\sqrt{2\pi}} \sqrt{x} K_{\frac{1}{2}}(x)$$

$$= \tfrac{1}{2} x \int_1^\infty e^{-xt}\, dt$$

$$= \tfrac{1}{2} e^{-x}.$$

(5) $T_{\frac{1}{2},0}(x)$.

$$T_{\frac{1}{2},0}(x) = \frac{1}{2\sqrt{2}} \frac{1}{\Gamma^2(\frac{5}{4})} x^{\frac{3}{2}} \int_1^\infty e^{-xt} (t-1)^{\frac{1}{2}}\, dt$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\Gamma^2(\frac{5}{4})} x^{\frac{3}{2}} e^{-x} \int_1^\infty e^{-xz^2} z^2\, dz,$$

if
$$t - 1 = z^2.$$

Hence
$$T_{\frac{1}{2},0}(x) = \frac{2\sqrt{2\pi}\, e^{-x}}{\Gamma^2(\frac{1}{4})}.$$

(6) $T_{0,\frac{1}{2}}(x)$.

$$T_{0,\frac{1}{2}}(x) = \frac{1}{2\sqrt{2}} \frac{x^{\frac{3}{2}}}{\Gamma^2(\frac{5}{4})} \int_1^\infty e^{-xt} (t+1)^{\frac{1}{2}}\, dt,$$

or, taking $x(t+1) = \frac{1}{2}u^2$, we have

$$T_{0,\frac{1}{2}}(x) = \frac{4\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})} e^x \int_{2\sqrt{x}}^\infty \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}} u^2\, du,$$

whence integrating by parts,

$$T_{0,\frac{1}{2}}(x) = \frac{4}{\Gamma^2(\frac{1}{4})} \{2e^{-x}\sqrt{x} + \sqrt{2\pi}\, \tfrac{1}{2}(1 - \alpha_{2\sqrt{x}})\}.$$

(7) We may note two formulae here which may be of service:

$$T_{0,\tau_2}(x) = \frac{1}{2^{\tau_2+1}} \frac{x^{\tau_2+1}}{\Gamma^2(\frac{1}{2}\tau_2+1)} \int_1^\infty e^{-xt} (t+1)^{\tau_2}\, dt,$$

and putting $x(t+1) = u$, we have

$$T_{0,\tau_2}(x) = \frac{1}{2^{\tau_2+1}} \frac{e^x}{\Gamma^2(\frac{1}{2}\tau_2+1)} \int_{2x}^\infty e^{-u} u^{\tau_2}\, du \quad \text{..............(xlv)},$$

which throws back the computing of $T_{0,\tau_2}(x)$ on the *Tables of the Incomplete* Γ-*Function*.

(8) Again

$$T_{\tau_1,0}(x) = \frac{1}{2^{\tau_1+1}} \frac{x^{\tau_1+1}}{\Gamma^2(\frac{1}{2}\tau_1+1)} e^{-x} \int_1^\infty e^{-x(t-1)} (t-1)^{\tau_1}\, dt$$

$$= \frac{1}{2^{\tau_1+1}} \frac{1}{\Gamma^2(\frac{1}{2}\tau_1+1)} e^{-x} \int_0^\infty e^{-u} u^{\tau_1}\, du$$

$$= \frac{1}{2^{\tau_1+1}} \frac{\Gamma(\tau_1+1)}{\Gamma^2(\frac{1}{2}\tau_1+1)} e^{-x} \quad \text{.................................(xlvi)}.$$

Thus $T_{\tau_1,0}(x)$ may be easily computed by aid of tables of the complete Γ-function and Newman and Glaisher's Tables of the Exponential. Putting in (xlvi) $\tau_1=0$, we have

$$T_{0,0}(x)=\tfrac{1}{2}e^{-x}, \text{ as in (1).}$$

Putting in (xlv) $\tau_2=0$, we have the same result. Again putting $\tau_1=\frac{1}{2}$, we find

$$T_{\frac{1}{2},0}(x)=\frac{1}{2\sqrt{2}}\frac{\Gamma(\frac{3}{2})}{\Gamma^2(\frac{5}{4})}e^{-x}=\frac{2\sqrt{2\pi}\,e^{-x}}{\Gamma^2(\frac{1}{4})},$$

which agrees with (5). Once more putting $\tau_1=-\frac{1}{2}$, we have

$$T_{-\frac{1}{2},0}(x)=\frac{1}{\sqrt{2}}\frac{\Gamma(\frac{1}{2})}{\Gamma^2(\frac{3}{4})}e^{-x}=\sqrt{\frac{\pi}{2}}\frac{1}{\Gamma^2(\frac{3}{4})}e^{-x},$$

which agrees with (2).

(9) $T_{-\frac{1}{2},\frac{1}{2}}(x)$.

$$T_{-\frac{1}{2},\frac{1}{2}}(x)=\tfrac{1}{2}x\int_1^\infty e^{-xt}(t-1)^{-\frac{1}{2}}(t+1)^{\frac{1}{2}}\,dt.$$

Now
$$K_0(x)=\int_1^\infty e^{-xt}(t-1)^{-\frac{1}{2}}(t+1)^{-\frac{1}{2}}\,dt,$$

or
$$e^{-x}K_0(x)=\int_1^\infty e^{-x(t+1)}(t-1)^{-\frac{1}{2}}(t+1)^{-\frac{1}{2}}\,dt.$$

Accordingly
$$\frac{d}{dx}(e^{-x}K_0(x))=-\int_1^\infty e^{-x(t+1)}(t-1)^{-\frac{1}{2}}(t+1)^{\frac{1}{2}}\,dt,$$

and
$$T_{-\frac{1}{2},\frac{1}{2}}(x)=-\tfrac{1}{2}xe^{x}\frac{d}{dx}(e^{-x}K_0(x))$$
$$=\tfrac{1}{2}x\left(K_0(x)-\frac{dK_0(x)}{dx}\right).$$

But*
$$\frac{dK_0(x)}{dx}=-K_1(x),$$

therefore
$$T_{-\frac{1}{2},\frac{1}{2}}(x)=\tfrac{1}{2}x(K_1(x)+K_0(x)).$$

(10) $T_{\frac{1}{2},-\frac{1}{2}}(x)$.

$$T_{\frac{1}{2},-\frac{1}{2}}(x)=\tfrac{1}{2}x\int_1^\infty e^{-xt}(t-1)^{\frac{1}{2}}(t+1)^{-\frac{1}{2}}\,dt.$$

Now
$$K_0(x)=e^{-x}\int_1^\infty e^{-x(t-1)}(t-1)^{-\frac{1}{2}}(t+1)^{-\frac{1}{2}}\,dt,$$

or
$$\frac{d}{dx}(K_0(x)e^{x})=-\int_1^\infty e^{-x(t-1)}(t-1)^{\frac{1}{2}}(t+1)^{-\frac{1}{2}}\,dt.$$

Thus
$$T_{\frac{1}{2},-\frac{1}{2}}(x)=-\frac{x}{2}e^{-x}\frac{d}{dx}(e^{x}K_0(x)),$$

or, finally,
$$T_{\frac{1}{2},-\frac{1}{2}}(x)=\tfrac{1}{2}x(K_1(x)-K_0(x)).$$

To find $T_{-\frac{1}{2},\frac{1}{2}}(x)$ and $T_{\frac{1}{2},-\frac{1}{2}}(x)$ we may use the values of $K_0(x)$ and $K_1(x)$ provided by Aldis in the *R. Soc. Proc.* Vol. LXIV. pp. 219—221.

* Watson, *Theory of Bessel Functions*, p. 79 (7).

(11) $T_{0,1}(x)$.

If τ_2 be a positive integer

$$T_{0,\tau_2}(x)=\frac{1}{2^{\tau_2+1}}\frac{x^{\tau_2+1}}{\Gamma^2(\frac{1}{2}\tau_2+1)}e^x\int_1^\infty e^{-x(t+1)}(t+1)^{\tau_2}dt$$

$$=\frac{(-1)^{\tau_2}}{2^{\tau_2+1}}\frac{x^{\tau_2+1}}{\Gamma^2(\frac{1}{2}\tau_2+1)}e^x\frac{d^{\tau_2}}{dx^{\tau_2}}\int_1^\infty e^{-x(t+1)}dt$$

$$=\frac{(-1)^{\tau_2}}{2^{\tau_2+1}}\frac{x^{\tau_2+1}}{\Gamma^2(\frac{1}{2}\tau_2+1)}e^x\frac{d^{\tau_2}}{dx^{\tau_2}}\left(\frac{e^{-2x}}{x}\right)$$

$$=\frac{1}{2^{\tau_2+1}}\frac{\Gamma(\tau_2+1)}{\Gamma^2(\frac{1}{2}\tau_2+1)}e^{-x}\left(1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\dots+\frac{(2x)^{\tau_2}}{\tau_2!}\right)\quad\dots\text{(xlvii)}.$$

If $\tau_2=1$,

$$T_{0,1}(x)=\frac{1}{\pi}e^{-x}(1+2x).$$

(12) $T_{1,0}(x)$.

We have at once from (xlvi)

$$T_{1,0}(x)=\frac{1}{\pi}e^{-x}.$$

(13) $T_{1,1}(x)$.

$$T_{1,1}(x)=T'_{\tau_1+\frac{1}{2}}(x)=T_{\frac{3}{2}}(x)=\tfrac{1}{4}e^{-x}(1+x).$$

See *Biometrika*, Vol. XXI, p. 183, ftn.

(14) $T_{\frac{1}{2},1}(x)$.

We will prove first a recurrence formula of some service in tabulating Double Bessel Functions.

$$T_{\tau_1,\tau_2+1}=\frac{1}{2^{\tau_1+\tau_2+2}}\frac{x^{\tau_1+\tau_2+2}}{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+\frac{3}{2})}\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}(t-1+2)\,dt$$

$$=\frac{1}{2^{\tau_1+\tau_2+2}}\frac{x^{\tau_1+\tau_2+2}}{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+\frac{3}{2})}$$

$$\times\left\{\int_1^\infty e^{-xt}(t-1)^{\tau_1+1}(t+1)^{\tau_2}dt+2\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}dt\right\},$$

or

$$T_{\tau_1,\tau_2+1}=T_{\tau_1+1,\tau_2}(x)+\frac{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+1)}{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+\frac{3}{2})}xT_{\tau_1,\tau_2}(x)\quad\dots\dots\text{(xlviii)}.$$

Put $\tau_1=\frac{1}{2}$, $\tau_2=0$, and we have

$$T_{\frac{1}{2},1}(x)=T_{\frac{3}{2},0}(x)+\frac{\Gamma^2(\frac{5}{4})}{\Gamma^2(\frac{7}{4})}xT_{\frac{1}{2},0}(x).$$

But by (8)

$$T_{\frac{3}{2},0}(x)=\frac{1}{4\sqrt{2}}\frac{\Gamma(\frac{5}{2})}{\Gamma^2(\frac{7}{4})}e^{-x},\qquad T_{\frac{1}{2},0}(x)=\frac{1}{2\sqrt{2}}\frac{\Gamma(\frac{3}{2})}{\Gamma^2(\frac{5}{4})}e^{-x}.$$

Accordingly

$$T_{\frac{1}{2},1}(x)=\frac{\sqrt{\pi}}{4\sqrt{2}\,\Gamma^2(\frac{7}{4})}(\tfrac{3}{4}+x)e^{-x}.$$

(15) $T_{1,\frac{1}{2}}(x)$.

We may rearrange (xlviii) in the form

$$T_{\tau_1+1,\tau_2}(x) = T_{\tau_1,\tau_2+1}(x) - \frac{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+1)}{\Gamma^2(\frac{1}{2}(\tau_1+\tau_2)+\frac{3}{2})}\, xT_{\tau_1,\tau_2}(x) \quad \ldots\ldots\text{(xlix)}.$$

Put $\tau_1 = 0$, $\tau_2 = \frac{1}{2}$, and we have

$$T_{1,\frac{1}{2}}(x) = T_{0,\frac{3}{2}}(x) - \frac{\Gamma^2(\frac{5}{4})}{\Gamma^2(\frac{7}{4})}\, xT_{0,\frac{1}{2}}(x).$$

But by (7)

$$\begin{aligned} T_{0,\frac{3}{2}}(x) &= \frac{1}{2^{\frac{3}{2}}}\frac{e^x}{\Gamma^2(\frac{7}{4})}\int_{2x}^{\infty} e^{-u} u^{\frac{3}{2}}\, du \\ &= \frac{1}{2^{\frac{3}{2}}}\frac{e^x}{\Gamma^2(\frac{7}{4})}\left(\int_0^{\infty} e^{-u} u^{\frac{3}{2}}\, du - \int_0^{2x} e^{-u} u^{\frac{3}{2}}\, du\right) \\ &= \frac{1}{2^{\frac{3}{2}}}\frac{e^x}{\Gamma^2(\frac{7}{4})}\,\Gamma(\tfrac{5}{2})\,(1 - I'_{2x}(\tfrac{5}{2})), \end{aligned}$$

where I' is the incomplete Γ-function ratio*.

Similarly $$T_{0,\frac{1}{2}}(x) = \frac{1}{2^{\frac{1}{2}}}\frac{\Gamma(\frac{3}{2})}{\Gamma^2(\frac{5}{4})}\, e^x (1 - I'_{2x}(\tfrac{3}{2})).$$

Thus $$T_{1,\frac{1}{2}}(x) = \frac{\sqrt{\pi}\, e^x}{4\sqrt{2}\,\Gamma^2(\frac{7}{4})}\,\{\tfrac{3}{4}(1 - I'_{2x}(\tfrac{5}{2})) - x(1 - I'_{2x}(\tfrac{3}{2}))\}.$$

The values of $I'_{2x}(\frac{5}{2})$ and $I'_{2x}(\frac{3}{2})$ must be found from the *Tables of the Incomplete* Γ-*Function*.

We may, if we please, work from the probability integral of the normal curve, for

$$T_{1,\frac{1}{2}}(x) = \frac{1}{2^{\frac{3}{2}}}\frac{x^{\frac{5}{2}}}{\Gamma^2(\frac{7}{4})}\int_1^{\infty} e^{-xt}(t-1)(t+1)^{\frac{1}{2}}\, dt.$$

Hence writing $x(t+1) = \frac{1}{2}u^2$ we have

$$T_{1,\frac{1}{2}}(x) = \frac{1}{2^{\frac{3}{2}}}\frac{e^x}{\Gamma^2(\frac{7}{4})}\int_{2\sqrt{x}}^{\infty} e^{-\frac{1}{2}u^2}\frac{(u^4 - 4xu^2)}{2\sqrt{2}}\, du,$$

and then integrating twice by parts we find

$$T_{1,\frac{1}{2}}(x) = \frac{3}{8\Gamma^2(\frac{7}{4})}\left\{\sqrt{x}\,e^{-x} + \sqrt{\frac{\pi}{2}}\left(1 - \frac{4x}{3}\right) e^x\, \tfrac{1}{2}(1 - \alpha_{2\sqrt{x}})\right\}.$$

Having regard to the values of $T_{0,-\frac{1}{2}}(x)$ in (3) and $T_{0,\frac{1}{2}}(x)$ in (6) this may be written

$$T_{1,\frac{1}{2}}(x) = \frac{1}{3}\frac{\Gamma^2(\frac{1}{4})}{\Gamma^2(\frac{3}{4})}\, T_{0,\frac{1}{2}}(x) - \frac{4x}{9}\, T_{0,-\frac{1}{2}}(x),$$

so that $T_{1,\frac{1}{2}}(x)$ can be found from $T_{0,\frac{1}{2}}(x)$ and $T_{0,-\frac{1}{2}}(x)$.

* The symbol I' is here used to distinguish the integral ratio $\dfrac{\int_0^z e^{-u} u^{q-1}\, du}{\Gamma(q)} = I'_z(q)$, from the quantity $I(u, p) = \dfrac{\int_0^{u\sqrt{p+1}} e^{-v} v^p\, dv}{\Gamma(p+1)}$ actually tabled in the work *Tables of the Incomplete* Γ-*Function*, 1922. To obtain $I'z(q)$ from those tables we must take $u = z/\sqrt{q}$ and $p = q - 1$.

It has not been my purpose in this paper to enter into the general mathematical theory of Double Bessel Functions; that may no doubt be of interest to the pure mathematician. My purpose here is solely to deal with one type of Double Bessel Function, the $\mathfrak{K}_{\tau_1, \tau_2}(x)$, or in the more suitable form of our purpose the $T_{\tau_1, \tau_2}(x)$ function. This function arises naturally in the consideration of some important statistical problems, and my only purpose in this paper is to develop those properties of $T_{\tau_1, \tau_2}(x)$ which are necessary, if we wish to compute tables by aid of which it will be possible to give a rapid practical answer to the problems in question.

In a second practical part of this paper I hope to provide the tables required and indicate by illustrations the type of problems they are designed to solve.

MISCELLANEA.

(i) Adjustments for the Moments of J-shaped Curves.

By W. PALIN ELDERTON, C.B.E., F.I.A.

When statistics expressible by the exponential $y = y_0 e^{-x/\sigma}$ are stated in groups for each equal subrange h of x, the successive groups are $\int_0^h y_0 e^{-x/\sigma}\,dx$; $\int_h^{2h} y_0 e^{-x/\sigma}\,dx$; etc.; or $y_0\sigma(1-e^{-h/\sigma})$; $y_0\sigma(1-e^{-h/\sigma})e^{-h/\sigma}$; $y_0\sigma(1-e^{-h/\sigma})e^{-2h/\sigma}$; etc. These terms may also be regarded as a geometrical progression*, the first term being $y_0\sigma(1-e^{-h/\sigma})$ and the common ratio $e^{-h/\sigma}$. It follows that if we treat the areas as a geometrical progression extending to infinity, calculate the moments on this assumption and read the result as graduated terms of a geometrical progression, we shall reach correctly graduated areas, and we can subsequently write down the equation to the curve with little trouble.

Other points are however involved. Let us write the geometrical progression as ka^x and put $A = (1-a)^{-1}$, then the moments about its mean are

2nd moment		$A^2 - A$,
3rd	"	$2A^3 - 3A^2 + A$,
4th	"	$9A^4 - 18A^3 + 10A^2 - A$,

and if we work out β_1 and β_2 we get $4 + h^2/\mu_2'$ and $9 + h^2/\mu_2'$ respectively.

Using the exponential, the moments, etc. about the mean are: $\mu_2 = \sigma^2$, $\mu_3 = 2\sigma^3$, $\mu_4 = 9\sigma^4$, $\beta_1 = 4$, $\beta_2 = 9$.

Hence when we calculate moments, assuming that the statistics form a geometrical progression, whereas they are really areas from a curve, and seek to choose the type of curve from Pearson's criteria in his system, we shall reach a persistent error. For this purpose the β_1 and β_2 found from the statistics should be reduced by h^2/μ_2.

This rule can be used as an approximation in all J-shaped curves and will be found to give satisfactory results.

So far we have assumed that we know the start of the curve and that all the bases of the areas are of equal size. If this does not apply we can, in the case of an exponential curve, fit the curve, excluding the first (incomplete) term, and regard that term as related to an appropriate base extrapolated from the graduation of the remainder. This is an arbitrary arrangement but has practical advantages.

In other J-shaped curves in similar circumstances the first step would be to assume an exponential, to find therefrom approximately the base of the first incomplete group, and then assume that the area is concentrated at the middle point. This will generally give good results: the assumption of the exponential overstates the base and the assumption of half-way assumes a less rapidly falling curve than the J-shaped forms of Types I and III. There is therefore a balance of error.

* "Geometrical progression" is used throughout to describe a discrete series and exponential curve to describe a continuous one.

Turning to the statistical side, the example on p. 106 of *Frequency Curves and Correlation* gives $\mu_2=2{\cdot}045$, $\beta_1=4{\cdot}629$, $\beta_2=9{\cdot}502$. These figures come from the unadjusted moments, and deducting ·49 from the above values for β_1 and β_2 we reach 4·14 and 9·01. The theoretical values when an exponential curve is to be used are 4 and 9.

If we apply the rule as an approximation in other J-shaped cases we find that in the example on p. 109, where a twisted J-shaped curve is given, $\mu_2=4{\cdot}260$, $\beta_1={\cdot}761$, $\beta_2=2{\cdot}640$, and the adjustment leads to $\beta_1={\cdot}527$ and $\beta_2=2{\cdot}412$. Hence $5\beta_2-6\beta_1-9$ becomes $-{\cdot}098$ instead of $-{\cdot}368$. The theoretical criterion would lead us to expect $5\beta_2-6\beta_1-9=0$.

These examples are not, of course, complete evidence, but they show that the suggestion may lead to accurate results, and it has the merit of simplicity. The rule with regard to the adjustment of the β's by h^2/μ_2 may be combined with the approximations given on p. 106 of *Frequency Curves and Correlation*, where it is mentioned that the mean is overstated, when μ_3 is positive, by about $\frac{h^2}{12\sigma^2}\sigma$, and the second moment about the *true* mean (i.e. the mean as corrected by $h^2/(12\sigma)$) is understated by about $\frac{h^2}{12\sigma^2}\sigma^2$. Since σ is unknown, these quantities will be (as they are small) $\frac{h^2}{12\mu_2'}\sqrt{\mu_2'}$ and $\frac{h^2}{12\mu_2'}\mu_2'$. In this form the dimensions are maintained, and we see that the degree of approximation is measured by $h^2/(12\mu_2')$. If h be taken as a unit and the moments found in terms of h, i.e. in working units, the corrections are $1/12\sqrt{\mu_2'}$ and $\frac{1}{12}$, as stated in the work just referred to.

(ii) Note on Mr Palin Elderton's Corrections to the Moments of J-curves.

By K. PEARSON.

Mr Elderton has deduced his empirical rule from the exponential curve and suggested that it may give good results for all J-curves. Such curves include those of finite range, and those having constants far removed from those of the exponential curve. The simplicity of the rule is so intriguing, that I thought I must try it on one or two curves far removed from the exponential, these curves having known β's, and computable frequencies.

I propose also to compare the accuracy of the results with moment corrections by other methods.

First Distribution. Consider the curve

$$y=y_0\left(\frac{6{\cdot}808+x}{3{\cdot}192-x}\right)^{0{\cdot}5}$$

This is a Type XII J-curve of limited range, for x runs from $-6{\cdot}808$ to $+3{\cdot}192$, or the range is 10 units, with an asymptote at the end of the range. The curve is what Mr Elderton terms a twisted J-shaped curve, i.e. it rises vertically at the $x=-6{\cdot}808$ end. We may write the curve in the form

$$y=y_0\left(\frac{x}{10-x}\right)^{0{\cdot}5},$$

and its constants are then

$$\text{Mean}=7{\cdot}5,\quad \sigma=2{\cdot}5,\quad \beta_1=1,\quad \beta_2=3.$$

We need not trouble about y_0.

Dividing the range 10 into 10 equal subranges, we find by the *Tables of the Incomplete Beta-Function* the following frequencies for a total of 10,000:

Subrange	0—1	1—2	2—3	3—4	4—5	5—6	6—7	7—8	8—9
Frequency	138	267	368	468	577	705	870	1109	1540

Taking moments about the centre of the group 705, I find for the crude moments coefficients

$$\nu_1' = 1{\cdot}9165, \quad \nu_2' = 9{\cdot}5977, \quad \nu_3' = 25{\cdot}6039, \quad \nu_4' = 134{\cdot}9077,$$

leading, after transference to the *crude mean*, to

$$\bar{x}' = 7{\cdot}4165, \quad \mu_2' = 5{\cdot}9247, \quad \mu_3' = -15{\cdot}49947, \quad \mu_4' = 109{\cdot}669{,}528.$$

To get μ_3' positive we change the direction of the axis of x and measure from the other end of our curve, thus

$$\bar{x}'' = 2{\cdot}5835, \quad \mu_2'' = 5{\cdot}9247, \quad \mu_3'' = 15{\cdot}49947, \quad \mu_4'' = 109{\cdot}669{,}528.$$

We have now to follow the rule and subtract $\frac{h^2}{12\sigma^2}\sigma$ from $\bar{x}''$. But we do not know σ, and are compelled to use $\sqrt{\mu_2''} = 2{\cdot}434{,}071$. Thus $\frac{h^2}{12\sqrt{\mu_2''}} = \frac{1}{29{\cdot}208{,}852}$, whereas the true μ_2 would give $\frac{1}{30} = {\cdot}033{,}333$ as against ${\cdot}034{,}236$.

The mean by Elderton's Rule is thus: $2{\cdot}549{,}167$.

The correction is thus not large enough; it would have been smaller still had we used the true but supposed unknown σ.

The β's from the unadjusted moments are

$$\beta_1'' = 1{\cdot}155{,}141, \quad \beta_2'' = 3{\cdot}124{,}304.$$

Elderton's Rule bids us subtract h^2/μ_2. Here we can use the corrected μ_2*. To ascertain this we must go back to $\nu_2' = 9{\cdot}5979$ and subtract from it the square of the distance of the *corrected* mean from 5·5, i.e. $(7{\cdot}4165 - 5{\cdot}5)^2 = (1{\cdot}9165)^2 = 3{\cdot}672{,}972$, to this we are to add $\frac{1}{12}$. Thus

$$\sigma^2 = 9{\cdot}5979 - 3{\cdot}672{,}972 + {\cdot}083{,}333 = 6{\cdot}008{,}261,$$

or

$$\sigma = 2{\cdot}451{,}175.$$

Further, $h^2/\sigma^2 = {\cdot}166{,}4375$. Accordingly we have

$$\beta_1' = {\cdot}988{,}7035, \quad \beta_2' = 2{\cdot}957{,}8665.$$

We can now collect these results and compare them with the values obtained by various methods.

TABLE I. *Results for Distribution, Curve I.*

Character	True Values	Unadjusted Values	Elderton's Adjustments	Pearse's Adjustments	Martin's Adjustments
Mean	2·500,000	2·583,500	2·549,167	2·499,970	2·529,440
Standard Deviation	2·500,000	2·434,071	2·451,175	2·499,079	2·472,198
β_1	1·000,000	1·155,141	·988,7035	·994,249	1·048,328
β_2	3·000,000	3·124,304	2·957,8665	2·987,057	3·033,454

We have shown how the values provided by Elderton's Rules are obtained. Miss Pearse's adjustments are given in *Biometrika*, Vol. XX^A. pp. 314—355, and in *Tables for Statisticians and Biometricians*, Part II. pp. clxxxvii—ccvi, with the requisite Tables XXXVIII—XLI.

* Mr Elderton tells me he would have used the uncorrected μ_2', but this is smaller than the corrected μ_2, and accordingly since the β's are already overcorrected, it would have increased the discrepancy.

I have to thank Mr E. S. Martin for the arithmetical work. In case any reader should care to follow out the work, I give his chief results. Writing down the frequencies in reverse order, we have

$$n_1 = 3958,\quad n_2 = 1540,\quad n_3 = 1109,\quad n_4 = 870,\quad n_5 = 705,\quad n_6 = 577.$$

Turning to Table XXXVIII, he found for

$$q = \cdot 4,\ n_6 = 511\cdot 90;\quad q = \cdot 5,\ n_6 = 576\cdot 59;\quad q = \cdot 6,\ n_6 = 621\cdot 01.$$

Clearly q must be taken as ·5. Table XLI then gave the K's without interpolation as

$$K_1 = -2814\cdot 3025,\quad K_2 = 2124\cdot 0472,\quad K_3 = -1783\cdot 9437,\quad K_4 = 1634\cdot 1329.$$

Now the first frequency was excluded, and moments taken for the remainder about the start of the range of second frequency n_2'. Using the formulae, *Tables for Statisticians*, p. cxc, the values

$$\mu_1' = 1\cdot 499{,}970,\quad \mu_2' = 8\cdot 495{,}305,\quad \mu_3' = 47\cdot 041{,}306,\quad \mu_4' = 299\cdot 256{,}473$$

were found.

Transferred to the centre we obtain

$$\mu_2 = 6\cdot 245{,}395,\quad \mu_3 = 15\cdot 562{,}793,\quad \mu_4 = 116\cdot 510{,}025,$$

and so σ and the β's.

It will be noticed that the whole of this moment calculating work must also be done when using Elderton's Rules. But the adjustments are briefer in the latter than looking up q and the K's from Miss Pearse's Tables. It will not as a rule be requisite to interpolate.

The last column of Table I gives the results of Mr Martin's not yet published method. By this method there are two unknowns to be previously found before computing the moments, namely the degree of asymptoticity as determined by Miss Pearse's q, and further the position of the asymptote—lying in the first subrange—is assumed to be unknown. This double ignorance renders the method less exact than Miss Pearse's which supposes the position of the asymptote known.

It is clear that both the Pearse* and the Martin methods give in this case more accurate results than Elderton's. It remains then for the judgment of the individual investigator to determine whether the increased accuracy is or is not worth the somewhat increased labour.

Second Distribution. Our first curve was one with low β-values. I thought it desirable to take an extreme case in the opposite direction, namely a J-shaped curve with high β-values. I sought a curve with $\beta_1 = 10$ and $\beta_2 = 20$, roughly, which would have fairly easily determined constants, and of which the areas could be found without double interpolation into the *Tables of the Incomplete Beta-Function*. A Type VI J-curve was taken with equation

$$y = \frac{100{,}000 \times 2\cdot 166{,}966}{(x-1)^{\cdot 5}\, x^{16\cdot 5}}.$$

This gives a total frequency of 100,000, with the following constants:

$$\text{Mean (from } x = 1) = \cdot 035{,}714,\qquad \sigma = \cdot 053{,}342,$$

$$\beta_1 = 11\cdot 206{,}897,\qquad \beta_2 = 21\cdot 844{,}013.$$

By aid of the transformation $x = 1/(1-z)$, we can convert the curve into an incomplete β-function, i.e. $B_z(\cdot 5, 15)$, and thus find the areas. But difficulties present themselves, if we are to have a workable number of sub-frequencies. The curve is extremely leptokurtic, and if we confine ourselves to the number of some 25 sub-frequencies, the subrange h must be of magnitude about ·04. This causes about 72 °/ₒ of the frequency to fall on the first subrange, and there is a long tail wherein if we only proceed by units of frequency it is difficult to know where exactly to place

* The Pearse method would doubtless have given still closer results, if we had used, as we ought also abruptness coefficients, at the far end of the range, where the curve is perpendicular to the x-axis.

them. To add to the difficulties interpolation by usual methods into the β-function Tables at the upper end of the curve is unsatisfactory. After certain modifications of the β-function have been made*, and the first sub-frequency checked by actual expansion, the following system was obtained:

x	Frequency	x	Frequency	
1·00—1·04	71796	1·48—1·52	23	
1·04—1·08	15022	1·52—1·56	15	
1·08—1·12	6438	1·56—1·60	10	
1·12—1·16	3108	1·60—1·64	6	
1·16—1·20	1600	1·64—1·68	4	
1·20—1·24	861	1·68—1·72	3	
1·24—1·28	480	1·72—1·76	2	
1·28—1·32	275	1·76—1·80	A 1	B 1
1·32—1·36	161	1·80—1·84	1	1
1·36—1·40	97	1·84—1·88	0	0
1·40—1·44	59	1·88—1·92	1	0
1·44—1·48	37	1·92—1·96	0	1

A and *B* are two alternative schemes for the arrangement of the tail. But they provide so little final change of significance when tried by the Pearse method, that it is clear that the three final units would have to be scattered far more widely apart than appeared reasonable to get a closer approach to the observed higher moment values. As scattering increases the β's, it was of no advantage to try the Scheme B, see Table II, with Elderton's rules as they are already too large. The reader must remember that with Pearse's method we neglect in the first place the first frequency and find moments about $x=1{\cdot}04$, but with Elderton's method we find moments about $x=1$. I give the final results in Table II without supplying the intermediate links. I may note that Pearse's q lies between ·4 and ·5. If we interpolate for it, we find $q={\cdot}478,766$. Our last column in Table II is given to indicate whether it is worth the trouble of interpolating for q and the K's.

TABLE II. *Results for Distribution, Curve II.*

Character	True Values	Crude Values	Elderton's Adjustments	Pearse's Adjustments		
				Scheme A $q={\cdot}5$	Scheme B $q={\cdot}5$	Scheme A Interpolated q
Mean from $x=1$	·035,714	·042,721	·040,051	·035,721	·035,741	·035,584
σ	·053,342	·049,934	·053,365	·053,241	·053,249	·053,296
β_1	11·206,897	13·874,989	13·313,147	10·809,219	10·832,196	10·780,903
β_2	21·884,013	24·332,380	23·770,538	20·373,483	20·455,015	20·036,753

Elderton's Rule gives the best result for the standard deviation†, but the deductions are seen to be insufficient in the case of the mean and the two β's, when h forms so considerable a part of σ. We should need to go to a higher approximation. Pearse's results indicate that we gain nothing of real significance by interpolating for q, a process which of course increases the labour

* I have to thank Mr E. C. Fieller for aid in this matter.

† His rule would give a worse result for the standard deviation had he obtained a better result for the mean.

of adjustment. Again, we may note with satisfaction that little difference is made by increasing the scatter of the tail, or the problem of where best to concentrate the last tail unit is after all not of great importance. Pearse's adjustments for the third and fourth moments give too low values for the β's, but, taken as a whole, for these β's as well as for the mean they yield more accurate results than Elderton's method. It is, of course, an application to a most extreme case, and there is little doubt that Elderton's method would give improved results were the corrective terms taken to a higher approximation. We must not, however, forget that it is precisely those distributions which give an overwhelming frequency in the first subrange (incomes, house property, etc.) that cause the greatest trouble with our graduation. For such curves h/σ is not a very small quantity, and we cannot afford to retain only the lowest power of the quantity appearing in the adjustment.

(iii) A General Expression for the Moments of Certain Symmetrical Functions of Normal Samples.

By R. C. GEARY, M.Sc.

In a paper entitled "The Moments of the Distribution of Normal Samples of Measures of Departure from Normality," R. A. Fisher* has developed a technique for the calculation of moments of functions of the type $k_p/k_2^{p/2}$, where

$$\left.\begin{aligned} k_1 &= \frac{1}{n}\sum_{i=1}^{n} x_i = m_1 \\ k_2 &= \frac{1}{n-1}\sum_{i=1}^{n} \delta_i^2 = \frac{nm_2}{n-1} \\ k_3 &= \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n} \delta_i^3 = \frac{n^2 m_3}{(n-1)(n-2)} \\ &\text{etc.,} \end{aligned}\right\} \quad \text{.........................(i),}$$

with

$$\delta_i = x_i - m_1 = \left(1 - \frac{1}{n}\right) x_i - \sum_{j \neq i}' \frac{x_j}{n} \quad \text{...............................(ii),}$$

and where the x_i represent the measures of n elements drawn at random from a normal universe. Fisher has given the exact formulae for some of the lower moments of $k_3/k_2^{3/2}$ and k_4/k_2^2. The object of this note is to show how these results may be derived by an alternative process and to write down a general formula for these moments. In what follows, the moments of the functions $m_p/m_2^{p/2}$ will be considered: the lower moments of the corresponding Fisher functions can easily be calculated therefrom. Square brackets, [ ], indicate universal arithmetic mean value, about a fixed origin, not about the sampling mean. In the present problem the universal mean and standard deviation may be presumed zero and unity respectively, without loss of generality.

First, a general formula for the moments of m_p, calculated from samples of n drawn from a *normal universe*, will be calculated. The method is that which C. C. Craig† has used to show how the semi-invariants of moments of samples drawn from *any universe* may be derived and by which he has calculated the exact values of the first four moments of m_3 and m_4.

Craig's method, as applied to the normal case, starts with the following identity in t:

$$[e^{tx}] \equiv e^{\frac{1}{2}t^2} \quad \text{...(iii),}$$

* *Proceedings of the Royal Society*, A, 130 (1931), p. 16 *seq.*

† "An Application of Thiele's Semi-Invariants to the Sampling Problem." *Metron*, Vol. VII, N. 4 (1928), p. 3 *seq.*

from which it follows, since the x_i are independent, that

$$\left[e^{\sum\limits_{i=1}^{n}\delta_i t_i}\right] = e^{\frac{1}{2}\left\{\Sigma t_i^2 - \frac{1}{n}(\Sigma t_i)^2\right\}} \quad \text{.................................(iv)},$$

upon substitution for the δ_i as given in (ii).

It is immediately evident that, in the normal case, $[m_p^r]$ is zero if p and r are both odd numbers. It will only be necessary to consider the case of rp even.

Now

$$[m_p^r] = \frac{1}{n^r}[(\Sigma\delta_i^p)^r] = \frac{1}{n^r}\left[\underset{\Sigma r_i = r}{S}\frac{r!}{r_1!\ldots r_e!}\delta_1^{pr_1}\ldots\delta_e^{pr_e}\right] = \frac{1}{n^r}\underset{\Sigma r_i = r}{S}\frac{r!}{r_1!\ldots r_e!}\binom{n}{e}[\delta_1^{pr_1}\ldots\delta_e^{pr_e}] \quad \text{...(v)}.$$

The means in the last expression can now be calculated by identifying the coefficients of $t_1^{pr_1}\ldots t_e^{pr_e}$ on both sides of (iv):

$$\frac{[\delta_1^{pr_1}\ldots\delta_e^{pr_e}]}{pr_1!\ldots pr_e!} = \frac{1}{2^q}\sum_{s=0}^{s=q}\left(-\frac{1}{n}\right)^{q-s}\frac{(2q-2s)!}{(q-s)!}\underset{\Sigma s_i = s}{S}\frac{1}{s_1!(pr_1-2s_1)!\ldots s_e!(pr_e-2s_e)!} \quad \text{......(vi)},$$

with $\boxed{rp = 2q}$

Hence, from (v) and (vi),

$$[m_p^r] = \frac{r!}{2^q n^r}\underset{\Sigma r_i = r}{S}\binom{n}{e}\frac{pr_1!\ldots pr_e!}{r_1!\ldots r_e!}\sum_{s=0}^{s=q}\left(-\frac{1}{n}\right)^{q-s}\frac{(2q-2s)!}{(q-s)!}\underset{\Sigma s_i = s}{S}\frac{1}{s_1!(pr_1-2s_1)!\ldots s_e!(pr_e-2s_e)!}$$
$$\text{......(vii)}.$$

Fisher* has shown that in the normal case a simple relation subsists between the moments $[k_p^r]$ and the $[k_p^r/k_2^q]$. His method of proof, depending on the properties of differential operators, is somewhat complicated. The following method indicates the genesis of the relationship.

By an orthogonal transformation of the original variables x_i, of which one of the transformed variables is $X_n = \frac{1}{\sqrt{n}}\Sigma x_i$, followed by a transformation into generalised polar coordinates of the remaining $n-1$ variables $X_1, \ldots, X_{n-1}$, of the differential element

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n \prod_{i=1}^{n} e^{-\frac{1}{2}x_i^2}\,\delta x_i,$$

it is easy to show that the mean m_1, the second moment m_2 ($=\frac{1}{n}\rho^2$, where ρ is the radius of the polar coordinates) and the $n-2$ polar angles are all independent. Now it is clear that the functions $m_p/m_2^{p/2}$, upon transformation, are explicit functions of the polar angles only and are therefore independent of m_2. Hence

$$\left[\frac{m_p^r}{m_2^q}\times m_2^q\right] = [m_p^r] = \left[\frac{m_p^r}{m_2^q}\right]\times[m_2^q] \quad \text{..............................(viii)}.$$

From the known distribution of m_2†, the last factor is as follows:

$$[m_2^q] = \frac{(n-1)(n+1)(n+3)\ldots(n+2q-3)}{n^q} \quad \text{..........................(ix)}.$$

From (vii), (viii) and (ix) the required expression for $[m_p^r/m_2^q]$ can be written down at once.

The property that m_2 and $m_p/m_2^{p/2}$ are independent may also be used to demonstrate certain simple relationships between two-dimensional semi-invariants of the type $S_{kl}(m_2, m_p)$ when l is kept constant and k is allowed to vary. The probable existence of these relationships was surmised by C. C. Craig‡ from the form of the lower semi-invariants. The $S_{kl}(m_2, m_p)$ are given by the identity in α, β:

$$[e^{\alpha m_2 + \beta m_p}] \equiv e^{\sum\limits_{k+l=1}^{\infty}\Sigma S_{kl}\frac{\alpha^k\beta^l}{k!\,l!}} \quad \text{..................................(x)}.$$

* *Op. cit.* p. 27.

† "Student," "The Probable Error of a Mean," *Biometrika*, Vol. VI. (1908) p. 1 *seq.*

‡ *Op. cit.* p. 61.

The first term may be written

$$\sum_{k+l=n}^{\infty}\sum M_{kl}\frac{\alpha^k\beta^l}{k!\,l!},$$

with

$$M_{kl}=[m_2^k m_p^l]=\frac{[m_2^{k+lp/2}]}{[m_2^{lp/2}]}\times[m_p^l],$$

since m_2 and $m_p^l/m_2^{lp/2}$ are independent for normal samples. Hence, from (ix),

$$M_{kl}=\frac{\dfrac{n-1+lp}{2}\cdot\dfrac{n+1+lp}{2}\cdots\dfrac{n+2k-3+lp}{2}}{\left(\dfrac{n}{2}\right)^k}\times[m_p^l],$$

from which it follows that, if α is so small that $2\alpha<n$,

$$\sum_{k+l=n}^{\infty}\sum M_{kl}\frac{\alpha^k\beta^l}{k!\,l!}=\sum_{l=0}^{\infty}\frac{\beta^l}{l!}\left(1-\frac{2\alpha}{n}\right)^{-\frac{n-1+lp}{2}}\times[m_p^l]=\left(1-\frac{2\alpha}{n}\right)^{-\frac{n-1}{2}}\sum_{l=0}^{\infty}\frac{\beta^l}{l!}\left[\left\{m_p\left(1-\frac{2\alpha}{n}\right)^{-p/2}\right\}^l\right]$$

$$=e^{-\frac{n-1}{2}\log\left(1-\frac{2\alpha}{n}\right)+\sum\limits_{l=1}^{\infty}S_{0l}\left(1-\frac{2\alpha}{n}\right)^{-lp/2}\frac{\beta^l}{l!}}\qquad\text{.......(xi).}$$

Expanding the logarithmic and binomial terms in the exponent of the last expression and comparing the coefficient of $\alpha^k\beta^l$ with the corresponding coefficients in the second side of (x), it will be seen that

$$\left.\begin{aligned}S_{k0}&=\frac{\overline{k-1}\,!}{2}(n-1)\left(\frac{2}{n}\right)^k\\ \text{and}\qquad S_{kl}&=S_{0l}\frac{pl\,(pl+2)\ldots(pl+2k-2)}{n^k},\quad l\neq 0\end{aligned}\right\}\qquad\text{.......(xii).}$$

These results are in agreement with those for $S_{kl}(m_2, m_p)$, $k, l=0, 1, 2, 3, 4$ and $p=3, 4$, from which Craig surmised the general results given at (xii).

(iv) A Statistical Study of the Daucus Carota L.

By WILLIAM DOWELL BATEN

(University of Michigan).

The object of this article is to compare two samples, each of one thousand, of the Wild Carrot taken from the roadside in Michigan and Indiana. The sample from Michigan was taken near Ann Arbor during the summer of 1930, while that from Indiana was taken near Terre Haute during the summer of 1931.

The Daucus Carota, or Wild Carrot, is a weed which grows profusely over the north-eastern and north-central parts of the United States. I have found it growing in Michigan, Indiana, Kentucky, West Virginia, Virginia, District of Columbia, Maryland, Delaware, Pennsylvania, New York and Ohio. This plant is from one foot to five or six feet in height and has its flowers arranged in umbels. At the ends of the tall and rigid stems are enlargements or knobs from which flower arms or rays grow. At the end of each ray is a composite flower made up of many tiny white flowers of different sizes. The entire inflorescence containing flower arms with their flowers is from one to four or five inches in diameter and as a whole resembles delicate white lace. This resemblance is no doubt the source of the name Queen Anne's Lace. Around the knobs, at the ends of the stems, grow the rays which are in rows, there being more rays near the base. Flower arms at the centre of the cluster are much shorter than the others and contain few

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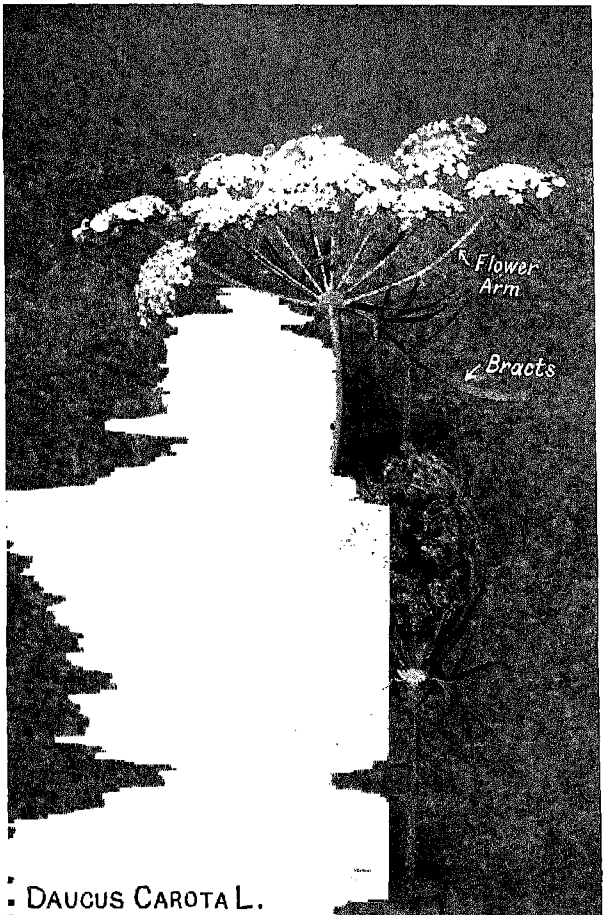


DAUCUS CAROTA L.

flowers which are sometimes pink and purple. The accompanying Plate I shows a flower cluster and also a cluster after the rays have turned inward and the seeds are maturing.

Beneath the bottom row of the flower arms are found green bracts which resemble sepals. They are slender and are made up of pointed branches which vary in number from one to seven or more. These branched leaves hug close to the lower rays while the cluster is young, but grow downward after it reaches maturity.

The following presents the chief characteristics of the distributions of the number of bracts from both samples and also those for the distribution of the number of flower arms. The significance of the means of these distributions are determined, together with the linear correlation between the number of bracts and the number of rays for the sample from Indiana.

1. Chief characteristics of the distributions of bracts.

The following tables give the frequency distributions of the number of bracts from the samples from Michigan and Indiana.

Number of bracts	Michigan Frequencies	Indiana Frequencies
4	1	0
5	7	0
6	8	0
7	41	0
8	303	98
9	224	143
10	140	159
11	127	205
12	93	201
13	52	189
14	1	3
15	2	2
16	1	0
Totals	1000	1000

	Michigan	Indiana
Mean	9·463 bracts	10·857 bracts
Mode	8 "	11 "
Median	9·123 "	10·985 "
Range	13 "	8 "
Standard deviation ...	1·71016 "	1·61510 "
Skewness	0·560702	−0·215047

$$\text{Significance of means} = \frac{\text{Difference of Means}}{\text{Probable Error of Difference of Means}}$$
$$= 27\cdot 88$$

The two distributions differ in several ways. The range for the sample from Michigan is 13 bracts, while the range for the Indiana sample is eight. There were no flower clusters on the Indiana plants which had less than eight bracts, while there were 57 plants from Michigan which had less than eight bracts per cluster. Most of the clusters from Michigan had nine or less bracts per cluster, while most of those from Indiana had 11 or more bracts; yet there was one cluster from Michigan which had 16 bracts while the highest number from the other sample

was 15 bracts. 72·4 per cent. from Michigan had 10 or *less* bracts per cluster, while 75·9 per cent. from Indiana had 10 or *more* bracts per cluster. 27·6 per cent. from Michigan had 11 or more while 60 per cent. from Indiana had 11 or more. 56 clusters from Michigan had 13 or more bracts per cluster, while 194 from Indiana had 13 or more bracts per cluster.

A very good idea of how these two distributions differ is also manifested by skewness; that for the Michigan sample was plus ·560702, while that for Indiana was minus ·215047. One distribution is skew to the right and the other is skew to the left.

Histograms in Diagram I help the eye to distinguish these differences to some extent.

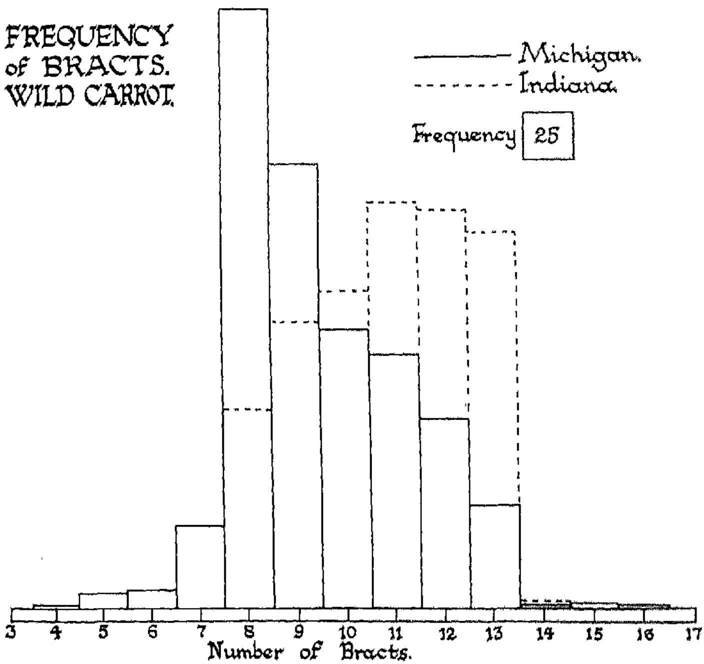


Diagram I.

The significance of the means shows that the two samples were not the result of random sampling. The probability that one mean would differ from the other so much suggests that it is almost impossible for these samples to be taken from the same parent population at random. This significance is nearly 28 times the probable error of the difference of the means, which shows clearly that the samples are not consistent with each other.

2. Characteristics of the distributions of the flower arms.

The following table gives the frequencies relating to the flower arms in the clusters of the flowers of the Wild Carrot.

Number of rays	Michigan Frequencies	Indiana Frequencies	Number of rays	Michigan Frequencies	Indiana Frequencies
13	1	0	53	14	36
18	1	0	54	18	58
20	1	0	55	19	26
22	3	0	56	17	50
23	9	0	57	5	37
24	7	0	58	4	31
25	9	0	59	9	20
26	11	0	60	4	41
27	19	0	61	1	21
28	24	1	62	2	28
29	22	0	63	5	12
30	36	4	64	0	17
31	28	0	65	3	8
32	41	4	66	1	22
33	32	4	67	0	13
34	36	7	68	1	17
35	40	3	69	2	14
36	45	5	70	0	9
37	33	5	71	0	5
38	48	22	72	0	8
39	35	9	73	0	4
40	57	28	74	0	7
41	35	13	75	0	2
42	40	34	76	1	6
43	30	23	77	1	2
44	34	35	78	0	2
45	34	17	79	0	1
46	34	36	80	0	4
47	23	22	81	0	2
48	26	53	82	0	5
49	21	24	83	0	2
50	33	44	84	0	2
51	26	44	89	0	1
52	19	49	105	0	1
			Totals	1000	1000

	Michigan	Indiana
Mean	40·512 f. arms	53·509 f. arms
Standard deviation ...	9·1880 "	10·1214 "
Skewness	·3795	·5142
Probable error of mean ...	·1933 f. arms	·2162 f. arms

Significance = 44·71

The means for the above distributions show clearly that the plants from Indiana have larger flower clusters on the average, which could not be detected by the casual observer or detected as one sees the plants along the roadside. In appearance the two flower clusters seem to be alike. There were no clusters from the Indiana sample that had 27 or less flower arms, while there were 58 from the Michigan sample. There were no clusters from Michigan which had more than 77 flower arms, while there were 20 from Indiana which had more than 77. There were only two clusters from Michigan with more than 69 rays, while there were 63 clusters from Indiana with more than 69 rays. Two-thirds of the distribution from Michigan contained *less* than 45 rays, while eight-tenths of that from Indiana contained 45 or *more* rays. Only 56 clusters from Michigan had more than 55 rays, while more than one-third of the sample from Indiana had

FREQUENCY of FLOWER ARMS. WILD CARROT.

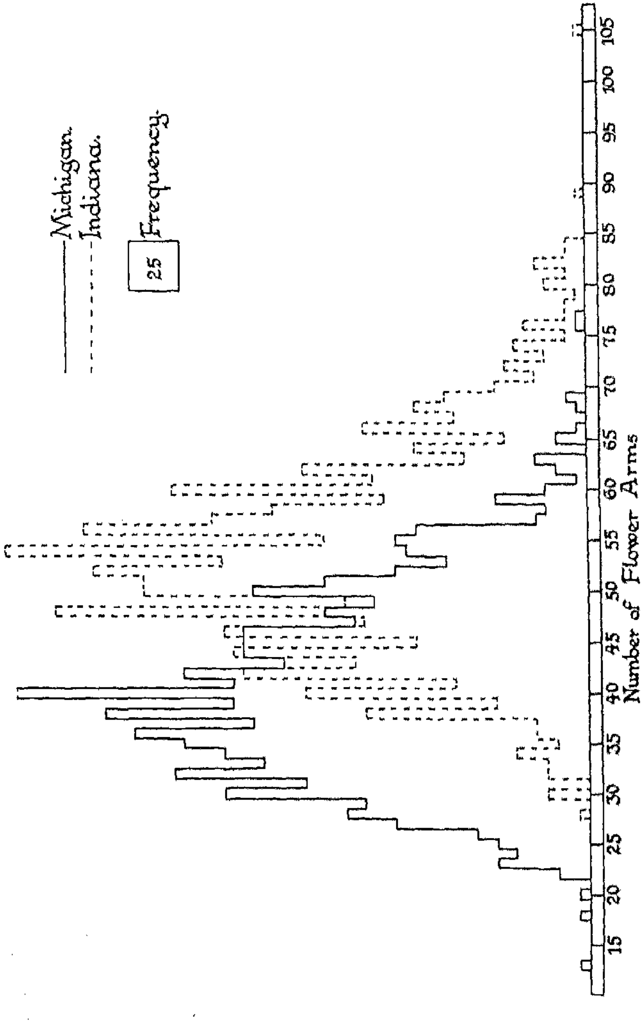


Diagram II.

more than 55 rays. More than nine-tenths of the distribution from Michigan lies below the mean of the Indiana distribution. More than nine-tenths of the distribution from Indiana lies above the mean for the Michigan distribution.

Histograms in Diagram II show clearly how the distributions differ, as to range, means, and the nature of the distributions at the ends. The large frequencies of the distribution from Michigan correspond in a measure to the small frequencies for that from Indiana and *vice versa*. The two distributions nearly coincide at 44 and 47. The facts that have been stated show clearly that the Indiana plants produced larger flower clusters than the plants from Michigan.

The significance of the means is 44·71, which certainly shows that the two samples were not due to random sampling from the same parent population. Just why these samples differ so widely I cannot say.

The Table on p. 189 shows that the frequencies for even numbers of flower arms are greater than those for odd numbers. This is true for the clusters from Michigan and Indiana. The fact is well exhibited on examining the histograms on page 190. For the Indiana sample the frequencies for the numbers in the first line are found in the second line.

37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62
6, *22*, 9, *28*, 13, *34*, 23, *35*, 17, *36*, 22, *53*, 24, *44*, 44, *49*, 36, *58*, 26, *50*, 37, *31*, 20, *41*, 21, *28*

This rise at the even numbers and fall at the odd numbers can be easily seen by examining the frequency polygons of original data. Histograms in Diagram II show this very clearly. In the large majority of the cases the frequencies for the even numbers are larger than the frequencies for the odd. The sum of the frequencies for the even numbers is 610, while the sum of the frequencies for the odd numbers is 390.

Just why there are larger frequencies for the even numbers of rays I cannot say. The fact that this plant is a dicotyledon may have something to do with it. Botanists perhaps can explain this phenomenon.

FREQUENCY of FLOWER ARMS in GROUPS of FIVES. WILD CARROT.

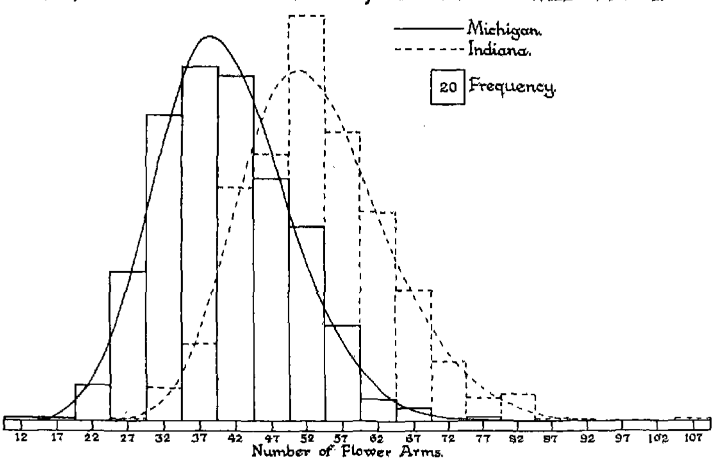


Diagram III.

Dividing the distribution into groups of twos does not remove many of the irregularities caused by the predominance of the plants with even numbers of flower arms. Grouping by threes eliminates almost all of the irregularities, yet gives distributions with two modes.

In Diagram III the two series of data are divided into groups of fives. In each case a Type 3 curve has been fitted to the data. The following table presents the data in groups of fives together with the computed frequencies from the Type III curves.

Observed and computed Frequencies of Rays in Clusters of the Wild Carrot.

Classes	Frequency for Michigan sample		Frequency for Indiana sample	
	Observed	Computed	Observed	Computed
10—14	1	·3	0	
15—19	1	3	0	..
20—24	20	24	0	
25—29	85	78	1	1
30—34	173	160	19	12
35—39	201	216	44	53
40—44	196	204	133	123
45—49	138	154	152	186
50—54	110	89	230	198
55—59	54	45	164	168
60—64	12	18	119	118
65—69	7	6	74	72
70—74	0	2	34	38
75—79	2	1	13	18
80—84	0	...	15	8
85—89	0	—	1	2
105—109	0	—	1	3*
Totals	1000	999·3	1000	1000

* Greater than 89.

The histograms for the distributions in groups of fives together with the corresponding Type III curves again show that the plants from Indiana produce flower clusters with a larger number of rays.

The Type III curve fits the Michigan sample better than a Type III curve does the Indiana sample. The tall column near the mean no doubt causes this poor fit.

3. Correlation coefficient between bracts and rays.

While examining the clusters the question arose as to whether clusters with a large number of rays also had a large number of bracts. While counting the rays and bracts, clusters were found which contained 13 bracts and 46 rays, also others were found with 13 bracts and 82 rays; also some with eight bracts and 30 rays and others with eight bracts and 69 rays. The correlation coefficient was considered to be the answer to this question. The Pearson linear correlation coefficient between the number of bracts and the number of flower arms for the sample from Indiana is

$$r = ·624†.$$

† ·630, if Sheppard's corrections be used for the standard deviations.

Correlation Table. Flower Arms and Bracts.

Number of flower arms per cluster from sample from Indiana

Number of bracts per cluster	25—29	30—34	35—39	40—44	45—49	50—54	55—59	60—64	65—69	70—74	75—79	80—84	85—89	105—109	Totals
15	—	—	—	—	—	—	—	—	—	1	—	1	—	—	2
14	—	—	—	—	—	—	—	2	—	—	1	—	—	—	3
13	—	—	—	2	8	30	37	38	37	20	9	8	—	—	189
12	—	1	1	5	21	41	48	40	25	8	3	6	1	1	201
11	—	2	3	19	29	66	48	28	8	2	—	—	—	—	205
10	—	3	13	31	29	49	22	10	1	1	—	—	—	—	159
9	—	4	14	39	42	32	7	1	2	2	—	—	—	—	143
8	1	9	13	37	23	12	2	0	1	—	—	—	—	—	98
Totals	1	19	44	133	152	230	164	119	74	34	13	15	1	1	1000

This coefficient does show that there is a rather definite relation between the number of bracts and the number of flower arms per cluster. This means that on the average clusters with a small number of bracts will also have a small number of flower arms, and those with a large number of bracts will on the average have a large number of flower arms. On examining the data it was soon seen that there were only three clusters with eight bracts which had more than 54 rays. There were two which had 15 bracts and these had 70 or more rays. Most of the plants with eight bracts contained 44 or less rays per cluster, while most of those with 13 bracts contained 60 or more rays. The above table shows how the clusters were distributed for the number of bracts.

The following equation, obtained by the method of least squares, gives the regression straight line of number of flower arms on bracts:

$$\tilde{y} = 10{\cdot}505 + 3{\cdot}961x^{*},$$

where $\tilde{y}$ represents the mean number of flower arms in the curve and x represents the number of bracts per cluster. This line is plotted in Diagram IV.

The following table gives the average number of flower arms per cluster for the respective number of bracts per cluster.

Number of bracts	Average number of rays
8	43·22
9	46·69
10	49·08
11	53·27
12	58·62
13	62·19
14	67·00
15	77·00

These points have been plotted in Diagram IV and lie very close to the regression straight line. The correlation of ·624 is far from perfect correlation yet is much further from no correlation. The above shows that the plants with a small number of bracts also have a small number of flower arms and the plants with a large number of bracts have a large number of flower arms. The above table shows that the average number of rays for all clusters which had eight bracts was 43·22 rays, etc.

* For number of bracts on flower arms the regression line is $\tilde{x} = 5{\cdot}597 + {\cdot}0983\,y$.

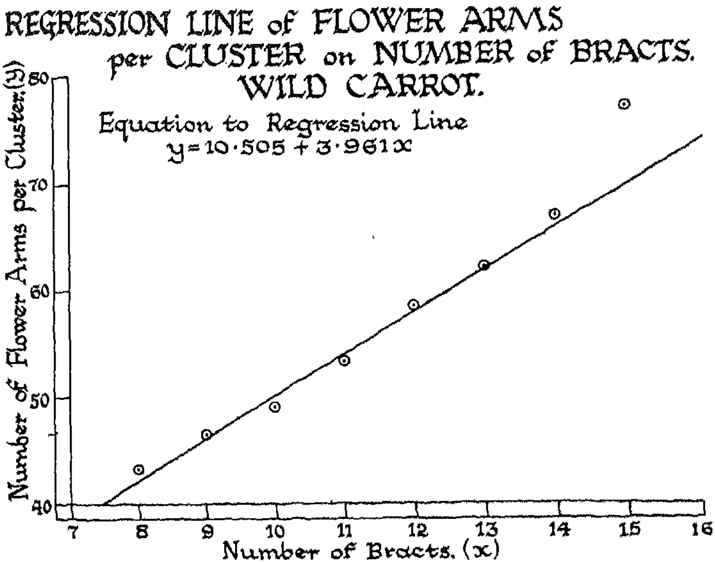


Diagram IV.

The above study was made to show that natural phenomena may be analyzed statistically to a great advantage and that this type of study brings out many interesting details which might be overlooked.

Seeds which were taken from the two environments have been planted under the same conditions in the Botanical Gardens of the University of Michigan. Samples of one thousand will again be compared. The experiment may extend over several years*.

The following table gives the number of flower arms per bract for the sample from Indiana, for the plants with 8, 9, 10, 11, 12, 13, 14 and 15 bracts respectively.

	Average number of flower arms per bract
Plants with 8 bracts	5·15
" " 9 "	5·18
" " 10 "	4·91
" " 11 "	4·84
" " 12 "	4·87
" " 13 "	4·78
" " 14 "	4·79
" " 15 "	5·13

* M. K. Pyóamebckar has studied this problem. *Bulletin of Applied Botany of Genetics and Plant-breeding* (Russian Journal), Vol. xxvi. 1931, pp. 194—262.

This table shows that for the plants with certain numbers of bracts the number of flower arms on the average is about five flower arms per bract. When all the flower arms and all the bracts were considered the number of flower arms per bract was 4·93, which is on the average about five.

When all of the flower arms and all the bracts were considered for the Michigan sample the number of flower arms per bract was 4·28, which shows again that the two samples differ widely.

(v) On a Property of the Mean Ranges in Samples from a Normal Population and on some Integrals of Prof. T. Hojo.

BY PROF. V. ROMANOVSKY, Tashkend.

I.

Studying the paper of Prof. K. Pearson "On the Mean Character and Variance of a Ranked Individual and on the Mean and Variance of the Intervals between Ranked Individuals" in *Biometrika*, Vol. XXIII. pp. 364—397, I discovered a property of the mean ranges in samples taken from a normal population which deserves some attention.

This property may be formulated as follows.

Let us put
$$\alpha_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx \quad \text{.........(1)};$$

then
$$w_m = 2m \int_0^1 x\alpha_x^{m-1} d\alpha_x, \quad \text{.........(2)},$$

representing the mean range in samples of size m from an infinite normal population with zero as mean and unity as variance, verifies identically the relation

$$\frac{w_{2m+1}}{4m+2} - C_m^1 \frac{w_{2m}}{4m} + C_m^2 \frac{w_{2m-1}}{4m-2} - \ldots + (-1)^m \frac{w_{m+1}}{2m+2} = 0 \qquad \left(C_m^h = \frac{m!}{h!\,(m-h)!}\right) \text{......(3)},$$

for $m = 1, 2, 3, \ldots$.

The demonstration of this identity is very simple.

Let us introduce the quantity

$$\lambda_m = \frac{w_{m+1}}{2m+2} = \int_0^1 x\alpha_x^m d\alpha_x,$$

and consider the integral
$$I_m = \int_0^1 x\alpha_x^m (1-\alpha_x)^m d\alpha_x \quad \text{.........(4)}.$$

From (1) we obtain
$$\alpha_x = \tfrac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}x^2} dx = \tfrac{1}{2} + u_x,$$

putting
$$u_x = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}x^2} dx,$$

and, therefore,
$$\alpha_x^m (1-\alpha_x)^m = (\tfrac{1}{2}+u_x)^m (\tfrac{1}{2}-u_x)^m = (\tfrac{1}{4}-u_x^2)^m$$
is an even function of x. Now

$$I_m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(\tfrac{1}{4}-u_x^2)^m e^{-\frac{1}{2}x^2} dx,$$

and it appears at once that I_m, as the integrand represents an uneven function, is identically zero:

$$I_m = 0.$$

But, from (4),

$$I_m = \int_0^1 x\alpha_x^m d\alpha_x - C_m^1 \int_0^1 x\alpha_x^{m+1} d\alpha_x + \ldots + (-1)^m \int_0^1 x\alpha_x^{2m} d\alpha_x,$$

and we see, remembering the definition of λ_m, that

$$\lambda_m - C_m^1 \lambda_{m+1} + C_m^2 \lambda_{m+2} - \ldots + (-1)^m \lambda_{2m} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

or

$$\frac{w_{m+1}}{2m+2} - C_m^1 \frac{w_{m+2}}{2m+4} + C_m^2 \frac{w_{m+3}}{2m+6} - \ldots + (-1)^m \frac{w_{2m+1}}{4m+2} = 0,$$

q.e.d., because this is only a slightly different form of (3).

We may remark that (5) may be written in the condensed form

$$\Delta^m \lambda_m = 0 \qquad (m = 1, 2, 3, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots(5 \textit{ bis}).$$

This relation can be applied to the verification of the tables of w_m calculated by Mr Tippett. Another application is this: for the calculation of λ_m we need only to calculate directly $\lambda_1, \lambda_3, \lambda_5, \ldots$, for $\lambda_2, \lambda_4, \lambda_6, \ldots$ we shall find from

$$\lambda_1 - \lambda_2 = 0, \quad \lambda_2 - 2\lambda_3 + \lambda_4 = 0,$$
$$\lambda_3 - 3\lambda_4 + 3\lambda_5 - \lambda_6 = 0, \text{ etc.}$$

II.

Having discovered the relation (5), I naturally tried to find similar relations for the integrals T, I, R, S, K of Prof. T. Hojo (*Biometrika*, Vol. XXIII. pp. 325–326), which, as it can be assuredly supposed, required much laborious calculation. Now it can be shown that this work could be reduced to a half owing to some simple recurrent relations between Prof. T. Hojo's integrals.

The integrals in question are obviously of two types:

$$H_{m,p} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha_x^m e^{-px^2} dx \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and

$$G_{m,p,q} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha_x^m \alpha_{qx} e^{-px^2} dx, \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where $m = 1, 2, 3, \ldots$, $p > 0$ and q are any real numbers. For simplicity we shall denote them as H_m and G_m, omitting other indices which will be supposed given and constant.

Let us take the first of these integrals.

We may write

$$\begin{aligned} \alpha_x^{m+1} (1-\alpha_x)^m &= (\tfrac{1}{2} + u_x)^{m+1} (\tfrac{1}{2} - u_x)^m \\ &= \tfrac{1}{2} (\tfrac{1}{4} - u_x^2)^m + u_x (\tfrac{1}{4} - u_x^2)^m \\ &= \tfrac{1}{2} \alpha_x^m (1-\alpha_x)^m + u_x (\tfrac{1}{4} - u_x^2)^m. \end{aligned}$$

As u_x is an uneven function of x, we must have

$$\int_{-\infty}^{\infty} u_x (\tfrac{1}{4} - u_x^2)^m e^{-px^2} dx = 0,$$

and therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha_x^{m+1} (1-\alpha_x)^m e^{-px^2} dx = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha_x^m (1-\alpha_x)^m e^{-px^2} dx,$$

or, expanding both integrands by Newton's theorem and integrating term by term,

$$H_{2m+1} - C_m^1 H_{2m} + \ldots + (-1)^m H_{m+1} = \tfrac{1}{2} [H_{2m} - C_m^1 H_{2m-1} + \ldots + (-1)^m H_m] \ \ldots\ldots(8),$$

which may be written also as

$$\Delta^m H_{m+1} = \tfrac{1}{2} \Delta^m H_m \ldots\ldots\ldots\ldots\ldots\ldots(8 \textit{ bis}).$$

This relation is well verified by the integrals T, I, S and K of Prof. T. Hojo, which shows the exactness with which these integrals were calculated for Table I of his paper.

Quite similarly, starting from the identity

$$\alpha_x^m (1-\alpha_x)^m \alpha_{qx} = \tfrac{1}{2} (\tfrac{1}{4} - u_x^2)^m + u_{qx} (\tfrac{1}{4} - u_x^2)^m,$$

where $u_{qx} = \alpha_{qx} - \frac{1}{2}$, we find

$$G_{2m} - C_m^1 G_{2m-1} + \ldots + (-1)^m G_m = \tfrac{1}{2} [H_{2m} - C_m^1 H_{2m-1} + \ldots + (-1)^m H_m] \ \ldots\ldots(9),$$

or

$$\Delta^m G_m = \tfrac{1}{2} \Delta^m H_m \qquad (m = 1, 2, 3, \ldots) \ \ldots\ldots\ldots\ldots(9 \textit{ bis}).$$

The relation (8) shows that $H_1, H_2, H_4, H_6, \ldots$ being known, we shall have $H_3, H_5, H_7, \ldots$ from (8). And when all the H's are calculated, we need only to calculate directly $G_1, G_3, G_5, \ldots$ in order to have $G_2, G_4, G_6, \ldots$ from (9).

I shall conclude with two further relations which can be of use. Let

$$A_{m,n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \alpha_x{}^m x^n e^{-px^2} dx,$$

m and n being any positive integers and p any positive real number. We easily find the relations

$$\Delta^m A_{m\ 2n+1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

$$\Delta^m A_{m+1,2n} = \tfrac{1}{2}\Delta^m A_{m,2n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

the differences being taken in respect to m.

Evidently many relations of the types considered in this note can be established which will be useful in researches like those of Professors K. Pearson and T. Hojo.

(vi) Note on the Shrinkage of Physical Characters in Man and Woman with Age, as an illustration of the use of χ^2, P Methods.

BY PAMELA C. V. LESSER.

It is known that when large series of measurements are taken on adult men and women the chief physical characters tend sensibly, if but slightly, to decrease with age, and further in Stature and Brain-Weight this shrinkage appears to be greater in woman than man*. This shrinkage is too slight to be adequately determined on short series, though even there it will be found to exist. The present note is not intended either to measure the relative shrinkage in man and woman, or to determine whether they are really due to the same causes. Its purpose is to indicate on comparatively small samples how it would be possible on more numerous data, to answer the problem of whether shrinkage in physical characters can be attributed to the same set of causes acting with the same intensity in man and woman.

Considering first only Brain-Weight and Stature, I have taken my data for the former from measurements on Bavarians of the two sexes†, choosing the age of 20 years as clearing the primes of both, and as the graphs of brain-weight indicate‡, the start of a slight but continuous shrinkage. The data for stature were taken ultimately from Retzius' "Ueber das Hirngewicht der Schweden," *Biologische Untersuchungen*, N.F. Bd. IX (1900), but I have used the tabulation provided by Pearl§. There was no occasion to adopt the same races for the two investigations, and it would not have been feasible to do so as statures to ages were not given for the Bavarians in Pearl's paper. Pearson's diagrams|| for stature and age indicate that shrinkage with age begins about 20 years for both sexes.

A. We will consider in the first place Brain-Weights.

Here for men the mean brain-weight of the younger group is 1364·05, and of the older group 1347·67, showing a shrinkage of 16·38 grs., which could not be demonstrated as significant on these numbers.

* See K. Pearson, "On our present Knowledge of the Relationship of Mind and Body," *Annals of Eugenics*, Vol. I. (1926) pp. 387, 390.

† Bischoff's data from *Das Gehirngewicht des Menschen*, 1880, as tabled by Raymond Pearl, *Biometrika*, Vol. IV. p. 100.

‡ Pearson, *loc. cit.* p. 390.

§ Pearl, *loc. cit.* p. 88.

|| *Loc. cit.* p. 387.

TABLE I. *Brain-Weights for Age Groups in Bavarians*

Men (weight in grams)

Age Group	1000–1149	1150–1199	1200–1249	1250–1299	1300–1349	1350–1399	1400–1449	1450–1499	1500–1599	1600–1699	Totals
50–80	4	9	16	27	25	35	22	11	9	6	164
20–49	6	7	38	47	59	78	54	34	31	11	365

Women (weight in grams)

Age Group	800–949	950–1049	1050–1099	1100–1149	1150–1199	1200–1249	1250–1299	1300–1349	1350–1399	1400–1599	Totals
50–80	3	6	8	19	11	14	14	3	6	1	85
20–49	0	2	14	22	45	54	55	23	13	10	238

For the women the younger group has a mean of 1244·88 and the older group one of 1189·21, indicating a shrinkage of 55·67 grs., a larger amount than in the case of the men. Dealing first with the men, we may suppose:

(i) That the parent population (from which we consider both samples to be drawn) has its relative frequencies given by the sum of the columns. The corresponding value of χ^2 is

$$\chi^2_{\text{col.}} = \underset{s}{S} \frac{NN'}{N+N'} \frac{\left(\frac{n_s}{N} - \frac{n_s'}{N'}\right)^2}{\frac{n_s + n_s'}{N+N'}},\text{*}$$

where N and N' are the sizes of the two samples, and n_s, n_s' the corresponding frequencies in the sth category. We find

$$\chi^2_{\text{col.}} = 8{\cdot}8479.$$

As in our case we have taken 10 categories we have

$$P_{\chi^2_{\text{col.}}} = {\cdot}452.$$

(ii) That the parent population be that which gives the highest probability of the two samples being drawn from the same population. We will write the corresponding χ^2† as

$$\chi^2_{\text{min.}} = \frac{NN'}{N+N'} \left(\underset{s}{S}\left(\frac{n_s}{N} \sim \frac{n_s'}{N'}\right)\right)^2,$$

and find

$$\chi^2_{\text{min.}} = 3{\cdot}3451, \text{ and } P_{\chi^2_{\text{min.}}} = {\cdot}946.$$

Thus our data for men are inadequate and do not enable us to assert that brain-weight shrinks with age in the males. They do, however, indicate that $\chi^2_{\text{min.}}$ and $\chi^2_{\text{col.}}$ may give widely different measures of the probability of the two samples coming from the same parent population.

Turning to the women, we find

$$\chi^2_{\text{col.}} = 36{\cdot}4146, \text{ and } \chi^2_{\text{min.}} = 19{\cdot}6177,$$

corresponding respectively to

$$P_{\chi^2_{\text{col.}}} = {\cdot}0002, \text{ and } P_{\chi^2_{\text{min.}}} = {\cdot}021.$$

* *Biometrika*, Vol. VIII. p. 250. † *Biometrika*, Vol. XXIV. p. 459.

Thus by the first method of $\chi^2_{\text{col.}}$ we should be prepared to accept the hypothesis that the old and young women have not been drawn from the same population, i.e. that the shrinkage with age is demonstrated. But by the second method which gives the parent population of maximum probability we should be more doubtful of the truth of this hypothesis, and some might, with the limit $P > \cdot 02$, be prepared to consider that they might well be drawn from the same parent population. This example again illustrates how important it is that we should actually take into account what is the parent population we are supposed to be dealing with.

We have now at our disposal four χ^2's, namely

	$\chi^2_{\text{col.}}$	$\chi^2_{\text{min.}}$
Men	8·8479	3·3451
Women	36·4146	19·6177

and we will denote the female values by adding a dash.

We next ask if the $\chi^2_{\text{col.}}$'s or the $\chi^2_{\text{min.}}$'s will the better enable us to determine whether the like causes are at work in the cases of men and women. We may do this in two ways*, either by considering the improbability of a ratio greater than χ'^2/χ^2 or a difference greater than $\frac{1}{2}(\chi'^2 - \chi^2)$. Our results give

$$\chi'^2_{\text{col.}}/\chi^2_{\text{col.}} = \frac{36\cdot 4146}{8\cdot 8479} = 4\cdot 1156,$$

$$\chi'^2_{\text{min.}}/\chi^2_{\text{min.}} = \frac{19\cdot 6177}{3\cdot 3451} = 5\cdot 8646,$$

and

$$\tfrac{1}{2}(\chi'^2_{\text{col.}} - \chi^2_{\text{col.}}) = 13\cdot 7833,$$

$$\tfrac{1}{2}(\chi'^2_{\text{min.}} - \chi^2_{\text{min.}}) = 8\cdot 1363.$$

From Table II of *Biometrika*† we deduce

$$P(\chi'^2_{\text{col.}}/\chi^2_{\text{col.}}) = 2I_{\cdot 19548}(4\cdot 5,\ 4\cdot 5) = \cdot 0469,$$

$$P(\chi'^2_{\text{min.}}/\chi^2_{\text{min.}}) = 2I_{\cdot 14567}(4\cdot 5,\ 4\cdot 5) = \cdot 0147.$$

From these results we should conclude that on the basis of choosing for both men and women the most probable parent populations, it is extremely likely that the shrinkage of brain in women is not due to the same causes as in the case of men. But if we took the parent population to have relative frequencies determined by the columnar marginal totals, this conclusion would be very doubtful. Considering that our data actually do show a significant difference between the brain-weights of young and old women, and do not show the like for men, the former conclusion certainly appears the more reasonable.

Turning now to the difference method we find, by Table I in *Biometrika*‡,

$$P(\tfrac{1}{2}\chi'^2_{\text{col.}} - \tfrac{1}{2}\chi^2_{\text{col.}}) = 2(\cdot 5 - \cdot 499{,}915) = \cdot 0002,$$

$$P(\tfrac{1}{2}\chi'^2_{\text{min.}} - \tfrac{1}{2}\chi^2_{\text{min.}}) = 2(\cdot 5 - \cdot 493{,}968) = \cdot 0121.$$

Probably (since $P < \cdot 02$) we should conclude from both these results that a real difference exists between the cases of men and women. But the discordance of $P(\frac{1}{2}\chi'^2_{\text{col.}} - \frac{1}{2}\chi^2_{\text{col.}}) = \cdot 0002$ and $P(\chi'^2_{\text{col.}}/\chi^2_{\text{col.}}) = \cdot 0469$, and the accordance of $P(\frac{1}{2}\chi'^2_{\text{min.}} - \frac{1}{2}\chi^2_{\text{min.}}) = \cdot 0121$ with $P(\chi'^2_{\text{min.}}/\chi^2_{\text{min.}}) = \cdot 0147$, certainly suggest that it is better to use the most likely parent population than that obtained from the samples combined.

B. We next take the case of Stature. Here, as in the case of brain-weights, adequate data indicate a definite if small shrinkage with age§. The mean stature of the men from 50 to 80 years is 168·831, and of the men from 20 to 50 169·504, indicating a shrinkage of 0·67 cms.

* *Biometrika*, Vol. XXIV. pp. 304—330. † *Loc. cit.* p. 347.

‡ *Loc. cit.* p. 344.

§ See diagrams, *Annals of Eugenics*, Vol. II. p. 100, and Vol. III. p. 291, as confirming those of the same journal, Vol. I. p. 387.

In the case of the women the like groups give 157·113 and 158·417, or a shrinkage of 1·3 cms. These values accord with experience from more numerous data, but standing alone cannot be considered as significantly demonstrating anything. As my purpose is not to demonstrate anything but only to illustrate the use of methods, I have not sought for long series. The shrinkage is, of course, less in stature or span than in weight or in the dynamic characters. It is particularly notable in the case of vital capacity*.

Table II gives the results for stature in a small group of Swedes†.

TABLE II.

Men (stature in cms.)

Age Group	144–152	153–158	159–161	162–164	165–167	168–170	171–173	174–176	177–179	180–182	183–191	Totals
50—80	3	8	11	18	26	26	28	13	12	6	3	154
20—50	4	4	18	29	46	45	38	32	32	8	6	262

Women (stature in cms.)

Age Group	132–143	144–146	147–149	150–152	153–155	156–158	159–161	162–164	165–167	168–170	171–176	Totals
50—80	2	1	8	9	24	21	12	15	12	2	0	106
20—50	2	5	6	5	20	19	18	23	16	9	4	127

We proceed first to find the $\chi^2_{\text{col.}}$ and $\chi^2_{\text{min.}}$ for men. They are

$$\chi^2_{\text{col.}} = 8·8532, \text{ leading to } P = ·546.$$

$$\chi^2_{\text{min.}} = 3·5031, \text{ leading to } P = ·964.$$

Hence, proceeding by either method, we conclude that no evidence can be drawn from these small samples of a shrinkage in stature with age.

Turning to the women we find

$$\chi^2_{\text{col.}} = 14·6957, \text{ leading to } P = ·145,$$

$$\chi^2_{\text{min.}} = 8·7271, \text{ leading to } P = ·559.$$

We therefore draw the same conclusion for women as for men, but remark that the probabilities are in both cases for women very much less than for men. These four results again emphasize the importance of considering what parent population is under consideration; the most likely parent population giving markedly higher probabilities of no distinction between old and young in both men and women.

We now consider whether the values of χ^2's reached show any difference in the case of men and women.

We have, if dashed letters refer to women,

$$\chi'^2_{\text{col.}}/\chi^2_{\text{col.}} = \frac{14·6957}{8·8532} = 1·6599,$$

$$\chi'^2_{\text{min.}}/\chi^2_{\text{min.}} = \frac{8·7271}{3·5031} = 2·4912.$$

* See *Annals of Eugenics*, Vol. II. p. 105, and Vol. III. p. 296.

† *Biometrika*, Vol. IV. pp. 88—89.

The corresponding probabilities are for the ratio test:

$$P_{(\chi'^2_{\text{col.}}/\chi^2_{\text{col.}})} = 2I_{\cdot 37395}(5, 5) = \cdot 437,$$
$$P_{(\chi'^2_{\text{min.}}/\chi^2_{\text{min.}})} = 2I_{\cdot 28648}(5, 5) = \cdot 166.$$

The χ^2's of the most likely populations give a lesser probability that the causes are accordant in men and women than the columnar populations, but in the case of both we cannot assert that women differ from men.

Proceeding now to the difference test we have

$$\left.\begin{array}{l} \tfrac{1}{2}(\chi'^2_{\text{col.}} - \chi^2_{\text{col.}}) \;= 2\cdot 9212 \\ \tfrac{1}{2}(\chi'^2_{\text{min.}} - \chi^2_{\text{min.}}) = 2\cdot 6120 \end{array}\right\} \text{ leading respectively to}$$
$$P_{(\frac{1}{2}\chi'^2_{\text{col.}} - \frac{1}{2}\chi^2_{\text{col.}})} = 2(\cdot 5 - \cdot 334{,}215) = \cdot 332,$$
$$P_{(\frac{1}{2}\chi'^2_{\text{min.}} - \frac{1}{2}\chi^2_{\text{min.}})} = 2(\cdot 5 - \cdot 308{,}648) = \cdot 383.$$

By this method there is little to choose between the two types of parent populations. From both we should conclude that we could not assert on the data any difference between men and women. This result might have been more or less anticipated, as our data were too sparse to distinguish in either sex between young and old. It will be noted that the ratio test gives in this case results varying more with the parent population selected than the difference test does, but no stress can be laid on this.

It is intended to discuss the problem suggested in this note more fully later on the ample data from Galton's first Anthropometric Laboratory. Meanwhile it seemed worth while to work out from that material a special case because it illustrates how divergent may be the conclusions to be drawn from the minimum χ^2 parent population and the columnar totals population. We deal with the case of Vital Capacity and Age.

Men (Vital Capacity in cm.^{3})

Age Group	50–139	140–154	155–169	170–184	185–199	200–214	215–229	230–244	245–259	260–274	275–289	290 and over	Totals
20—31	50	53	114	244	272	469	456	414	248	218	114	153	2805
31 and over	141	155	159	338	319	447	352	304	133	151	64	59	2622

$$\chi^2_{\text{min.}} = \frac{NN'}{N+N'}\left[S\left(\frac{n_s}{N} \sim \frac{n_s'}{N'}\right)\right]^2 = 149\cdot 4902.$$
$$\chi^2_{\text{col.}} = S\left(\frac{NN'}{n_s+n_s'}\left(\frac{n_s}{N} - \frac{n_s'}{N'}\right)^2\right) = 248\cdot 2215.$$

By both methods $P = \cdot 000{,}000\ldots\ldots$, but there are several more zero's before we come to a significant figure in $P_{\chi^2_{\text{col.}}}$ then in $P_{\chi^2_{\text{min.}}}$. Undoubtedly there is a shrinkage in vital capacity with age in men.

Women (Vital Capacity in cm.^{3})

Age Group	27·5–82·4	82·5–92·4	92·5–102·4	102·5–112·4	112·5–122·4	122·5–132·4	132·5–142·4	142·5–152·4	152·5–162·4	162·5–172·4	172·5–182·4	182·5 and over	Totals
18—29	22	17	42	49	86	108	126	100	71	59	41	41	762
29 and over	82	46	89	99	130	111	110	81	69	42	25	25	909

$$\chi^2_{\text{min.}} = 71\cdot 1734 \text{ and } \chi^2_{\text{col.}} = 92\cdot 2205.$$

In both cases $P = \cdot 000{,}000\ldots\ldots$, although again the $P_{\chi^2_{\text{min.}}}$ will give a somewhat greater probability than the $P_{\chi^2_{\text{col.}}}$. Both demonstrate with overwhelming probability that women's Vital Capacity shrinks with age. But however overwhelming the probability, it is greater in the case of men.

We now turn to the fundamental problem: Can this shrinkage in men and women be attributed to the same set of physical or physiological causes?

We apply first the ratio test. We have

$$\chi'^2_{\text{min.}}/\chi^2_{\text{min.}} = \frac{149\cdot 490217}{71\cdot 173415} = 2\cdot 1004,$$

$$\chi'^2_{\text{col.}}/\chi^2_{\text{col.}} = \frac{248\cdot 221460}{92\cdot 220480} = 2\cdot 6916.$$

These lead us to*

$$P_{(\chi'^2_{\text{min.}}/\chi^2_{\text{min.}})} = 2I_{\cdot 32254}(5\cdot 5,\ 5\cdot 5) = \cdot 2342,$$
$$P_{(\chi'^2_{\text{col.}}/\chi^2_{\text{col.}})} = 2I_{\cdot 27089}(5\cdot 5,\ 5\cdot 5) = \cdot 1154.$$

While the ratio of the $\chi^2_{\text{min.}}$'s gives, as we might expect, a higher degree of probability than the ratio of the $\chi^2_{\text{col.}}$'s, there is nothing in either to suggest that it would not be reasonable to treat the shrinkage in men and women as due to the same set of causes. A glance, however, at the curve for n categories on p. 308 of *Biometrika*, Vol. XXIV, shows us that the point corresponding to the two χ^2's lies inside the curve, or that the difference test will provide a much more stringent test than the ratio test.

We have accordingly†

$$\tfrac{1}{2}(\chi'^2_{\text{min.}} - \chi^2_{\text{min.}}) = 39\cdot 158{,}396,$$
$$\tfrac{1}{2}(\chi'^2_{\text{col.}} - \chi^2_{\text{col.}}) = 78\cdot 000{,}485,$$

and
$$P_{(\frac{1}{2}(\chi'^2_{\text{min.}} - \chi^2_{\text{min.}}))} = 2\left[\cdot 5 - S_6(\tfrac{1}{2}\chi'^2_{\text{min.}} - \tfrac{1}{2}\chi^2_{\text{min.}})\right]$$
$$= 2\left[\cdot 5 - \text{something considerably greater than } -\cdot 499{,}993\right].$$

Thus we have
$$P_{(\frac{1}{2}(\chi'^2_{\text{min.}} - \chi^2_{\text{min.}}))} < \cdot 000{,}012.$$

In the same manner
$$P_{(\frac{1}{2}(\chi'^2_{\text{col.}} - \chi^2_{\text{col.}}))} < \cdot 000{,}012,$$

and considerably less than $P_{(\frac{1}{2}(\chi'^2_{\text{min.}} - \chi^2_{\text{min.}}))}$.

It cannot be doubted that if Table I for $S_m(x)$ had been carried further, we should have found both P's less than 1 in 1,000,000 at least. We conclude therefore that with the more stringent test the shrinkage of vital capacity with age is not due to the same causes in man and woman, but it is a secondary sexual character very possibly due to differences not only in their physical environments, but in their physiological life.

This example illustrates the extreme importance of applying in each case the more stringent test whether it be the χ^2's ratio or their difference. We may well overlook an important conclusion, if we do not bear this in mind. In this particular example there is no marked diversity in result if we use $\chi^2_{\text{min.}}$ or $\chi^2_{\text{col.}}$. But this will not always be so, and in the case of shrinkage of brain-weight in women our conclusion will be far less assertive in the case of $P_{\chi^2_{\text{min.}}}$ than in that of $P_{\chi^2_{\text{col.}}}$.

* Using Table II, *Biometrika*, Vol. XXIV. p. 348, with $v = n = 12$.

† Using Table I, *Biometrika*, Vol. XXIV. p. 345, where we see $S_6(39\cdot 158{,}396)$ and $S_6(78\cdot 000{,}485)$ lie well beyond the limits of the table.

(vii) On the Distribution of Student's Ratio for Samples of Three Drawn from a Rectangular Distribution.

BY VICTOR PERLO, M.A., Columbia University.

Let samples be drawn from a continuous distribution of finite range, the chance of a value lying in a given interval within this range being proportional to the length of the interval. The distributions of most of the important statistical measures for this type of population are not known. This paper presents the distribution of t for samples of three, and some comparisons with Student's distribution.

The statistical measure t is defined, for samples of n, as $(\bar{x}-m)\sqrt{n}/s$, where $\bar{x}$ is the sample mean, m is a trial value of the true mean, and $s^2=\frac{\Sigma(x-\bar{x})^2}{n-1}$. For samples of three, then,

$$t=(\bar{x}-m)\sqrt{3}/s.$$

If we regard a sample as a point in three dimensional space, we find by methods similar to those used by H. L. Rietz in determining the distribution of the standard deviation of similar samples*, that the distribution of t is determined by that of the angle between a cube's diagonal and the radius vector drawn from the cube's centre to a point within the cube, which reduces to the problem of computing the volume within the cube of a conical shell with axis the cube's diagonal and vertex the cube's centre. We get for the distribution ordinate:

$$\frac{-9}{4(t+1)(t^2-4)}\left(\frac{1}{t+1}+\frac{3t}{t^2-4}\right)+\frac{3^{\frac{3}{2}}(t^2+2)}{(t^2-4)^{\frac{5}{2}}}\tan^{-1}\frac{\sqrt{t^2-4}}{\sqrt{3}(t+2)}\quad(\infty\geqslant t\geqslant\tfrac{1}{2})\;\ldots\ldots\ldots(1),$$

$$\frac{\sqrt{3}}{2(4-t^2)\sqrt{1-t^2}}\left(1-\frac{9t^2}{4-t^2}\right)+\frac{3^{\frac{3}{2}}(2+t^2)}{(4-t^2)^{\frac{5}{2}}}\tanh^{-1}\sqrt{\frac{1-t^2}{4-t^2}}\quad(\tfrac{1}{2}\geqslant t\geqslant 0)\;\ldots\ldots\ldots(2).$$

For $2\geqslant t\geqslant\frac{1}{2}$, the last term of (1) becomes

$$\frac{3^{\frac{3}{2}}(t^2+2)}{(4-t^2)^{\frac{5}{2}}}\tanh^{-1}\frac{\sqrt{4-t^2}}{\sqrt{3}(t+2)}$$

to preserve reality.

The distribution is continuous with continuous derivatives except at $t=\pm\frac{1}{2}$, where the derivative has points of discontinuity. This function may well be compared with "Student's" distribution for samples of three drawn from a normally distributed population. Plotting shows it to be greater than "Student's" at the ends and the middle, and less elsewhere, that is, more leptokurtic. Also interesting is a comparison with the extremely simple expression for the probability ordinate for samples of two, as derived by Rider,

$$\frac{1}{2(1+|t|)^2}\,\dagger.$$

Let P be the probability of a sample having $|t|\geqslant|t_0|$, where t_0 is some fixed value of the argument. Geometrically it is twice the volume of a cone of angle ϕ_0 $(\phi=f(t))$ within the cube (i.e. the cone about the cube's diagonal with vertex the cube's centre plus its opposite). Integrating the expressions (1) and (2) over the appropriate intervals, dropping the subscript, and assuming t positive, we get

$$P=\frac{-9}{2(t+1)(t^2-4)}+\frac{3^{\frac{3}{2}}t}{(t^2-4)^{\frac{3}{2}}}\tan^{-1}\frac{\sqrt{t^2-4}}{\sqrt{3}(t+2)}\quad(\infty>t\geqslant\tfrac{1}{2})\;\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$P=1+\frac{\sqrt{3}t\sqrt{1-t^2}}{4-t^2}-\frac{3^{\frac{3}{2}}t}{(4-t^2)^{\frac{3}{2}}}\sqrt{\frac{1-t^2}{4-t^2}}\quad(\tfrac{1}{2}\geqslant t\geqslant 0)\;\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

* H. L. Rietz, Note on the Distribution of the Standard Deviation, etc., *Biometrika*, vol. XXIII. 1931, pp. 424—426.

† Paul R. Rider, On the Distribution of the Ratio of Mean to Standard Deviation, etc., *Biometrika*, vol. XXI. 1929, pp 124—143, especially pp. 140—141. The formula in Rider's paper is incorrectly stated as $\frac{1}{2(1-|t|)^2}$. Corresponding errors appear in the cumulated probability expressions.

For calculations of the ordinate and P rational approximations to (1) and (3) are obtained by expanding the inverse tangent in powers of its argument. For (1) we get

$$\frac{8t+9}{3(t+1)^2(t+2)^2}+\frac{4-t}{12(t+1)^2(t+2)^4}+\frac{t^2+2}{15(t+2)^5}-\frac{(t^2+2)(t-2)}{63(t+2)^6}+\dots+(-1)^n\frac{(t^2+2)(t-2)^{n-2}}{(2n+1)(t+2)^{n+3}3^{n-1}}+\dots$$

(3) gives in a similar expansion

$$P=\frac{3}{2}\cdot\frac{(2t+3)}{(t+1)(t+2)^2}-\frac{t}{3(t+2)^3}+\frac{t(t-2)}{15(t+2)^4}-\dots+(-1)^n\frac{t(t-2)^{n-1}}{3^{n-1}(2n+1)(t+2)^{n+2}}+\dots$$

For $t>2$, the first three terms of these series give results correct to at least four places.

Let p be the probability that t exceeds a given value obtained from "Student's" distribution corresponding to the true P found above. The fiduciary limits most frequently applied in testing for significant deviations of sample means from true means are ·05 and ·01. The following table shows values of t for which p or P take on those values.

t	p	P
4·30	·0500	·0774
5·74	·0292	·0500
9·90	·0100	·0204
14·85	·0045	·0100

The limit of P/p as t approaches infinity is found by comparing the first terms of the expansions in powers of t^{-1} of (3) and the expression for p, $1-\frac{t}{\sqrt{2+t^2}}$. The limiting ratio obtained is $\frac{\pi\sqrt{3}}{2}=2·7207$.

(viii) The Distribution of $\sqrt{\beta_1}$ in Samples of Four from a Normal Universe.

By A. T. McKAY, M.Sc.

The estimated value of $\sqrt{\frac{3}{4}\beta_1}$ for samples of 4 is the statistic

$$\theta=\frac{\sqrt{3}\sum_1^4(x_r-\bar{x})^3}{\left[\sum_1^4(x_r-\bar{x})^2\right]^{\frac{3}{2}}}\quad\dots\dots(1).$$

Now
$$\sum_1^4(x_r-\bar{x})^3=\{(x_1-\bar{x})^3+(x_2-\bar{x})^3\}+\{(x_3-\bar{x})^3+(x_4-\bar{x})^3\}\quad\dots\dots(2),$$

and factorising each curled bracket separately we find

$$(x_1+x_2-2\bar{x})A+(x_3+x_4-2\bar{x})B,$$

where A and B represent the remaining factors. But obviously $(x_1+x_2-2\bar{x})=-(x_3+x_4-2\bar{x})$, hence this term is a factor of the expression (2). By taking a different grouping of terms in (2) we may readily show that

$$\sum_1^4(x_r-\bar{x})^3=\tfrac{3}{8}(x_1+x_2-x_3-x_4)(x_1+x_3-x_2-x_4)(x_1+x_4-x_2-x_3)\quad\dots\dots(3).$$

In equation (1) let us now employ the following orthogonal transformation

$$\left.\begin{aligned}y_0&=(x_1+x_2+x_3+x_4)/2\\y_1&=(x_1+x_2-x_3-x_4)/2\\y_2&=(x_1+x_3-x_2-x_4)/2\\y_3&=(x_1+x_4-x_2-x_3)/2\end{aligned}\right\}\quad\dots\dots(4),$$

which yields
$$\theta_1=\frac{y_1y_2y_3}{\left[\frac{1}{3}(y_1^2+y_2^2+y_3^2)\right]^{\frac{3}{2}}}\quad\dots\dots(5).$$

Hence in virtue of the well-known orthogonal property of the normal function, we see that our problem reduces to seeking the distribution of the statistic θ_1 in samples of *three* from a normal universe. By partial differentiation of (5) we find that there is an absolute maximum of θ_1 when $y_1 = y_2 = y_3$, thus the distribution of θ_1, and therefore of θ, has termini at ± 1. This means that the distribution of $\sqrt{\beta_1}$ terminates at $\pm 2/\sqrt{3}$. Since the distribution is symmetrical about zero we have only even moments, which are given by

$$\mu_{2n}(\theta_1) = \frac{3^{3n}}{(2\pi)^{\frac{3}{2}}} \iiint_{-\infty}^{\infty} \left[\frac{y_1 y_2 y_3}{(\Sigma y^2)^{\frac{3}{2}}}\right]^{2n} e^{-\frac{1}{2}\Sigma y^2}\, dy_1\, dy_2\, dy_3 \quad \text{.....................(6).}$$

Changing to polar coordinates we derive

$$\mu_{2n}(\theta_1) = \frac{3^{3n}}{(2\pi)^{\frac{3}{2}}} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} (\cos\theta \sin^2\theta \cos\phi \sin\phi)^{2n} \sin\theta\, r^2 e^{-r^2/2}\, dr\, d\theta\, d\phi \quad \text{.........(7).}$$

The integrand of (7) is separable, and the use of standard formulae readily yields

$$\mu_{2n}(\theta_1) = \frac{3^{3n}}{2\pi} \frac{\Gamma^3\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{6n+3}{2}\right)} \quad \text{.....................................(8).}$$

Hence $\mu_2(\theta_1) = \frac{9}{35}$ and $\mu_4(\theta_1) = \frac{729}{5005}$, and thus making allowance for the factor of $\sqrt{\frac{4}{3}}$ in equation (1), $\mu_2(\sqrt{\beta_1}) = \frac{12}{35}$ and $\mu_4(\sqrt{\beta_1}) = \frac{1296}{5005}$ giving $B = \frac{315}{143}$. These values are confirmed from an expression of R. A. Fisher's*.

Let us now proceed to find the distribution of θ_1. From equation (5), we see that it is necessary to integrate

$$\frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}\Sigma y^2}\, dy_1\, dy_2\, dy_3 \quad \text{...(9)}$$

over the field of integration for which

$$w < \frac{3^{\frac{3}{2}} y_1 y_2 y_3}{(\Sigma y^2)^{\frac{3}{2}}} < w + \delta w \quad \text{.......................................(10),}$$

or transforming to polar coordinates, we require to integrate

$$\frac{1}{(2\pi)^{\frac{3}{2}}} r^2 e^{-r^2/2}\, dr \sin\theta\, d\theta\, d\phi \quad \text{....................................(11)}$$

over the field for which

$$w < \frac{3^{\frac{3}{2}}}{2} \sin^2\theta \cos\theta \sin 2\phi < w + \delta w \quad \text{...............................(12),}$$

where $0 \leqslant r \leqslant \infty$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$.

Now since (12) is independent of r, the latter variable can be integrated out in (11), hence the subject of integration is thus

$$\frac{1}{4\pi} \sin\theta\, d\theta\, d\phi \quad \text{..(13).}$$

Regarding w as positive, which is merely equivalent to treating the positive half of the distribution, and writing $\cos\theta = x$, we have to integrate

$$\frac{1}{\pi}\, dx\, d\phi \quad \text{...(14)}$$

over the field for which

$$w < \frac{3^{\frac{3}{2}}}{2} x(1 - x^2) \sin\phi < w + \delta w \quad \text{................................(15),}$$

* R. A. Fisher, *Proc. Roy. Soc.* Ser. A, Vol. 130, No. A, 812, or E. S. Pearson, *Biometrika*, Vol. XXII. Parts III. and IV. 1931 (Miscellanea).

where now $0 \leqslant x \leqslant 1$ and $0 \leqslant \phi \leqslant \pi/2$. Performing the integration with respect to ϕ, we see that we have to effect the integration

$$\frac{1}{\pi}\int \frac{\partial \phi}{\partial w}\,dx \qquad \text{.....(16)}$$

over the field $0 \leqslant x \leqslant 1$ conditioned by

$$2w - 3^{\frac{3}{2}}x(1-x^2)\sin\phi \geqslant 0 \qquad \text{.....(17)}$$

The equation (16) thus becomes

$$\frac{1}{\pi}\int \frac{\partial}{\partial w}\sin^{-1}(2w/3^{\frac{3}{2}}x(1-x^2))\,dx \qquad \text{.....(18)},$$

where the limits for x are determined from the fact that x runs from 0 to 1, subject to the condition that

$$0 \leqslant 2w/3^{\frac{3}{2}}x(1-x^2) \leqslant 1 \qquad \text{.....(19)}.$$

Designating the distribution function sought by $f(w)$, i.e. the expression (18), we have then

$$f(w) = \frac{2}{\pi}\int_{x_1}^{x_2} \frac{dx}{\sqrt{27x^2(1-x^2)^2 - 4w^2}} \qquad \text{.....(20)},$$

where x_1 and x_2 are to be determined from (19).

By changing the variable in (20) we have

$$f(w) = \frac{1}{\pi}\int_{t_1}^{t_2} \frac{dt}{\sqrt{t\,\{27t(1-t)^2 - 4w^2\}}} \qquad \text{.....(21)},$$

where t_1 and t_2 are to be found from

$$\left.\begin{array}{l} 0 \leqslant 4w^2/27t(1-t)^2 \leqslant 1 \\ 0 \leqslant t \leqslant 1 \end{array}\right\} \qquad \text{.....(22)}.$$

Thus, from (21), we conclude that $f(w)$ is a complete elliptic integral, for the limits are singularities of the integrand*.

In equation (22) we observe that the element between the two inequality signs has a minimum value at $t = \frac{1}{3}$, hence for our limits of the integral we require the two roots of the cubic

$$27t(1-t)^2 - 4w^2 = 0 \qquad \text{.....(23)}$$

which lie nearest to, and on each side of, the value $t = \frac{1}{3}$. This will be readily seen from a rough graph. Solving the cubic (23) by the usual method, the appropriate roots are found to be

$$\left.\begin{array}{ll} t_1 = \frac{4}{3}\cos^2(\frac{1}{3}\cos^{-1}w + \pi/3) & 0 < t_1 < \frac{1}{3} \\ t_2 = \frac{4}{3}\cos^2(\frac{1}{3}\cos^{-1}w + 2\pi/3) & \frac{1}{3} < t_2 < 1 \end{array}\right\} \qquad \text{.....(24)}.$$

In (21) let us now make the substitution to a new variable y defined by

$$t = \frac{4}{3}\cos^2(\frac{1}{3}\cos^{-1}y + 2\pi/3),$$

then

$$f(w) = \frac{2}{\pi\,3^{\frac{3}{2}}}\int_{-w}^{+w} \frac{\sin(\frac{1}{3}\cos^{-1}y + 2\pi/3)}{\sqrt{(1-y^2)(y^2-w^2)}}\,dy \qquad \text{.....(25)},$$

or putting $y = \sqrt{1-x^2}$ we have

$$f(w) = \frac{2}{\pi\,3^{\frac{3}{2}}}\int_{-\Omega}^{+\Omega} \frac{\sin(\frac{1}{3}\sin^{-1}x + 2\pi/3)}{\sqrt{(1-x^2)(\Omega^2-x^2)}}\,dx \qquad \text{.....(26)},$$

where $\Omega^2 = 1 - w^2$. Expanding the sine term and noting the disappearance of the "odd" part we have

$$f(w) = \frac{2}{3\pi}\int_0^{\Omega} \frac{\cos(\frac{1}{3}\sin^{-1}x)}{\sqrt{(1-x^2)(\Omega^2-x^2)}}\,dx \qquad \text{.....(27)}.$$

* Whittaker and Watson, *Modern Analysis* (1920), see § 22·32 *et seq*. We might infer from this that $f(w)$ will prove to be a hypergeometric function.

Writing in this $x = \sin\theta$ and $\Omega = \sin\alpha$,

$$f(w) = \frac{2}{3\pi}\int_0^{\alpha} \frac{\cos\theta/3\, d\theta}{\sqrt{\sin^2\alpha - \sin^2\theta}} \quad\text{.......................}(28)$$

$$= \frac{2}{3\pi}\int_0^{2\alpha} \frac{\cos\phi/6\, d\phi}{\sqrt{2(\cos\phi - \cos 2\alpha)}} \quad\text{.......................}(29).$$

But the latter is Mehler's Integral* for the Legendre Function, hence

$$f(w) = \tfrac{1}{3} P_{-\frac{1}{3}}(\cos 2\alpha) \quad\text{.......................}(30)$$

$$= \tfrac{1}{3} F(\tfrac{1}{3}, \tfrac{2}{3}; 1; (1 - w^2)) \quad\text{.......................}(31),$$

where $F(a, b; c; x)$ is the usual hypergeometric function notation.

Just as a check on our analysis we may now seek the moments of the distribution by proceeding from (31),

$$\mu_{2n}(\theta_1) = \tfrac{2}{3}\int_0^1 w^{2n} F(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1 - w^2)\, dw \quad\text{.......................}(32)$$

$$= \tfrac{2}{3}\int_0^1 \sin^{2n}\theta \cos\theta\, F(\tfrac{1}{3}, \tfrac{2}{3}; 1; \cos^2\theta)\, d\theta \quad\text{.......................}(33).$$

Expanding the hypergeometric function and integrating term by term we find

$$\mu_{2n} = \frac{1}{3}\, \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2n+3}{2}\right)}\, F\left(\tfrac{1}{3}, \tfrac{2}{3}, \frac{2n+3}{2}, 1\right) \quad\text{.......................}(34)$$

$$= \frac{1}{3}\, \frac{\Gamma^2\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{6n+7}{6}\right)\Gamma\left(\frac{6n+5}{6}\right)} \quad\text{.......................}(35).$$

Whence by employing the Gamma Function Triplication formula† with argument $(n + \frac{1}{2})$ we derive the result stated in (8).

From equation (31) we conclude that the distribution of $\sqrt{\beta_1}$ in samples of 4 from a normal universe is

$$\phi(x) = \frac{1}{2\sqrt{3}}\, F(\tfrac{1}{3}, \tfrac{2}{3}, 1; 1 - \tfrac{3}{4}x^2) \quad\text{.......................}(36).$$

Now it is known that when $a + b - c = 0$, the hypergeometric function $F(a, b; c; t)$ is convergent when $|t| < 1$ and divergent when $t = 1$, whence we conclude that the distribution $\phi(x)$ has the following properties:

(i) Symmetrical.

(ii) A cusp at infinity when $x = 0$.

(iii) A finite ordinate of $1/2\sqrt{3} = 0{\cdot}288675$ at the termini $x = \pm 2/\sqrt{3} = \pm 1{\cdot}1547$.

(iv) $\sigma^2 = \frac{12}{35}$, $B_2 = \frac{213}{143}$.

The graph of the curve is shown in the figure, together with the Pearson Type curve, which results by using the second and fourth moments of the true distribution. The calculation of the latter is as follows:

$$\mu_2 = \tfrac{12}{35}, \quad B_2 = \tfrac{213}{143},$$

* Whittaker and Watson, *loc. cit.* §§ 15·291 and 15·22.

† Whittaker and Watson, *loc. cit.* § 12·15.

whence using Elderton's notation

$$m=\tfrac{24}{19}=1{\cdot}263158,\quad a=6/\sqrt{19}=1{\cdot}376494,\quad y_0=0{\cdot}58372,\quad a^2=1{\cdot}894737,$$

$$y=0{\cdot}58372\,(1-x^2/1{\cdot}894737)^{1{\cdot}263158} \qquad \text{.....(37)},$$

with termini at $x=\pm 1{\cdot}376494$.

Approximation to the Curve.

By use of Stirling's approximation in the general term of the hypergeometric series and comparing the approximation derived therefrom with the true coefficients obtained by direct calculation, it may be shown that

$$F(\tfrac{1}{3},\tfrac{2}{3};1,z)=\psi(z)+\epsilon_1(z)-\alpha\epsilon_2(z) \qquad \text{.....(38)},$$

where

$$\alpha=0{\cdot}275663,\quad \psi(z)=1-\alpha\log(1-z),$$

$$\epsilon_1(z)=(0{\cdot}001488z+0{\cdot}000120z^2+\ldots),\quad \epsilon_2(z)=\sum_1^\infty z^n\,\frac{(1-e^{-2/9n})}{n}.$$

Now for $0\leqslant z\leqslant 1$, z^n and $(1-e^{-2/9n})/n$ are always positive and decrease steadily as n increases, hence by Cauchy's Integral test,

$$\epsilon_2(z)<z\,(1-e^{-2/9})+\int_1^\infty z^t\,(1-e^{-2/9t})\,\frac{dt}{t} \qquad \text{.....(39)}.$$

By use of the mean value theorem and a change of variable we find

$$\epsilon_2(z)<0{\cdot}19920z+z\int_0^{2/9}(1-e^{-y})\,\frac{dy}{y} \qquad \text{.....(40)}.$$

The latter integral may be evaluated from the British Association Tables (Vol. I. Table VII) with the result

$$\epsilon_2(z)<0{\cdot}4097z \qquad \text{.....(41)},$$

hence

$$\alpha\epsilon_2(z)-\epsilon_1(z)<0{\cdot}1115z \qquad \text{.....(42)}.$$

Thus the percentage error in taking $\psi(z)$ as an approximation is numerically less than or equal to

$$11{\cdot}15z/\{1-\alpha\log(1-z)\} \qquad \text{.....(43)},$$

so that

$$\lim_{z\to 0 \text{ or } 1}\{\psi(z)-F(\tfrac{1}{3},\tfrac{2}{3};1,z)\}\to 0 \qquad \text{.....(44)}.$$

The expression (43) can be shown to have a maximum value of 6·22 % at $z=0{\cdot}8443$, so we may conclude that $\psi(z)$ errs in excess by less than 6·22 % of its own value. Thus with a percentage error of at most 6·22 % the distribution function $\phi(x)$ of equation (36) is given by

$$\phi(x)\sim 0{\cdot}311568-0{\cdot}366466\log_{10}|x| \qquad \text{.....(45)}.$$

The latter is, of course, considerably more accurate when x is very small or very near the limit of the range.

By integrating equation (45) over the entire range of the distribution we obtain the value 1·087 instead of unity. Thus the total error of the entire area is 8·7 %, so that when integrating over a subrange this error can be simply apportioned.

The Rectangular Universe.

Whenever the distribution of a statistic in samples of n is determined, it is always of considerable interest to enquire to what extent the form of the derived distribution is dependent on that of the parent. In our case, for example, we shall consider the distribution of $\sqrt{\beta_1}$ in samples of 4 from a rectangular universe. We proceed by means of a sampling experiment, using the first 200 samples of 4 given by Shewhart*. Since the means and standard deviations are also

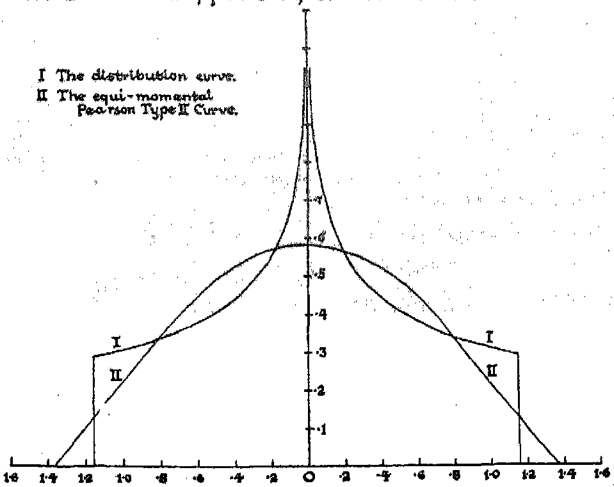
* W. A. Shewhart, *Economic Control of Quality of Manufactured Product* (Macmillan, 1931), Appendix II, Table B.

recorded in his Table E, the necessary calculation was not so difficult. The final frequency distribution is shown in the table below.

Range for $\mid\sqrt{\beta_1}\mid$	Frequency (Rectangular)	Theor. Freq. (Normal)	χ^2
·0— ·1	44	31·40	5·05
·1— ·2	22	24·28	·21
·2— ·3	22	20·77	·07
·3— ·4	18	18·57	·02
·4— ·5	17	16·92	·00
·5— ·6	15	15·64	·02
·6— ·7	10	14·65	1·47
·7— ·8	13	13·88	·06
·8— ·9	11	13·34	·41
·9—1·0	9	12·61	1·03
1·0—1·1	14	12·16	·28
1·1—1·15	5	5·78	·11
Totals	200	200·00	8·73

Column 3 gives approximate values of the theoretical frequencies found by reading from the Curve I in the figure and using Simpson's Rule. This is sufficiently accurate for our purpose. With a $\chi^2 = 8{\cdot}73$ and 11 degrees of freedom we find that $P(>\chi^2) > {\cdot}65$, thus the fitting is a very plausible one.

The Distribution of $\sqrt{\beta_1}$ in samples of four from a Normal Universe.



Summary and Conclusions.

1. The distribution of $\sqrt{\beta_1}$ in samples of four from a normal universe is determined and proves to be a symmetrical curve having finite ordinates at the termini and an infinitely distant cusp on the axis of symmetry.

2. This new, inverted T, type distribution is interesting in that it shows, among other things, that the normal universe can give rise to a derived distribution which cannot be approximated to by a Pearson Type Curve. It provides a warning, therefore, against approximating to theoretical distributions by the mere use of moments without first assuring, by means of sampling experiments or otherwise, that the approximation curve selected has the same general character as the true curve.

3. The results of a sampling experiment suggest that for a rectangular universe the distribution of $\sqrt{\beta_1}$ in samples of four could most reasonably be the same as that for the normal universe.

(ix) Note on Mr McKay's paper.

I should not expect that a *single* Pearson curve would describe satisfactorily the distribution of any statistical coefficient based upon a sample of *four*. One has had enough experience in the distribution of the product-moment coefficient* and the correlation coefficient† in very small samples from a normal population to realise the truth of this. On the other hand I should be surprised if the Pearson curves would not give a reasonable approximation as the samples increased from 15 to 25‡.

Even in Mr McKay's case the divergence is not so excessive as his diagram would suggest. We may look upon Mr McKay's curve not as a *single* curve, but as a curve and its mirrored image, in precisely the same manner as I have treated the distribution of the "centre of the range" in samples of size n drawn from a rectangular parent population§. In that case we have a cusp at a *finite* distance from the origin on the axis of symmetry, and mirror curves, each of which is a Pearson curve of Type IX. These mirror curves are in that case the accurate solution, not an approximation.

Accordingly in the present case of the distribution of $\sqrt{\beta_1}$ in samples of four, it seems reasonable to use mirror curves as Mr McKay does in his accurate solution, and fit each of them to a Pearson curve by using the second or fourth moment coefficients about the asymptotic axis. The combined mirror curves will then have the same first four moment coefficients about the axis of symmetry. We are not provided with the third moment coefficient of the half McKay curve so that one must be content with the second or fourth. The appropriate Pearson curve is Type VIII, i.e. $y=y_0\left(\frac{a}{x}\right)^m$, where x ranges from 0 to a.

We have for moment coefficients about $x=0$‖:

$$\mu_2'=a^2\frac{1-m}{3-m},\quad \mu_4'=a^4\frac{(1-m)}{5-m},$$

and for unit area

$$y_0=(1-m)/2a\,¶.$$

Accordingly

$$\frac{\mu_4'}{\mu_2'^2}=\frac{(3-m)^2}{(1-m)(5-m)}=\frac{s^2}{s^2-4},$$

if $s=3-m$.

* *Biometrika*, Vol. XXI. pp. 170—180.

† *Biometrika*, Vol. XI. pp. 387—388.

‡ See the last two memoirs just cited.

§ *Biometrika*, Vol. XXIII. pp. 394—396.

‖ *Phil. Trans.* Vol. 216 A, p. 438.

¶ The factor ½ is introduced, because the area is for *half* the mirror curves combined.

But $\mu_4'/\mu_2'^2 = \beta_2$ of McKay's curve $= 315/143$, hence

$$s^2 = \beta_2 (s^2 - 4),$$

or

$$s^2 = 315/43 \text{ and } s = 2{\cdot}7065{,}8113.$$

Thus

$$m = {\cdot}2934{,}1887.$$

Further, the standard deviation of McKay's curve is $\left(\frac{12}{35}\right)^{\frac{1}{2}}$ and accordingly we find

$$a^2 = \frac{12}{35} \times \frac{2{\cdot}7065{,}8113}{{\cdot}7065{,}8113} = 1{\cdot}3133{,}2501^5,$$

or

$$a = 1{\cdot}1460{,}0393.$$

We have here the chief characteristics of the accurate McKay curve in approximative values.

Lastly

$$y_0 = \frac{1-m}{2a} = \frac{{\cdot}7065{,}8113}{2 \times 1{\cdot}1460{,}0393} = {\cdot}3082{,}8041.$$

Thus the fit by the first four moments consists in the mirror curves of Type VIII form:

$$y = {\cdot}308{,}280 \left(\frac{1{\cdot}146{,}004}{|x|}\right)^{{\cdot}2934{,}1887}$$

We can, however, get a still better result by an appeal *à priori* to a principle which determines the range of $\sqrt{\beta_1}$ in samples from any parent population.

Consider a parent population with a range b and let samples of size n be drawn from it. Now consider the following scheme: suppose $n-2$ of the sample values are at $x=0$ taken as at the end of the range, one value at $x=c$ $(c<b)$ and a third at $x=b$. Then we have

$$\sqrt{\beta_1} = \mathop{S}_{s=1}^{n} \frac{(x_s - \bar{x})^3}{n} \Big/ \left(\mathop{S}_{s=1}^{n} \frac{(x_s - \bar{x})^2}{n}\right)^{\frac{3}{2}}.$$

Now $\bar{x} = (c+b)/n$ and accordingly we have

$$\begin{aligned}\sqrt{\beta_1} &= \sqrt{n}\, \frac{\{(c(n-1)-b)^3 + (b(n-1)-c)^3 - (c+b)^3(n-2)\}}{\{(c(n-1)-b)^2 + (b(n-1)-c)^2 + (c+b)^2(n-2)\}^{\frac{3}{2}}} \\ &= \frac{(n-2)\{(c^2+b^2)(n-1) - 3cb(c+b)\}}{\{(c^2+b^2)(n-1) - 2cb\}^{\frac{3}{2}}} \\ &= (n-2)\frac{\{n-1-3n\lambda\}}{(n-1-2n\lambda)^{\frac{3}{2}}}, \quad \text{where } \lambda = \frac{bc}{(b+c)^2}.\end{aligned}$$

When $c=0$, or we have one value at one end of the range and the rest at the other,

$$\sqrt{\beta_1} = \frac{n-2}{\sqrt{n-1}}.$$

This is the maximum value of $\sqrt{\beta_1}$. For if we move the value at one end closer to the $n-1$ values at the other, we merely shorten the range b, but get the same value of $\sqrt{\beta_1}$ which is independent of the range. If we put two at one end of the range and $n-2$ at the other, this is putting $c=b$, or $\lambda = \frac{1}{4}$, we have

$$\sqrt{\beta_1} = \frac{n-4}{\sqrt{2}\,(n-2)}, \text{ which is less than } \frac{n-2}{\sqrt{n-1}}.$$

Finally, if we start with one value at one end of the range and $n-1$ at the other, and move one of the latter out a distance c, then we have as above

$$\sqrt{\beta_1} = \frac{(n-2)\{n-1-3n\lambda\}}{(n-1-2n\lambda)^{\frac{3}{2}}},$$

but if we make this a maximum with λ we find

$$n - 1 - 2n\lambda = n - 1 - 3n\lambda,$$

or $\lambda = 0$, that is $c=0$, or to move a value from the end of the range reduces $\sqrt{\beta_1}$.

But the arrangement 1 and $n-1$ values at the ends of the range is independent of (i) the size of the range, which may be increased to infinity, or of (ii) the nature of the parent population. Thus under all circumstances the value of $\sqrt{\beta_1}$ must lie in the range $\pm\frac{n-2}{\sqrt{n-1}}$.

To this extent the parent population is indifferent. If $n=4$, then we have for the range $\pm 2/\sqrt{3}=1{\cdot}154{,}7005$, agreeing with Mr McKay's value.

Knowing our range we can make use of either the second or fourth moment coefficient as we have not a knowledge of the odd moments to fit our curve, or we might fit one of Mr Hansmann's curves* which use μ_2, β_2, and β_4. As my sole purpose is to indicate that a well-chosen Pearson curve can approximate to Mr McKay's curve, I will take a Type VIII curve and knowing the range fit from one even moment coefficient. While I should prefer the second moment coefficient, had the range been infinite, I prefer the fourth to the second when the range is limited. There is, however, very little difference between the values of y_0 and m found from the second and fourth moment coefficients, and both give results very close to the values found as above, when the range is supposed unknown. We have for the constants of the Type VIII curve:

	Range	y_0	m
Range unknown, using μ_2' and μ_4'	1·146,004	·308,280	·293,419
Range known, using μ_2'	1·154,701	·299,778	·307,892
Range known, using μ_4'	1·154,701	·295,291	·318,054

On the scale of Mr McKay's diagram there is scarcely anything to show between these curves. In the diagram below the curve as found from the known range and μ_4' is figured.

Thus the fit is by mirror curves of the Type VIII form:

$$y = {\cdot}295{,}291\left(\frac{1{\cdot}154{,}701}{|x|}\right)^{{\cdot}318{,}054}.$$

We have here the chief characteristics of the accurate McKay curve in approximative values, namely:

(i) Symmetrical.

(ii) A cusp at infinity when $x=0$.

(iii) A finite ordinate $=0{\cdot}295{,}291$ (instead of $0{\cdot}288{,}675$), at the termini $x=\pm 1{\cdot}154{,}701$.

(iv) $\sigma^2={\cdot}339{,}030$ instead of ${\cdot}342{,}857$, and μ_4 the same as the true value.

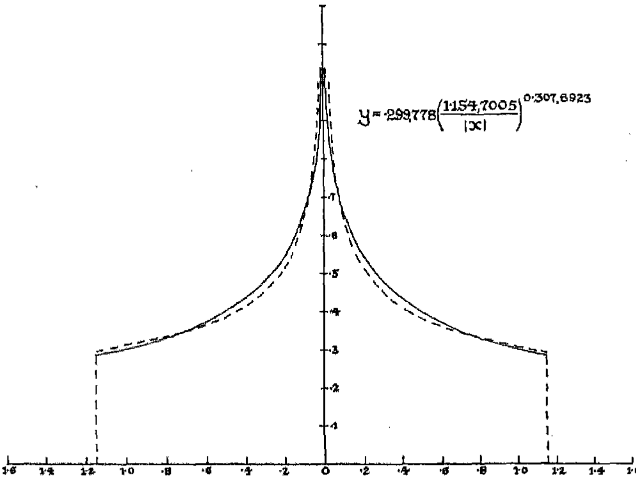
(v) $\log_{10}\phi(x)=\bar{1}{\cdot}808{,}1928-{\cdot}318{,}054\,(1+\log_{10}|x|)$.

An examination of the diagram shows that the mirror curves of Type VIII, while they provide no very accurate fit, are substantially more satisfactory than the attempt to fit a continuous curve of Type II to what is actually a mirror curve. Had we worked from the three moment coefficients μ_2', μ_3' and μ_4' of the McKay mirror curve about its origin we might have hoped for a still better fit, and we have at any rate encouragement for the suggestion that with larger, if not very large samples, the fitting of the mirrored curve, not the pair of mirrored curves, by a Pearson curve will give a practically reasonable graduation of $\sqrt{\beta_1}$†.

* Thesis for the London doctorate not yet published. Mr Hansmann fits higher order symmetrical curves by taking β_2 and β_4 of the observations to agree with the curve values.

† J. Pepper has shown, indeed, that with samples of 10 only, he got a very good fit to the $\sqrt{\beta_1}$ distribution even without the use of mirrored curves of finite range.

Comparison of Pearson Curve of Type VIII and its Mirror Curve with McKay's accurate Curve $y=\frac{1}{2\sqrt{3}}F(\frac{1}{3},\frac{2}{3},1,1-\frac{3}{4}x^2)$



Mr McKay's discussion of the Rectangular Parent Population is of much interest and value as showing that with very small samples the actual distribution of the parent population is of small importance, but at the same time it contains a warning to those who propose from small samples to deduce anything concerning the characteristics of the parent population. As I have endeavoured to indicate in a recent paper*, it may well need samples of upwards of 100 to safely infer whether the parent population is normal or rectangular. Thus the chief value of Mr McKay's comparison of samples from a rectangular parent population with the theoretical results from a normal parent population does not seem to me to lie in the fact that the latter will suffice to describe the former—the theoretical results of sampling from a rectangular population would no doubt equally well describe very small samples from a normal population—no, the chief value lies in the warning it gives that, notwithstanding we have in a small sampling found agreement with theoretical predictions as to small sampling from a type of parent population A, this provides no real evidence that the actual parent population was not of a wholly different type B.

K. P.

(x) Note on the Fitting of Frequency Curves.

One of the chief difficulties which beset the path of the inventor of a system of frequency curves is the too ready manner in which others may apply, or rather misapply them and so bring discredit on a system, the rules of which they have not followed, or more often misunderstood.

* See *Biometrika*, Vol. xxiv. p. 871.

I could cite many instances of this in the case of my own system of curves*, but a noteworthy illustration of it occurs in a dissertation for the Ph.D. of the University of Michigan by Mr Pae-Tsi Yuan†. It is entitled "On the Logarithmic Frequency Distribution and the Semi-logarithmic Correlation Surface." I am not concerned here with the question of whether Mr Yuan has contributed anything novel to the subject, which has been worn fairly threadbare by numerous previous writers. I deal only with the two points in which he refers to my own contributions to the topic. In a paper of 1895‡ I defined the "skewness" of a frequency distribution to be the ratio of the distance between the mean and the mode to the standard deviation of the distribution, and in 1905§ I showed that the logarithmic curve could not be of wide use, because the range of "skewness" it provides is limited, while in actual practice "skewness" can take any value whatever. Mr Yuan‖ remarks that while the skewness of the logarithmic curve with my definition is limited, this only indicates that my definition of skewness does not give a satisfactory measure of skewness, and advocates a_3 which is the symbol he prefers to use instead of $\sqrt{\beta_1}$. Now $\sqrt{\beta_1}$ may if any one prefers be used as a measure of asymmetry in frequency distributions, but that has nothing to do with my definition of "skewness." Whether you call (mean $\sim$ mode) divided by standard deviation "skewness" or not, the fact remains that the quantity in question is a physical character of frequency distributions, and is limited in the logarithmic curve and is not limited in frequency distributions in general. $\sqrt{\beta_1}$ is not limited in the logarithmic curve, but for every value of β_1 there is only one available value of β_2 and of the other higher β's. The curve connecting β_1 and β_2 has been plotted by Pretorius¶ and his graph is reproduced on p. 215. Unless the β_1 and β_2 of a distribution give a point lying close to the broken line (L) of this diagram we cannot get a good representation of the frequency. If the point does lie near that line, I will guarantee as good a fit with a Type VI curve to any actual observational series.

If the point (β_1, β_2) lies some distance from the (L) line, its fourth moment coefficient must be discordant with that provided by the logarithmic curve, and the graduation will fail to be as good as the Pearson curve. Now how does Mr Yuan illustrate the supposed superiority of fit of the logarithmic curve over a Pearson curve?

He compares the logarithmic curve with a Pearson Type III curve! Why not straight away compare it with a rectangle or a normal curve? The logarithmic curve lies in Type VI area and is nearer to a Type V than a Type III distribution.

For a practical illustration he takes the distribution for the weights of 1000 female students as follows:

Central Weights in lbs.

	74·45	84·45	94·45	104·45	114·45	124·45	134·45	144·45	154·45	164·45	174·45	184·45	194·45	204·45	214·45	Total
Frequency	2	16	82	231	248	196	122	63	23	5	7	1	2	1	1	1000

* See for example: Paul R. Rider, *Biometrika*, Vol. xxiv. p. 386, where no attention has been paid to the "abruptness" or to the limitation of the range. Or again: G. L. Edgett, *Metron*, Vol. ix. No. 2, pp. 31—52, who applies a wrong type (using a method similar to that suggested by me in 1895 and then found lacking in accuracy) and then asserts that this type is a bad fit. But illustrations—especially in practical memoirs—are really too frequent to be recorded here.

† Published in the *Annals of Mathematical Statistics*, Vol. iv. pp. 30—74. Edward Brothers, Ann Arbor, Mich.

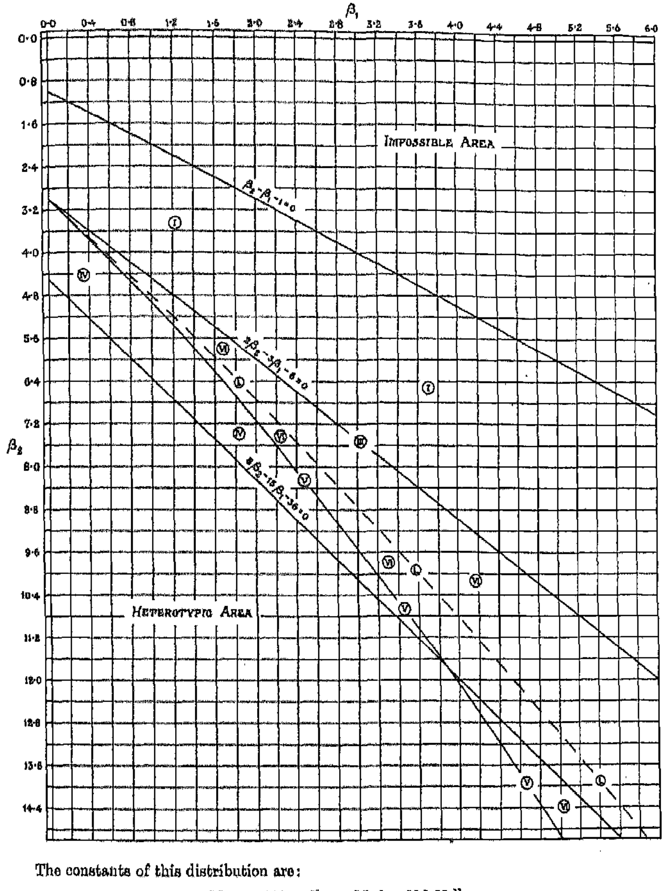
‡ *Phil. Trans.* Vol. 186 A, p. 370.

§ *Biometrika*, Vol. iv. p. 195.

‖ *Loc. cit.* p. 42.

¶ *Biometrika*, Vol. xxii. p. 147.

DIAGRAM SHOWING THE RELATION BETWEEN β_1 AND β_2 FOR THE LOGARITHMICALLY TRANSFORMED NORMAL CURVE.



The constants of this distribution are:

$$\text{Mean} = 118{\cdot}24 \text{ lbs.}, \quad \text{Mode} = 112{\cdot}68 \text{ lbs.}$$

$$\sigma = 1{\cdot}691{,}752^*, \quad \beta_1 = {\cdot}953{,}403, \quad \beta_2 = 5{\cdot}453{,}569.$$

A glance at Pretorius' diagram shows us that the point (β_1, β_2) is far away from the (L) line, and accordingly a logarithmic curve would give an entirely erroneous β_2 and an incorrect fourth

* Working units $= 16{\cdot}91752$ lbs.

moment coefficient. The Pearson curve appropriate to the data is Type IV, and the corresponding equation is

$$y = \frac{12 \cdot 112{,}672\, e^{+9 \cdot 076{,}367 \tan^{-1}(x/3 \cdot 855{,}7375)}}{\left(1+\left(\frac{x}{3 \cdot 855{,}7375}\right)^2\right)^{6 \cdot 180{,}879}} \text{ in working units,}$$

with origin at 3·411,005 working units, i.e. 34·11005 lbs. before the mean. The mode is at 2·854,550 working units from the origin.

The areas of this curve were calculated and the following results reached:

Observed Frequency		Logarithmic* Curve Areas		Pearson's Type IV Curve	
2	18	0	10	1·298	16·987
16		10		15·689	
82		97		88·175	
231		228		216·122	
248		255		202·803	
196		190		199·006	
122		114		113·705	
63		57		55·934	
23		27		25·701	
5	12	12	18	11·562	16·789
7		6		5·227	
1	5	2	4	1·137	4·085
2		1		·553	
1		1		·588	
1		0			

For 10 groups: $\chi^2 = 13 \cdot 1764$, $P(\chi^2) = \cdot 156$; $\chi^2 = 5 \cdot 5751$, $P(\chi^2) = \cdot 780$.

It is clear that the fit of the Type IV curve which provides the correct β_2 is superior to that of the logarithmic curve, as it naturally should be. Mr Yuan may of course graduate any data with a logarithmic curve if he considers the fit good enough for his purpose, but it is idle and illogical to pick out a wrong type from my system, and then magnify the value of the logarithmic curve at the expense of that system.

* Values on p. 50 of Mr Yuan's paper.

K. P.

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CONTENTS

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BIOMETRIKA

THE CRANIAL COORDINATOGRAPH, THE STANDARD PLANES OF THE SKULL, AND THE VALUE OF CARTESIAN GEOMETRY TO THE CRANIOLOGIST, WITH SOME ILLUSTRATIONS OF THE USES OF THE NEW METHOD*.

BY KARL PEARSON.

1. *Introductory.*

Many years ago when I had more leisure to give to craniometry than I have had recently, I became convinced of the unsatisfactory nature of the "standard" planes of the skull, and the need for a careful revision of the whole subject. In particular, when one had realised the absolute asymmetry of the skull in all parts and in all directions, it seemed irrational to suppose that the auricular axis was likely to lie in a "horizontal" plane. Assuming for a moment that there is such an entity as a "median sagittal" plane, or plane with regard to which the absolutely symmetrical skull should have mirror symmetry, then in the natural skull the ears would be shifted right and left, forward and backward, up and down with regard to this mirror plane of symmetry, and accordingly the auricular axis would make an angle with the mirror plane of symmetry. Further, all the fundamental points of the skull which should lie in the mirror plane, i.e. the "mid-sagittal" points, would in an actual skull be dispersed some to the right and some to the left of it, and the fiction of a median sagittal plane as a true *standard vertical plane* of the skull, seemed to vanish with the asymmetry of the skull.

Convinced that, with an asymmetrical system like the skull, the main object must still be to start from a fitly chosen median sagittal section as the standard median vertical plane, and that the standard horizontal plane must be perpendicular to this, I could only look upon the Frankfurt Horizontal Plane and the Transverse Vertical Plane, both passing through the auricular axis, and the Median Sagittal Plane (as determined by any three points†) as very temporary and inadequate expedients to obtain three mutually perpendicular standard cranial planes. These customary planes are not mutually rectangular. By construction the Frankfurt Horizontal Plane and the usual Transverse Vertical Plane are at right angles. Hence the third plane ought to be at right angles to both these planes, that is to say at right angles to the auricular axis. It is perfectly easy to draw the curve of intersection of such a

* The material points of this paper were given as a lecture before the Oxford University Anthropological Society on May 25, 1933.

† These three points are assumed to be in the "mirror plane of symmetry," for example Nasion, Bregma and Lambda as used in the Biometric Laboratory, or Nasion, Inion and Basion as suggested by Martin. See p. 227 below.

plane with the external surface of the skull, if we have an instrument for setting the skull with its auricular axis perpendicular to the plane of the craniometric table. That is possible by aid of the cranial coordinatograph. It only remains to settle through what point this plane perpendicular to the auricular axis shall be taken. If the skull had complete mirror symmetry then this plane should bisect the auricular axis, i.e. pass through the point—the Mid-porion—midway between the Right and Left Poria. In the random selection of crania I have used to illustrate this paper, this Mid-porion plane deviates so widely from any supposed median sagittal section, that no one would think of using it. By aid of my coordinatograph, Dr Morant kindly drew for me on a Hindu skull the Frankfurt Horizontal Plane *, the usual Transverse Vertical Plane, as determined by placing the skull on a Ranke craniophor, and the Mid-porion perpendicular plane. Plates I—III indicate the absurd results thus reached. The absurdity lies in the fact that in no ordinary skull is the auricular axis perpendicular to any reasonable Median Sagittal Plane, and the sooner we, as craniologists, realise this the better. The auricular axis makes an angle differing from a right angle with the Median Sagittal Plane and has no real claim to be selected as a "horizontal line." A very brief experience will convince an attentive observer that when the subject is holding his head "straight" the two ears are not usually on the same level, to say nothing of their equality in distance from any mid-line of the face †.

I have, perhaps, said enough to convince the reader that the fundamental crux of the determination of the standard planes of the skull lies in the discovery of an adequately satisfactory "median sagittal section," i.e. a standard vertical sagittal section or an approximate mirror plane. This must precede the determination of a standard horizontal plane, if only for the strong reason that with a truly symmetrical skull we can find twelve or more points which lie in the mirror plane, while the "horizontal plane" has only some four points for its determination, and these in no case so simply determinable as the positions of those we have spoken of as mid-sagittal points. The "crux" lies in this: we have twelve or more points which "should" lie in one plane. If they don't, what is an adequate representative of that plane? The mathematician would answer at once that a "good" substitute for it would be the plane that made the sum of the squares of the distances from it of these mid-sagittal points a minimum ‡. At the present stage of these investigations "weighting" the mid-sagittal points must be left on one side, and we solve our

* *Not* determined from the left Orbitale, but from the mean height of right and left Orbitalia.

† In very marked cases the unequal height of the ears is recorded as an "anatomical anomaly," see *Annals of Eugenics*, Vol. IV. pp. 285—6, Plates III and IV.

‡ Not infrequently such a plane is termed the "best" plane. But to accept this view we should first have to show that the deviations of these points from this plane followed the normal law of frequency, and that the standard errors of the deviations of the individual points were the same, otherwise the question of weighting the individual points arises. This suggests valuable, but very laborious work in determining the variations of what I have termed the mid-sagittal standard points from a mid-sagittal plane and a consideration of their individual frequency distributions. We should probably find ourselves finally thrust on the problem of whether the standard deviations varied from race to race. If they should do so, weighting would be very troublesome.

problem by stating that the standard mid-sagittal plane is the plane—in a mathematical sense—of close fit to the twelve or more mid-sagittal points.

It was with regard to problems of this kind, in particular cranial problems, that in 1901 I published a paper on "Lines and Planes of Closest Fit to Systems of Points in Space*," and showed that the determination of good fitting planes depended on finding the standard deviations and correlation coefficients of the coordinates of the points in space. The solution was reduced to a problem in solid Cartesian geometry. To apply it to the skull we must (i) determine some three rectangular planes associated with the skull—these I term the fundamental reference planes, and (ii) have some instrument which will rapidly provide us with the three coordinates of any point whatever of the skull in relation to these three planes. Such an instrument I term a cranial coordinatograph. One especially designed by me and made for me by Messrs Hawksley and Sons will be described below.

Now let us see where we stand. The skull may be looked upon as a system of indefinitely numerous points. By aid of the cranial coordinatograph we can at once form tables of the coordinates in space of any number of these points we please. The instrument enables us to construct plan and elevation drawings of these points. We can then proceed to deduce properties of the skull either by the methods of solid Cartesian geometry so familiar to the mathematician, or by the graphical rules of plan and elevation drawings so familiar to the engineer.

We are thus able to determine (i) the distance between any two points on the skull—the callipers may be dispensed with, although the tape will still be required; (ii) the equation to the line joining any two points and the angle made by this line with any other line or plane; (iii) the angle between any two planes as represented by their equations; (iv) the standard mid-sagittal plane as defined above, and—perhaps the most important of all determinations—whether (v) any two homologous points on the skull have true mirror symmetry. The cranial coordinatograph seems to me to throw open a new field in craniometry, much as the modern theory of statistics did some forty years ago. It adds solid analytical geometry to the technique of the craniologist, and provides a valuable addition to his *instrumentarium*.

I have no desire to screen the labour of calculation involved in the new processes suggested in this paper. The question is: Are the results worth that labour? Personally I think they are. It is something to gain a means whereby we can distinguish between the adequacy of various median sagittal planes, or obtain a measure of the inadequacy of the Frankfurt Horizontal Plane. And these are only two of the many problems solvable by the new method†.

* *Philosophical Magazine*, November, 1901, pp. 559—572.

† Apart from new problems we obtain other methods of solving old problems. We have the coordinates of Nasion, Alveolar Point and Basion; they give the equations to the sides of the fundamental facial triangle; and we can study its angular properties without using the trigonometer. Or again we have the coordinates of Nasion and Prosthion (or, if we prefer, Alveolar Point), and have the equation to their join. One of the direction cosines of this line gives the angle it makes with the Frankfurt Plane, that is the Profile Angle. We can thus dispense with the goniometer, which is at best a faulty instrument for it assumes that the line joining Nasion and Prosthion projects into a vertical line on the usual Transverse Vertical Plane, which in general it does not.

Before we can determine the coordinates of points on the skull we must fix on coordinate planes. These might of course be selected in any manner, but it is convenient to select for them already familiar planes. I call them the planes of reference, and the reader must distinguish them from the standard planes of the skull, which are something quite different. I take as planes of reference: (i) the Frankfurt Horizontal Plane as determined by balancing the Poria on the extreme points of the knife edges of a Ranke craniophor. The Orbitalia are then marked on right and left orbits and the skull rotated until the horizontal plane through the auricular axis (i.e. the join of the Poria* or the knife edge of the ear plugs) bisects the difference of the heights of the two Orbitalia†. This is our first reference plane. With the skull thus adjusted and the scriber at this height, the most posterior point in the occipital region in this plane (to be called the plane of $x=0$) is marked on the skull. It will be called *Kappa*. This point Kappa is usually on or close to the occipital protuberance. The Frankfurt Horizontal Plane as thus defined will pass either through both Orbitalia or above one lower orbital margin and below the other. See Plate I (*a*). (ii) While the skull is still adjusted to this slight modification of the usual Frankfurt Plane, the horizontal bar of the craniophor is brought down until it is in contact with the skull and moved forward or backward until its point meets the sagittal suture. This point is marked on the skull. It is the Apex in my terminology‡. The plane through the Apex and the Poria is that of the so-called Transeverse Vertical Plane, the plane for which the Transverse Contour is usually provided. It is to be our plane of (xz) or $y=0$. The planes (i) and (ii) meet in the auricular axis which is accordingly our axis of z. (iii) *Any* plane perpendicular to the auricular axis will serve as the third plane of reference. Now suppose the skull removed from the craniophor, and adjusted so that the auricular axis is perpendicular to a drawing-board, then the paper on that drawing-board may be taken as the plane of (xy) or $z=0$. The point in which the auricular axis meets this drawing-board will be our origin, or the origin is the plan of the two Poria. If we now project onto the drawing-board all points we please of the skull, we shall have their plans on the plane of $z=0$. These will give us their x and y coordinates. The plan of the Apex joined to the plan of the Poria will give the axis of x, and the line in this plane through the plan of the Poria perpendicular to the join of the the plans of Apex and Poria will be the axis of y. This axis of y should *very* closely pass through the plan of Kappa if our adjustments have been accurately made. I take the positive direction of the axis of x to be away

* I prefer to mark the Poria only after the skull is supported on the knife edges.

† The scriber set to the level of the left of the knife edges is first applied to one Orbitale, and the skull rotated until this Orbitale and the Poria are in one plane. The scriber is then applied to the second orbit; if its Orbitale is *above* the scriber, the scriber point is marked on the skull below the Orbitale, and the skull rotated till the scriber bisects the difference. If the scriber as first set is above the second Orbitale, then that Orbitale is brought up to the scriber, which when applied to the first orbit will now be below its Orbitale and the bisection must take place on that orbit. The ultimate plane, to be called that of (zy) or $x=0$, is our first plane of reference.

‡ The *Apex* must not be confused with the *Vertex*; the latter is the point of the skull at maximum perpendicular distance from the Frankfurt Horizontal Plane.

from the Apex, i.e. towards the base of the skull, and the positive direction of the axis of y away from Kappa or towards the face of the skull. I have chosen the plane of $y = 0$, or the Transverse Vertical Plane, as the plane for giving the elevations. The Frankfurt Horizontal Plane is the plane through the axis of y perpendicular to the plane of the drawing board, and, if the skull were truly symmetrical, the plane of the plans would be parallel to the mirror plane or a true Median Sagittal Plane. Accordingly all the mid-sagittal points would have the same elevations.

Plates IV—VI provide photographs of the plan and elevation drawings of six skulls—those of a Fuegian, an ancient Egyptian from Nubia, a modern Arab, a Negro from the Teita Hills district, a 17th century Londoner, and a modern Hindu. These skulls were chosen at random, and the diagrams in every case show us that the mid-sagittal points do not lie in a single plane perpendicular to the auricular axis, for if they did these points would have equal elevations*. A true mid-sagittal plane is a fiction in every one of these cases, and we are compelled to replace it by the idea of the "closest fitting" plane to the chief mid-sagittal points.

2. *The Cranial Coordinatograph.*

In order to obtain the plans and elevations of points on the skull (not confining ourselves as in our present illustrations to the mid-sagittal points) it is needful to devise an instrument which will serve three purposes:

(*a*) Set the line joining any two selected points of the skull perpendicular to a drawing board. In our present illustration this line is the auricular axis or join of the Poria, but we might find in other investigations that some other line would be of more service as axis of z. This can equally well be achieved by the coordinatograph.

(*b*) The instrument must be capable of measuring the height of any point above the drawing-board, that is we must read off easily upon it the elevations of any chosen cranial points.

(*c*) At the same time it must by a simple action record on the drawing-board the plan of the cranial point. These objects are all achieved by the present instrument.

Diagrammatically the instrument consists of three arms, two of which are capable of fine motion and to each of which a vernier is attached. See Fig. 1 on p. 222.

AB is a vertical rod on which slide the two arms D_1P_1 and D_2P_2, which can be brought into absolute contact such that the points P_1 and P_2 coalesce. CD_3 is a fixed arm, and carries a vertical cylinder in the axis of which is a needle point P_3. When the button at C is pressed, this needle comes down on the drawing-board and the inked rim of the cylinder makes a circle with a needle point in its centre. This

* The models based upon the plan and elevation drawings provided by the cranial coordinatograph had to be tilted for photography in order that the names of the points should be legible. Accordingly the reader will find it best when examining them to hold the page nearly vertical, when the models come more closely into correct perspective.

is the plan of the point, with which one or other of the points P_1 or P_2 is placed in contact. The three points P_1, P_2 and P_3 are in a vertical line perpendicular to the base BB' of the instrument. Thus the line $P_1P_2P_3$ is accurately parallel to the vertical rod AB. This rod is graduated in millimetres from zero at B. Thus with the fine adjustments and verniers the height of P_1 or P_2 or their difference in height can be determined with close approximation. The other part of the instrument, the cranial staddle or skull trivet—it would be misleading or ambitious to call it a craniophor—consists of a triangular plate carrying a saucer, and supported by three screws at the angles. The skull is placed on a bed of plasticine in the saucer, with the required line roughly adjusted to the vertical. The point P_2 of the

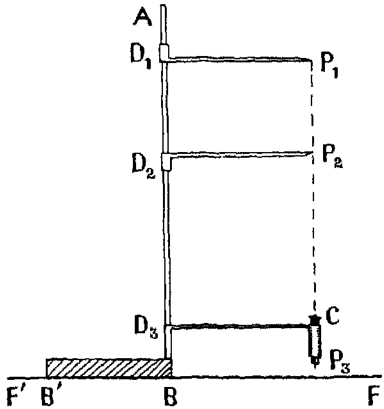


Fig. 1.

coordinatograph is brought into contact with the lower point of the line; the upper point P_1 is brought down to the approximate level of the upper point. Then by the turning of the three staddle screws and the fine adjustment movements of the arms D_1P_1 and D_2P_2, the skull is brought into contact at the given points—e.g. the Poria—with the terminals P_1 and P_2 of the arms of the coordinatograph. The adjustment does not take in practice much longer to carry out than it does to describe, and when it is completed the required line—e.g. the auricular axis—is perpendicular to the drawing-board—a touch on the button at C now records the position of the origin on the drawing-board. The upper arm is then raised above the skull out of the way, and the lower arm used; it is brought into contact successively with the points of the skull which are to have their coordinates found. The button C determines the plan, the scale on the vertical AB, with the vernier at D_2, gives the elevation of the point. The plan of the Apex gives the x-axis, and the line through the origin perpendicular to this gives the y-axis. Thus the three coordinates of any point can be found from the recorded elevations and the measurement of the plan drawing on the board.

The coordinatograph and the cranial staddle are made relatively heavy, so that they may not be moved too quickly either by over-haste or accident, and thus compel the user to readjust the skull and start afresh. Care has to be taken in adjusting the skull on its bed, that no required point lies immediately above or very close to one of the screw legs of the staddle. The base plate of the staddle is raised sufficiently above the drawing-board to allow of the plan-recorder D_3C passing beneath it as the plans of one or two points of the skull will frequently be beneath this plate. With practice in the use of the apparatus I think the adjustment of a skull and the determination of the three coordinates of some twenty points do not involve much more than an hour's labour.

Table I gives as an illustration the three coordinates of some fourteen points on six skulls referred to the reference planes we have discussed above. After a short discussion of mirror symmetry, I shall return to these sets of coordinates and indicate the type of problem in which their determination can be of service.

Plate VII (*a*) gives a photograph of the Cranial Coordinatograph in its first form. Slightly to the right we see a skull on the cranial staddle; to the left of it the two scriber arms of the coordinatograph are adjusted to the upper and lower Poria of the skull. The scale, the fine adjusting mechanisms of the scriber arms, with the verniers on the sloping faces of the cut away portions of the arm brackets are visible. Two instruments for recording plans are also in the picture. That on the right is of the Klaatsch type*, and is set for determining the plan of the glabella of the skull. That on the left is the plan-pricker from my osteometric instrument. The upper is the scriber arm, the lower arm marks the plan of its point by a circle (inked with a pad) with central needle point, corresponding to the cylinder-bearing arm of the diagram in Fig. 1. The use of such auxiliary instruments requires a double operation, the measurement of the elevation by the coordinatograph and the location of the plan by some form of vertical projector. This double instrumental setting has been got rid of by attaching an arm like that on my plan-pricker on the left to the base of the coordinatograph: see Plate VII (*b*). Thus, if either arm be set to a point on the adjusted skull, a tap on the plan-pricker gives the plan, and a reading of the scale on the vertical upright of the coordinatograph the elevation of the point in question.

3. *On the Symmetry of the Skull.*

I have indicated in my introductory remarks that I do not look upon the Frankfurt Horizontal Plane passing through the auricular axis as the best approach we can make to one of the fundamental planes of the skull. The first thing which strikes an observer of the human frame is its superficial approach to symmetry. This symmetry is not axial, but planar or mirror symmetry. Take any body whatever with a plane side and place that side against a mirror, and we see at once an example

* This does not differ essentially from Lissauer's Diagraph (*Archiv für Anthropologie*, Bd. xv. Supplement, S. 15 and Tafel XIV), but I was personally interested in Klaatsch's contour-tracings (of which I possess several originals), and I had the original projector used in the Biometric Laboratory made for me to his pattern by his instrument maker. I therefore speak of it as of "Klaatsch type" without claiming for him, or indeed for Lissauer, the invention of craniographs.

TABLE I.

Coordinates of Cranial Points (mms.).

Referred to Auricular Axis as axis of z and "Transverse" plane as plane of xz; $z=0$ is plane of drawing board.

Mid-sagittal Points*	(A) Fuegian			(B) Nubian			(C) Arab			(D) Negro			(E) English			(F) Hindu		
	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z	x	y	z
Alveolar Point ...	+ 60·8	+101·2	187·0	+ 44·6	+94·6	190·9	+ 46·6	+88·8	198·1	+ 41·5	+98·2	189·4	+ 42·8	+97·2	197·5	+ 37·8	+104·4	188·8
Nasal Spine	+ 36·9	+ 93·5	188·3	+ 24·1	+96·0	189·3	+ 28·8	+92·3	198·1	+ 23·5	+97·0	187·7	+ 27·4	+98·1	197·7	+ 25·3	+102·4	185·9
Nasion	− 20·5	+ 93·5	189·0	− 30·3	+92·8	189·3	− 29·4	+83·7	196·5	− 23·9	+93·0	187·8	− 30·5	+90·4	197·8	− 27·0	+ 89·2	184·2
Glabella	− 29·6	+ 96·9	188·6	− 41·0	+94·1	186·9	− 39·6	+84·8	196·0	− 35·7	+95·5	186·7	− 38·9	+94·5	196·8	− 36·3	+ 93·8	186·1
Bregma	−108·8	+ 20·4	185·8	−116·9	+10·4	190·1	−119·9	+ 4·1	190·6	−107·4	+24·2	183·1	−119·6	+16·4	199·1	−106·7	+ 10·9	188·6
Apex	−114·5	0·0	188·5	−118·3	0·0	189·7	−119·0	0·0	196·1	−112·1	0·0	184·3	−121·2	0·0	199·7	−107·6	0·0	187·4
Lambda	− 54·6	− 84·1	189·3	− 47·7	−95·4	189·1	− 49·9	−78·1	189·7	− 56·6	−82·1	185·0	− 66·0	−88·6	191·4	− 56·6	− 77·2	184·2
Kappa	0·0	− 89·0	190·5	0·0	−93·1	188·1	0·0	−71·2	189·8	0·0	−84·6	185·0	0·0	−86·9	198·0	0·0	− 82·2	185·4
Occipital Protuberance	+ 9·8	− 92·3	192·8	+ 16·4	−78·2	187·5	− 6·2	−75·5	190·6	+ 13·0	−73·5	187·4	+ 3·4	−83·0	197·0	+ 5·3	− 78·2	185·4
Inion (Inferior) ...	—	—	—	—	—	—	+ 6·9	−65·8	189·8	+ 13·9	−66·3	187·6	+ 8·1	−73·6	196·2	+ 8·5	− 69·3	186·8
Opisthion	+ 31·1	− 38·4	189·7	+ 30·0	−34·7	187·3	+ 29·8	−38·4	193·2	+ 25·9	−36·4	185·8	+ 17·3	−39·2	195·9	+ 26·9	− 41·5	186·6
Basion	+ 25·2	+ 1·6	187·4	+ 26·8	+ 2·9	187·9	+ 25·5	+ 4·1	193·1	+ 24·9	+ 1·6	185·4	+ 14·1	− 1·9	196·9	+ 24·7	− 0·5	188·7
Base of Palatal Spine	+ 38·0	+ 48·4	188·2	+ 28·4	+46·2	189·8	+ 30·8	+45·4	194·4	+ 28·8	+47·6	185·8	+ 21·3	+47·8	196·6	+ 27·7	+ 58·7	188·1
Mid-porion	0·0	0·0	190·0	0·0	0·0	189·85	0·0	0·0	194·8	0·0	0·0	186·2	0·0	0·0	196·0	0·0	0·0	186·0

* The Inferior Inion was not used in the case of (A) and (B). In the case of (A)—(F) the Kappa was so close to the x-axis that I have taken it on that axis.

of such mirror symmetry*. This apparent symmetry is the striking feature of the living head or the skull, and it is idle to neglect it and suppose we can determine any standard plane without regard to it. Of course as soon as we begin to measure we find that the skull is very far from symmetrical. Much work has been devoted in the Biometric Laboratory in recent years to the question of asymmetry in the human frame, and more will shortly be published. I do not propose to deal at present with the results of those investigations, although they actually upset some current beliefs. As far as the skull is concerned those investigations deal with the measurement of homologous distances or the size and shape of homologous bones. But we have to remember that the brain controls the development of the brain case as much or rather more than the case controls the brain. Let us start with a hypothetical brain of perfect mirror symmetry, and let it retain this symmetry from the earliest fœtal life. The homologous bones will spread from their ossification centres over the brain but they will spread unequally and not homologously; the resulting brain case might possibly have mirror symmetry, but this could not be ascertained in general from a measurement of homologous bones. We need to ascertain first at least some approach to a probable mirror plane. If we suppose homologous bones to grow "at random," according to a general law, but that there be no absolute equality of growth in homologous bones in definite directions, then they will meet and the sutures be formed in a more or less random manner, first the edge of a right side, then of a left side bone protruding into the territory of the other. The best we can do is to take some form of average of the sutures which should lie in the mirror plane. As we cannot attempt this for every point on the whole series of sutures, we do the best we can by fitting a close plane to a reasonably large number of definite points on these sutures. If this gives us the best plane available for the skull, it by no means follows that it would be with equal closeness the mirror plane of the brain or of the living head. Indeed the brain might be truly symmetrical, while the skull was asymmetrical. Without having definite evidence to produce, I think, however, that the living head is on the average more asymmetrical than the skull. Suppose we take a full-face portrait of a head, how shall we determine what is its mirror plane, and judge what it would look like if symmetrical?

I tested in the first place a rough outline sketch of Cromwell's death mask—Plate VIII (ii) (*b*). The nose decidedly deviated to the left cheek. What is the "best" mirror plane? The only thing to be done was to bisect the lip line and the external occular distance; these points of bisection were joined, and the drawing cut in half down the bisecting line. The two halves were traced in reverse and then the two right sides were joined up and the two left sides were joined up to form absolutely symmetrical faces. The results in Plate VIII (*ii*) (*a*) and (*c*) are absurd, but suggestive. The reader can choose (ii) (*a*) with the duplication of the famous wart over the right eye, or (ii) (*c*) without the wart at all. We see at once that the skewness of the nose leads to a marked diminution of that

* The mirror must be a silvered plate, and not the ordinary glass mirror, for the latter will show a vacant sheet between object and image.

organ, or to its exaggeration. In Plate VIII (i) I have applied the same treatment to a drawing of the Ashmolean bust of Cromwell.

Now this difficulty follows us when we ask what would a familiar face look like were it symmetrical. We take a full-face portrait and at once are met by the problem: Where is the mirror plane to be placed? I took the photograph of a colleague and after several trials finally settled that again it was best to take the dichotomic line through the bisections of mouth and external orbital distance. This line would not be a truly vertical line as the head was slightly inclined to the right: see Plate IX (*b*). To my horror the bulk of the nose and neck fell in one half of the divided photograph. These two halves had now to be reversed, and this was done by rephotographing them, and obtaining two prints, one with the film against the printing paper and the other with the glass. Joining my two halves together I obtained two perfectly symmetrical faces—Plate IX (*a*) and (*c*)—but at the cost of much reality. Any one who examines these "symmetricised" portraits will I think come to the conclusions that (i) grace does not necessarily connote facial symmetry, and (ii) a good deal of personal individuality is linked with facial asymmetry.

After this I felt able to deal with the problem of a symmetrical skull. Precisely the same process was repeated on the *Norma facialis* of an Egyptian skull. See Plate X. Allowing for the weakening of the reversed prints, I think, any one accustomed to handling skulls will be struck by the naturalness and individuality of the central photograph as compared with the symmetricised products to right and left of it.

But while it is relatively easy to create symmetrical portraits of the living face or of the *Norma facialis* of a skull, it is far less easy to determine the plane associated with an actual head or skull, by aid of which we can appreciate its asymmetry. As I have indicated in the introduction to this paper, there are a certain number of points which may be termed the mid-sagittal points, for they should lie, if the skull were symmetrical, in one plane, the mid-sagittal (and are not infrequently assumed to do so even in a natural skull). These points are the following: (i) Alveolar Point, (ii) Nasal Spine, (iii) Nasion, (iv) Glabella, (v) Bregma, (vi) Apex, (vii) Lambda, (viii) Kappa (see p. 220 above), (ix) Occipital Protuberance (Superior Inion), (x) Inferior Inion, (xi) Opisthion, (xii) Basion, and (xiii) Palatal Spine. As I have indicated, the plane of closest fit, as judged by minimum mean square deviation to these points, will be defined as the standard mid-sagittal plane. In the past the mid-sagittal plane has been arbitrarily determined by selecting three of the mid-sagittal points, regardless of whether such plane was or was not at right-angles to other so-called standard planes. Let me illustrate the difficulties heretofore current by citing the weightiest authority on the subject. I am inclined to think that in his case as in the case of many others the man who writes the biggest book is held to be the weightiest authority. If we cannot form an impression on the reader by the lucidity of our writings, we can at least impress him by the weightiness of our volumes. The very origin of the term weighty for an authority may be studied from this aspect.

Rudolf Martin in his *Lehrbuch der Anthropologie in systematischer Darstellung* (note that word *systematic*!) tells us in Bd. II, S. 582, after emphasising the necessity of planes of orientation for the skull, that:

"Ueber die Mediansagittal Ebene kann kein Zweifel bestehen; sie ist durch drei Punkte (Nasion, Inion und Basion) bestimmt. Zwar liegen diese nicht immer genau in einer Ebene, aber die Abweichungen sind so unbedeutend, dass sie in der Praxis vernachlässigt werden können."

In translation:

There can be no doubt about the Median Sagittal Plane. It is determined by three points (Nasion, Inion and Basion). These points do not lie always exactly in one plane, but their deviations are so unimportant, that in practice they may be neglected.

Why Martin selected out of the many points which should lie on the mirror plane the Nasion, Inion and Basion "*without doubt*" as those to determine the Median Sagittal Plane, he does not tell us, and I cannot tell the reader. I do know that one of his three points is one of the most difficult to determine on the skull, and no two writers seem to agree on how it is to be found! But let us go a step further, we are told that the Median Sagittal Plane is to be determined by these three points—when they are found—that seems clear enough. But alas! we are then informed that the plane which passes through these three points will not exactly pass through them, but the deviations may be neglected. Did Martin not know that a plane is fully determined by three points? On S. 582, this would appear to be so; but on S. 583, speaking of the horizontal plane of the skull, he says it ought to go through four points, but as the skull is asymmetrical the plane can only be taken through *three*, "for three points mathematically determine a plane."

In the Biometric Laboratory the Nasion, Bregma and Lambda have been taken for drawing the median sagittal contour, partly because these points are more or less clearly determined by the intersection of the cranial sutures, and partly because the vault of the skull seems for many purposes more important than the base.

Now let us return for a moment to Martin. His median sagittal section is to be taken (notwithstanding negligible deviations) through the three points—Nasion, Inion and Basion. We naturally look up his definitions of these points. They are:

S. 619. The Nasion is the meeting point of the sutura nasofrontalis (i.e. the suture between the nasal and frontal bones) with the *Median Sagittal Plane*. This plane therefore determines *the Nasion*.

S. 615. The Inion is the point in which the Lineae nuchiae superiores meet in the *Median Sagittal Plane*; if these lines are so feebly developed, that they do not reach the *Median Sagittal Plane*, they must be artificially produced till they do. Thus again the median sagittal plane determines the Inion.

S. 615. The Basion is the point in which the anterior border of the *foramen magnum* is met by the *Median Sagittal Plane*. Thus the most weighty of modern anthropologists defines the Median Sagittal Plane in terms of three cranial points, which according to him are only to be ascertained by an *a priori* knowledge of the Median Sagittal Plane. If this be "systematische Darstellung," is there not some need for a little mathematical logic—a little biometry to clear away these craniological fogs?

I have no desire to defend any particular plane which goes through three points as being the better representative of a plane which should pass through a dozen or more, but when we are told that there can be "kein Zweifel" as to what is the Median Sagittal Plane one is tempted to ask whether Martin's "ohne Zweifel" Plane is really superior to the nasion-bregma-lambda plane of the Biometric Laboratory. Having the coordinates of the thirteen mid-sagittal points for the six skulls selected at random which illustrate this paper, it was easy to write down the equations in those cases to Martin's Plane and the Biometric Laboratory Plane, and to measure (i) the angles between these planes, and the plane of "closest" fit to the thirteen points, and (ii) the mean square residuals of the mid-sagittal points from Martin's "ohne Zweifel" Plane and the Biometric Plane. The results are given in Table II, A and B.

TABLE II.

A. *Angles Plane of Maximum Symmetry makes with the Biometric and Martin's "Median Sagittal Planes."*

Skull	Biometric Laboratory Plane (Nasion, Bregma, Lambda)	Martin's "ohne Zweifel" Plane (Nasion, Inion, Basion)
Ancient Egyptian ...	0° 26′·6	1° 20′·0
Modern Arab ...	1° 36′·3	0° 55′·0
Negro (Teita) ...	1° 39′·5	5° 46′·7
Fuegian	1° 59′·6	7° 10′·5
17th century English	3° 14′·0	0° 47′·8
Bengal Hindu ...	3° 56′·7	6° 19′·2
Mean Angle	2° 8′·8	3° 43′·2

Or, Martin's Plane has on the average a 73 °/ₒ increase of angular deviation on the Biometric Plane.

B. *Mean Square Residuals for the two "Median Sagittal Planes."*

Skull	Biometric Laboratory Plane	Martin's Plane
Ancient Egyptian ...	1·5167	1·5909
Modern Arab ...	4·0284	2·5098
Negro (Teita) ...	5·7284	32·0419
Fuegian	8·5585	62·4137
English	27·3563	4·0257
Bengal Hindu ...	31·1950	35·5764
Mean Value	13·0055	23·0414

Or, Martin's Plane has on the average a 77 °/ₒ increase of Mean Square Residual on the Biometric Plane.

The sections A and B of Table II show us that neither the Biometric Laboratory Plane, nor Martin's "ohne Zweifel" Plane lies on the average very close to the

Plane of Minimum Deviation from the mid-sagittal points. This plane may be spoken of as the plane of nearest approach to the mirror plane of the skull or more shortly as the *Plane of Maximum Symmetry*. This latter plane we shall take as the First Standard Plane of the skull, or the Standard Median Vertical Plane. Naturally the Standard Horizontal Plane will be perpendicular to this plane, and we shall determine how far the Frankfurt Horizontal Plane is deficient in this respect. The Standard Transverse Vertical Plane will be again perpendicular to both our Standard Plane of Maximum Symmetry and to our Standard Horizontal Plane, and we shall determine how far this plane deviates from the usual Transverse Vertical Plane through the auricular axis perpendicular to the Frankfurt Horizontal. It will be seen that whether we judge by angular deviations from the planes of maximum symmetry or by the mean square deviations the nasion-bregma-lambda plane is on our present evidence much superior to the nasion-inion-basion plane. Accordingly we shall not consider it needful again to refer at length to the Median Sagittal Plane of Martin. When we speak of the "usual" Median Sagittal Plane, we shall mean that in which the long series of median sagittal contours issued by the Biometric Laboratory has been drawn, i.e. the nasion-bregma-lambda plane.

4. *Procedure for the Determination of the First Standard Plane or Plane of Maximum Symmetry of the Skull.*

I have already indicated the first stage of this procedure, the determination of the reference planes, and in particular the auricular axis which is to be set perpendicular to the plane of the drawing-board. But there is one point I should wish once more to emphasise. The auricular axis may be defined to be the line joining the extreme points of the knife edges on which the skull rests on the craniophor, *when these knife edges are properly adjusted*. This adjustment is sometimes defined as the process of bringing the tip of the knife edges to the *Poria*. But what are the Poria? Martin* defines the Porion "as that point on the upper border of the auricular passage which is vertically above the middle of the same." But how the middle of the auricular passage is to be found, he does not tell us, nor can I conceive how it is possible for an asymmetrical conicboidal space to have a "middle." Still less, if I could discover this "middle" could I take a "vertical" through it to meet the upper border of the auricular passage, because a vertical can only mean a line perpendicular to the horizontal plane, and that plane can only be found when the knife edge tips are already placed on the Poria. Thus according to Martin the Poria can only be found after the Frankfurt Horizontal Plane has already been determined. The very process of tilting the skull round on the knife edge tips to bring the Orbitalia to the height of the top of the knife edges causes the tips of the knife edges to slip along the upper border of the auricular passages. In my opinion the only way to determine the Poria is to mark them after the skull is adjusted on the craniophor to the Frankfurt Horizontal, the knife edges being withdrawn outwards to the very verges of the upper borders of the auricular passages. If, with some definition, other than Martin's, the Poria be marked before the Frankfurt Plane

* *Loc. cit.* S. 618.

is determined, then there is difficulty about successfully balancing the skull with its knife edge tips on these Poria; the tilting of the skull produces a constrained equilibrium and the knife edge tips tend to slip off the Poria and a minor catastrophe may result.

However, having marked in one way or another the Poria, it is fairly easy to adjust the skull by the three screws of the skull staddle so that the two Poria are in contact with two upper arm tips of the coordinatograph. A touch of the button marks the point, where the auricular axis meets the plane of the drawing-board, or is the plan of the Poria and the origin of coordinates. Before the coordinatograph is moved, the elevations of the Poria must be read off and recorded; a check on their accuracy is that their differences should equal the intraporial distance, which should be taken with the callipers, when the skull has been removed from the craniophor and before it is placed on the staddle. The Apex and the Kappa are now projected on to the drawing-board by aid of the coordinatograph, and their plans, joined to the plan of the Poria, give the axes of x and y respectively. These axes should be perpendicular. If they are not, the y-axis must be taken perpendicular to the x-axis, and the Kappa will have an x-coordinate differing from zero. Such a coordinate was too small to be measurable in the skulls dealt with by me. The plan and elevation of every other "mid-sagittal" point is obtained in the same manner, and the record for each skull will be a series of coordinates similar to those given in Table I.

The craniologist may not take it amiss if we remind him of the chief theorems in solid analytical geometry which are of value in the study of the skull. If (a_1, b_1, c_1) (a_2, b_2, c_2) (a_3, b_3, c_3) be the coordinates of three points, and

$$l_1x + m_1y + n_1z = p_1 \text{ and } l_2x + m_2y + n_2z = p_2$$

be the equations to two planes in the prepared form, i.e. such that

$$l_1^2 + m_1^2 + n_1^2 = 1, \quad l_2^2 + m_2^2 + n_2^2 = 1,$$

then:

(i) The equation to the line passing through (a_1, b_1, c_1) (a_2, b_2, c_2) is

$$\frac{x - a_1}{a_2 - a_1} = \frac{y - b_1}{b_2 - b_1} = \frac{z - c_1}{c_2 - c_1} \quad \text{..................(i)}.$$

(ii) The direction cosines of the lines are

$$\left.\begin{array}{l} L_{12} = (a_2 - a_1)/r_{12}, \quad M_{12} = (b_2 - b_1)/r_{12}, \quad N_{12} = (c_2 - c_1)/r_{12} \\ \text{where} \quad r_{12}^2 = (a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2 \end{array}\right\} \quad \text{.........(ii)}.$$

(iii) The angle between two lines, θ, is given by:

$$\cos\theta = L_{12}L_{34} + M_{12}M_{34} + N_{12}N_{34} \quad \text{..................(iii)}.$$

(iv) The angle ϕ between two planes as above is given by

$$\cos\phi = l_1l_2 + m_1m_2 + n_1n_2 \quad \text{..................(iv)}.$$

(v) The angle ψ between a line and a plane is given by

$$\sin\psi = L_{12}l_1 + M_{12}m_1 + N_{12}n_1 \quad \text{..................(v)}.$$

(vi) The plane through the points is given by

$$\begin{vmatrix} x-a_1 & y-b_1 & z-c_1 \\ a_2-a_1 & b_2-b_1 & c_2-c_1 \\ a_3-a_1 & b_3-b_1 & c_3-c_1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)},$$

where it is simplest to express the determinant numerically before expanding it.

(vii) Let x, y, z represent the coordinates referred to the three planes of reference already discussed of any of the "mid-sagittal" points and S denote a summation for all these points: let $\bar{x}, \bar{y}, \bar{z}$ be the mean coordinates of all these points, n in number; let

$$\sigma_x^2 = \frac{1}{n} S(x^2) - \bar{x}^2,\ \sigma_y^2 = \frac{1}{n} S(y^2) - \bar{y}^2,\ \sigma_z^2 = \frac{1}{n} S(z^2) - \bar{z}^2,$$

be the x, y and z squared standard deviations; and let

$$p_{zy} = \frac{1}{n} S(zy) - \bar{z}\bar{y},\ p_{xz} = \frac{1}{n} S(xz) - \bar{x}\bar{z},\ p_{yx} = \frac{1}{n} S(yx) - \bar{y}\bar{x}$$

be the product moment coefficients.

We must then solve the equation

$$\begin{vmatrix} \sigma_x^2 - \Sigma^2, & p_{yx}, & p_{xz} \\ p_{yx}, & \sigma_y^2 - \Sigma^2, & p_{zy} \\ p_{xz}, & p_{zy} & \sigma_z^2 - \Sigma^2 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(vii)},$$

or

$$\Sigma^6 - \Sigma^4(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) + \Sigma^2(\sigma_z^2\sigma_y^2 + \sigma_x^2\sigma_z^2 + \sigma_y^2\sigma_z^2 - p^2_{zy} - p^2_{xz} - p^2_{yx})$$
$$- (\sigma_x^2\sigma_y^2\sigma_z^2 - \sigma_x^2 p^2_{zy} - \sigma_y^2 p^2_{xz} - \sigma_z^2 p^2_{yx} + 2p_{zy}\,p_{xz}\,p_{yx}) = 0 \quad \ldots\ldots\text{(viii)},$$

and we shall obtain three values $\Sigma_1^2, \Sigma_2^2, \Sigma_3^2$ of Σ^2*. These values of Σ^2 are the minimum and the two maximum values of the mean square deviations of the n points from three planes†. The Σ_1^2 which gives the minimum value is the one we are seeking in the first place; it provides the plane of maximum symmetry, or the nearest approach we can get to a true mirror plane or median sagittal plane of the skull. The Standard Horizontal Plane and the Standard Transverse Vertical Plane must be defined with regard to this plane, and we shall deal with them later on.

To determine this plane we must solve the equations:

$$\left.\begin{array}{r} (\sigma_x^2 - \Sigma_1^2)\,L_1 + p_{yx} M_1 + p_{xz} N_1 = 0 \\ p_{yx} L_1 + (\sigma_y^2 - \Sigma_1^2)\,M_1 + p_{zy} N_1 = 0 \\ p_{xz} L_1 + p_{zy} M_1 + (\sigma_z^2 - \Sigma_1^2)\,N_1 = 0 \\ \text{subject to} \qquad L_1^2 + M_1^2 + N_1^2 = 1 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}.$$

L_1, M_1, N_1 are the direction cosines of the plane of maximum symmetry, and

$$P_1 = L_1\bar{x} + M_1\bar{y} + N_1\bar{z} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(x)}$$

or the equation of the plane is

$$L_1(x-\bar{x}) + M_1(y-\bar{y}) + N_1(z-\bar{z}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xi)}$$

and Σ_1^2 measures the mean square deviation of the mid-sagittal points from this plane.

* *Philosophical Magazine*, 1901, pp. 561—568.

† All the terms in brackets are positive, and the equation has three real roots.

5. *Illustration of Numerical Work.*

We will now give a numerical example of finding a plane of maximum symmetry, and then more briefly provide the equations giving Σ_1^2 and this and other planes for several skulls. We will take the Arab skull, section (C) of Table I, p. 224, and consider the 13 sets of mid-sagittal points, coordinates given in the following table:

TABLE III.

Coordinates for Arab Skull.

Point	x	y	z	$z'=z-193{\cdot}5$
1. Alveolar Point ...	+ 46·6	+88·8	198·1	+4·6
2. Nasal Spine	+ 28·8	+92·3	198·1	+4·6
3. Nasion	− 29·4	+83·7	196·5	+3·0
4. Glabella	− 39·6	+84·8	196·0	+2·5
5. Bregma	−119·9	+ 4·1	190·6	−2·9
6. Apex	−119·0	0·0	196·1	+2·6
7. Lambda	− 49·9	−78·1	189·7	−3·8
8. Occipital Protuberance	− 6·2	−75·5	190·6	−2·9
9. Kappa	0·0	−71·2	189·8	−3·7
10. Inferior Inion ...	+ 6·9	−65·8	189·8	−3·7
11. Opisthion	+ 29·8	−38·4	193·2	−0·3
12. Basion	+ 25·5	+ 4·1	193·1	−0·4
13. Baso Palatal Spine ...	+ 30·8	+45·4	194·4	+0·9
Sum	−195·6	+74·2	2516·0	+0·5

Sum ÷ 13;

(i) $\bar{x}=-15{\cdot}046{,}154$, $\bar{y}=+5{\cdot}707{,}692$, $\bar{z}=193{\cdot}538{,}4615$, $\bar{z}'=+{\cdot}038{,}4615$.

Squares;

(ii) $\bar{x}^2=226{\cdot}386{,}750$, $\bar{y}^2=32{\cdot}577{,}748$, ... $\bar{z}'^2={\cdot}001{,}479$.

Products:

(iii) $\bar{y}\bar{z}'=+{\cdot}219{,}526$, $\bar{z}'\bar{x}=-{\cdot}578{,}698$, ... $\bar{x}\bar{y}=-85{\cdot}878{,}813$.

Mean Squares of Coordinates from Table IV:

(iv) $S(x^2)/13=3002{\cdot}578{,}4615$, $S(y^2)/13=4259{\cdot}210{,}769$, $S(z'^2)/13=9{\cdot}540{,}769$.

Mean Products from Table IV:

(v) $S(yz')/13=181{\cdot}356{,}923$, $S(z'x)/13=29{\cdot}892{,}308$, $S(xy)/13=365{\cdot}822{,}308$.

(iv) minus (ii): $\sigma_x^2=2776{\cdot}191{,}711$, $\sigma_y^2=4226{\cdot}633{,}021$, $\sigma_z^2=9{\cdot}539{,}290$.

(v) minus (iii): $p_{yz}=181{\cdot}137{,}397$, $p_{zx}=30{\cdot}471{,}006$, $p_{xy}=451{\cdot}701{,}121$.

We are now in a position to write down the fundamental cubic as given by Equation (viii). It is

$$\Sigma^6-7012{\cdot}364{,}022\,\Sigma^4+1156{,}2972{\cdot}392{,}766\,\Sigma^2-1996{,}0121{\cdot}046{,}469=0.$$

The ratio of the coefficients of the last two terms is 1·726, and we test with $\Sigma^2=1{\cdot}730$ giving 22839, and 1·728 giving − 238·353. The root is close to 1·728.

Applying Newton's Rule we obtain 1·728,0208 and thence finally $\Sigma_1^2 = 1{\cdot}728{,}0207$. This is the mean square deviation of the thirteen mid-sagittal points from the Plane of Maximum Symmetry. The other two roots of the cubic will be discussed later.

TABLE IV.

Squares and Products of the Coordinates of the Arab Skull.*

	x^2	y^2	z'^2	yz'	$z'x$	xy
(1)	+2171·56	7885·44	21·16	+408·48	+214·36	+4138·08
(2)	829·44	8519·29	21·16	+424·58	+132·48	+2658·24
(3)	864·36	7005·69	9·00	+251·10	− 88·20	−2460·78
(4)	1568·16	7191·04	6·25	+212·00	− 99·00	−3358·08
(5)	14376·01	16·81	8·41	− 11·89	+347·71	− 491·59
(6)	14161·00	0·00	6·76	0·00	−309·40	0·00
(7)	2490·01	6099·61	14·44	+296·78	+189·62	+3897·19
(8)	38·44	5700·25	8·41	+218·95	+ 17·98	+ 468·10
(9)	0·00	5069·44	13·69	+263·44	0·00	0·00
(10)	47·61	4329·64	13·69	+243·46	− 25·53	− 454·02
(11)	888·04	1474·56	·09	+ 11·52	− 8·94	−1144·32
(12)	650·25	16·81	·16	− 1·64	− 10·20	+ 104·55
(13)	948·64	2061·16	·81	+ 40·86	+ 27·72	+1398·32
Sum	39033·52	55369·74	124·03	+2357·64	+388·60	+4755·69

Sum ÷ 13 3002·578,4615 4259·210,769 9·540,769 +181·356,923 +29·892,308 +365·822,308

The whole of these values (Tables III and IV) have been put down for the use of any craniologist who may desire to test the labour of finding a Plane of Maximum Symmetry.

We have now to determine the direction cosines L_1, M_1, N_1, of the Plane of Maximum Symmetry from Equations (ix). These give us:

$$2774{\cdot}463{,}690\,L_1 + 451{\cdot}701{,}121\,M_1 + 30{\cdot}471{,}006\,N_1 = 0,$$

$$451{\cdot}701{,}121\,L_1 + 4224{\cdot}905{,}000\,M_1 + 181{\cdot}137{,}397\,N_1 = 0,$$

$$30{\cdot}471{,}006\,L_1 + 181{\cdot}137{,}397\,M_1 + 7{\cdot}811{,}269\,N_1 = 0.$$

From the first two equations we find

$$\frac{L_1}{1} = \frac{M_1}{10{\cdot}418{,}268} = \frac{-N_1}{245{\cdot}492{,}616}$$

for the relative values, and since $L_1^2 + M_1^2 + N_1^2 = 1$ we have for the absolute values of the direction cosines

$$L_1 = {\cdot}0040{,}6975, \quad M_1 = {\cdot}0423{,}9970, \quad N_1 = -{\cdot}9990{,}9244,$$

and if these values be substituted in the third equation for L_1, M_1, N_1 above, the

* In actual practice the squares are put directly on the machine from Barlow's Tables, and the product multiplications are a continuous process on the machine.

left-hand side will be found to be $+ \cdot 0000{,}0087$ instead of zero, which checks the value for Σ_1^2 as we have only worked to six decimal places in the coefficients.

Finally substituting in the values for P_1 given by (x) we find $P_1 = -193 \cdot 181{,}851$ or the equation to the Plane of Maximum Symmetry is

$$-\cdot 0040{,}6975\,x - \cdot 0423{,}9970\,y + \cdot 9990{,}9244\,z = +193\cdot 181{,}851 \quad \ldots\ldots\text{(xii)}.$$

Knowing this plane we can determine the angles it makes with:

(*a*) the Frankfurt Horizontal Plane, i.e. $x = 0$. The cosine of this angle $= -\cdot 0040{,}6975$, or the angle is $90° \, 14' \cdot 0$. Thus the Frankfurt Horizontal Plane is not for this skull perpendicular to the Plane of Maximum Symmetry;

(*b*) the Transverse Vertical Plane, i.e. $y = 0$. The cosine of this angle $= -\cdot 0423{,}9970$ and the angle is $90° \, 21' \cdot 8$, or the Transverse Vertical Plane is not perpendicular to the Plane of Maximum Symmetry;

(*c*) the Plane perpendicular to the auricular axis, i.e. $z = 0$. The cosine of this angle is $\cdot 9990{,}9244$ and the angle is $2° \, 26' \cdot 5$. Thus the auricular axis for this skull is not perpendicular to the best plane we can adopt for mid-sagittal symmetry.

Again for this skull the Left Porion has for coordinates (0, 0, 255·7) and the Right Porion (0, 0, 133·9); thus the Interporial Distance equals 121·8, and the Mid-porion is (0, 0, 194·8). Thus the Mid-porion is 1·3 mm. above the mean height of the thirteen mid-sagittal points, and further the Plane of Maximum Symmetry does not pass through the Mid-porion, but meets the auricular axis at the point $z = 193 \cdot 36$. The perpendicular distances of the Poria from the Plane of Maximum Symmetry are Left Porion 62·29 and Right Porion 59·40, which indicate the extent the two ears are asymmetrically placed. We can now write down the Equation for the Median Sagittal Plane as based on Nasion, Bregma and Lambda for this skull. By Equation (vi) it is

$$\begin{vmatrix} x + 119\cdot 9, & y - 4\cdot 1, & z - 190\cdot 6 \\ -29\cdot 4 + 119\cdot 9, & 83\cdot 7 - 4\cdot 1, & 196\cdot 5 - 190\cdot 6 \\ -49\cdot 9 + 119\cdot 9, & -78\cdot 1 - 4\cdot 1, & 189\cdot 7 - 190\cdot 6 \end{vmatrix} = 0,$$

or,

$$\begin{vmatrix} x + 119\cdot 9, & y - 4\cdot 1, & z - 190\cdot 6 \\ 90\cdot 5, & 79\cdot 6, & 5\cdot 9 \\ 70\cdot 0, & -82\cdot 2, & -0\cdot 9 \end{vmatrix} = 0,$$

or, expanding,

$$413\cdot 34x + 494\cdot 45y - 13011\cdot 10z + 252{,}7447\cdot 881 = 0.$$

Dividing by the square root of the sum of the squares of the coefficients of x, y, z we have the equation to the plane in its prepared form, i.e. with the coefficients the direction cosines and the constant term the perpendicular from the origin,

$$-\cdot 031{,}7294x - \cdot 037{,}9556y + \cdot 998{,}7756z = +194\cdot 015{,}354.$$

It is possible now by inserting the coordinates of any of the mid-sagittal points to determine its distance from this plane. The following are the distances of these points for the Arab skull:

(1) Alveolar Point	+1·0070	(8) Occipital Protuberance	+ ·5864
(2) Nasal Spine ...	+ ·5750	(9) Kappa	+1·7453
(3) Nasion ...	·0000	(10) Inferior Inion ...	+2·1692
(4) Glabella ...	+ ·2175	(11) Opisthion	+ ·5399
(5) Bregma ...	·0000	(12) Basion	+2·1165
(6) Apex	−5·6204	(13) Base Palatal Spine ...	+2·5538
(7) Lambda ...	·0000	Sum of Squares of these Distances	52·3686

Mean Square Distance from Plane 4·0284.

Thus although the plane actually passes through *three* of the mid-sagittal points, its mean square deviation is more than double that of the Plane of Maximum Symmetry. While the mid-sagittal contour drawn in the Nasion-Bregma-Lambda Plane may serve many useful comparative purposes it clearly differs widely from any suitable mirror plane, and actually for this Arab skull the angle between this $\nu.\beta.\lambda.$ plane and the Plane of Maximum Symmetry is 1° 36′·3. It makes an angle of 91° 49′·1 instead of a right angle with the Frankfurt Horizontal Plane, an angle of 92° 50′·5 with the auricular axis and an angle of 92° 10′·5 with the Transverse Vertical Plane.

We may give as one furthur illustration the measure of prognathism as found from the angle between the Frankfurt Horizontal Plane and the line joining Alveolar Point to Nasion*. These points are (1) and (3), or for the Arab skull (+46·6, +88·8, +198·1) and (−29·4, +83·7, +96·5). The equation to the line joining them is

$$\frac{x-46\cdot6}{-29\cdot4-46\cdot6}=\frac{y-88\cdot8}{83\cdot7-88\cdot8}=\frac{z-198\cdot1}{196\cdot5-198\cdot1},$$

or

$$\frac{x-46\cdot6}{76\cdot0}=\frac{y-88\cdot8}{5\cdot1}=\frac{z-198\cdot1}{1\cdot6},$$

or the direction cosines (l, m, n) of this line are proportional to

$$76\cdot0, \qquad 5\cdot1, \qquad 1\cdot6,$$

giving for absolute value, since $l^2+m^2+n^2=1$,

$$+\cdot997{,}536, \qquad +\cdot066{,}940, \qquad +\cdot021{,}001.$$

The Sine of the angle this lines makes with the Frankfurt Horizonal Plane, i.e. $x=0$, is ·997,536 or the Profile Angle itself = 85° 58′·6.

If we ask how far does the Nasion-alveolar line lie outside the Plane of Maximum Symmetry we have to find the angle between the plane whose direction cosines are

$$-\cdot004{,}070, \qquad -\cdot042{,}400, \qquad +\cdot999{,}092,$$

and the line with direction cosines

$$+\cdot997{,}536, \qquad +\cdot066{,}940, \qquad +\cdot021{,}001.$$

* This is of course the measure of prognathism in the skull, but the racial order of prognathism in the living, owing to the thickness and protrusion of the lips, may be very different. Perhaps this fact is not always adequately emphasised.

The sine of the angle (see Equation (v)) is + ·014,0837, giving an angle of + 48′·4, or the Profile Line is skewed out of the Plane of Maximum Symmetry towards the right side by more than three-quarters of a degree.

We have illustrated sufficiently the general algebraic and numerical processes by which, when the cranial coordinatograph has given the coordinates, the properties of any skull can be discussed by the aid of the elementary formulae of analytical geometry of three dimensions.

6. *Analytical Geometry of six illustrative Crania.*

The following six illustrative skulls were chosen at random. It is not suggested that average results obtained from them would not be widely modified when considerable series of the same races are dealt with; they are solely given here as indicative of the type of problems the cranial coordinatograph enables the craniologist to attack.

We have the following numerical results:

(*A*) *Fuegian Skull** (Recent).

Elevation of R. Porion 253·8, L. Porion 126·2, Mid-porion 190·0, Bi-porionic Distance 127·6.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6 - 7971{\cdot}463{,}075\, \Sigma^4 + 1460{,}0879{\cdot}769{,}528\, \Sigma^2 - 2446{,}0978{\cdot}059{,}113 = 0.$$

$$\text{Value of } \Sigma_1^2 = 1{\cdot}676{,}844.$$

Plane of Maximum Symmetry:

$$-{\cdot}0095{,}7040\, x + {\cdot}0157{,}3352\, y + {\cdot}9998{,}3036\, z = +\,189{\cdot}0220{,}8197.$$

$\nu.\beta.\lambda.$ Plane†:

$$-{\cdot}044{,}7048\, x + {\cdot}010{,}2709\, y + {\cdot}998{,}9474\, z = +\,190{\cdot}677{,}8475.$$

Martin's Plane‡:

$$+{\cdot}122{,}4878\, x + {\cdot}038{,}6513\, y + {\cdot}992{,}9011\, z = +\,188{\cdot}966{,}1990.$$

(*B*) *Nubian Skull** (Ancient Egyptian from Kerma).

Elevation of R. Porion 247·0, L. Porion 132·7, Mid-porion 189·85, Bi-porionic Distance 114·3.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6 - 7662{\cdot}666{,}546\, \Sigma^4 + 1392{,}3838{\cdot}355{,}612\, \Sigma^2 - 1801{,}6485{\cdot}994{,}210 = 0.$$

$$\text{Value of } \Sigma_1^2 = 1{\cdot}294{,}8535.$$

Plane of Maximum Symmetry:

$$+{\cdot}005{,}4027\, x - {\cdot}006{,}2292\, y + {\cdot}999{,}9660\, z = +\,188{\cdot}664{,}6436.$$

* In the case of these crania the Inferior Inion was omitted.

† The usual "mid-sagittal" plane, i.e. that through Nasion (ν), Bregma (β) and Lambda (λ).

‡ The plane through Nasion, Inion, Basion, assumed by Martin to be "ohne Zweifel" the "best" mid-sagittal plane.

$\nu.\beta.\lambda.$ Plane:

$$-\cdot0112{,}3688\,x+\cdot0021{,}0153\,y-\cdot999{,}9347\,z=+188\cdot752{,}1300.$$

Martin's Plane:

$$+\cdot013{,}9371\,x-\cdot006{,}7188\,y+\cdot999{,}8803\,z=188\cdot231{,}5380.$$

(*C*) *Arab Skull* (Modern from Palestine).

Elevation of R. Porion 133·9, L. Porion 255·7, Mid-porion 194·8, Bi-porionic Distance 121·8.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7012\cdot364{,}022\,\Sigma^4+1156{,}2972\cdot392{,}766\,\Sigma^2-1996{,}0121\cdot046{,}469=0.$$

$$\text{Value of }\Sigma_1^2=1\cdot7280{,}2066.$$

Plane of Maximum Symmetry:

$$-\cdot0040{,}6975\,x-\cdot0423\cdot9970\,y+\cdot9990{,}9244\,z=+193\cdot181{,}851.$$

$\nu.\beta.\lambda.$ Plane:

$$-\cdot031{,}7294\,x-\cdot037{,}9556\,y+\cdot998{,}7756\,z=+194\cdot015{,}354.$$

Martin's Plane:

$$-\cdot010{,}3855\,x+\cdot035{,}5214\,y-\cdot999{,}3150\,z=-193\cdot086{,}908.$$

(*D*) *Teita Negro Skull.*

Elevation of R. Porion 130·6, L. Porion 241·8, Mid-porion 186·2, Bi-porionic Distance 111·2.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7350\cdot914{,}793\,\Sigma^4+1196{,}5899\cdot9308\,\Sigma^2-1369{,}2532\cdot1372=0.$$

$$\text{Value of }\Sigma_1^2=1\cdot1451{,}0147.$$

Plane of Maximum Symmetry:

$$-\cdot0220{,}1936\,x-\cdot0083{,}0922\,y+\cdot9997{,}2301\,z=+186\cdot384{,}314.$$

$\nu.\beta.\lambda.$ Plane:

$$-\cdot0508{,}8478\,x-\cdot0064{,}6706\,y+\cdot9986{,}8359\,z=+188\cdot167{,}4885.$$

Martin's Plane:

$$+\cdot076{,}1587\,x+\cdot014{,}4832\,y+\cdot996{,}9905\,z=+186\cdot761{,}565.$$

(*E*) *17th Century English Skull* (St Bride's Graveyard).

Elevation of R. Porion 258·7, L. Porion 133·3, Mid-porion 196·0, Bi-porionic Distance 125·4.

Cubic to determine Minimum Mean Square Residual Σ_1^2:

$$\Sigma^6-7887\cdot700{,}659\,\Sigma^4+1383{,}3943\cdot158{,}623\,\Sigma^2-4033{,}3720\cdot552{,}071=0.$$

$$\text{Value of }\Sigma_1^2=2\cdot9204{,}2324.$$

Plane of Maximum Symmetry:

$$+\cdot0081{,}2136\,x-\cdot0109{,}6533\,y+\cdot9999{,}0690\,z=+196\cdot739{,}7800.$$

$\nu.\beta.\lambda.$ Plane:

$$+\cdot 0528{,}8718\,x - \cdot 0461{,}5479\,y + \cdot 9975{,}3330\,z = +191\cdot 526{,}634.$$

Martin's Plane:

$$+\cdot 0178{,}5293\,x - \cdot 0011{,}2259\,y + \cdot 9998{,}3999\,z = +197\cdot 122{,}354.$$

(*F*) *Hindu Skull* (Modern, Bengal).

Elevation of R. Porion 130·4, L. Porion 241·6, Mid-porion 186·0, Bi-porionic Distance 111·2.

Cubic to determine Minimum Mean Square Residual Σ_1^2.

$$\Sigma^6 - 7399\cdot 968{,}994\,\Sigma^4 + 1162{,}8428\cdot 712{,}714\,\Sigma^2 - 2607{,}0720\cdot 672{,}768 = 0.$$

$$\text{Value of } \Sigma_1^2 = 2\cdot 2451{,}8824.$$

Plane of Maximum Symmetry:

$$-\cdot 0017{,}4154\,x - \cdot 0054{,}1230\,y + \cdot 9999{,}8384\,z = +186\cdot 605{,}594.$$

$\nu.\beta.\lambda.$ Plane:

$$+\cdot 0667{,}4424\,x - \cdot 0118{,}7277\,y + \cdot 9976{,}9948\,z = +180\cdot 915{,}098.$$

Martin's Plane:

$$-\cdot 1114{,}3673\,x - \cdot 0143{,}7877\,y + \cdot 9936{,}6750\,z = +184\cdot 759{,}760.$$

On the basis of these and similar results we will now proceed to some comparisons.

7. *Angular Relations.*

We have seen that it has been customary to treat the Frankfurt Horizontal Plane, the Transverse Vertical Plane through the auricular axis and a certain plane termed the Median Sagittal Plane as standard planes of the skull. Such standard planes should be mutually rectangular, but while the first two are at right angles, they are rarely perpendicular to the third. The third plane as defined by Martin, to judge from the present illustrative crania, seems very inferior to the $\nu.\beta.\lambda.$ plane (see our p. 228). We shall here then confine our attention to the latter plane. Table V provides the angles between our First Standard Plane of the skull—the Plane of Maximum Symmetry—as representing the mirror plane of the skull, and the planes which have been usually hitherto treated as standard planes, but which we treat merely as planes of reference.

Section (α) of the Table shows that the $\nu.\beta.\lambda.$ Mid-sagittal Plane in none of the six skulls approaches closely to the Plane of Maximum Symmetry, the average angle between the two planes being more than two degrees. In the English skull there is more than three degrees and in the Hindu nearly four degrees. Thus the $\nu.\beta.\lambda.$ plane cannot be looked upon as a close fit to the mid-sagittal points.

Section (β) shows us that the Frankfurt Horizontal Plane is *not* perpendicular to the Plane of Maximum Symmetry, which I personally think should be a prerequisite of a standard horizontal plane. The average deviation is over 30′.

TABLE V.

Angles between Standard Planes and Planes of Reference.

Angle between						
(α) Plane of Maximum Symmetry and "Median Sagittal" Plane (ν.β.λ.)	Ancient Egyptian (Nubia) 0° 24′·6	Arab (Modern) 1° 36′·3	Negro (Teita) 1° 39′·5	Fuegian 1° 59′·6	English (17th century) 3° 14′·0	Hindu (Bengal) 3° 56′·7
	These planes may differ widely, almost visibly from one another. Average Angle 2° 8′·45.					
(β) Plane of Maximum Symmetry and Frankfurt Horizontal Plane	Negro (Teita) 91° 15′·7	Fuegian 90° 35′·7	Arab (Modern) 90° 14′·0	Hindu (Bengal) 90° 6′·0	Ancient Egyptian (Nubia) 89° 41′·4	English (17th century) 89° 31′·9
	Average Deviation from perpendicularity = 30′·3					
(γ) "Mid-sagittal" Plane (ν.β.λ.) and Frankfurt Horizontal Plane	Hindu (Bengal) 93° 49′·6	Negro (Teita) 92° 55′·0	Fuegian 92° 34′·3	Arab (Modern) 91° 49′·1	Ancient Egyptian (Nubia) 89° 21′·4	English (17th century) 86° 58′·1
	Average Deviation from perpendicularity = 2° 27′·3					
(δ) Plane of Maximum Symmetry and Plane perpendicular to Auricular Axis	Hindu (Bengal) 0° 19′·6	Ancient Egyptian (Nubia) 0° 28′·8	English (17th century) 0° 47′·0	Fuegian 1° 5′·3	Negro (Teita) 1° 20′·0	Arab (Modern) 2° 26′·5
	Average Angle between Planes = 1° 4′·5					
(ε) "Mid-sagittal" Plane (ν.β.λ.) and Plane perpendicular to Auricular Axis	Ancient Egyptian (Nubia) 0° 39′·3	Fuegian 2° 37′·7	Arab (Modern) 2° 50′·5	Negro (Teita) 2° 56′·4	Hindu (Bengal) 3° 53′·2	English (17th century) 4° 1′·5
	Average Angle between Planes = 2° 49′·8					
(ζ) Plane of Maximum Symmetry and Transverse ("Vertical") Plane	English (17th century) 90° 37′·7	Negro (Teita) 90° 28′·5	Arab (Modern) 90° 21′·8	Ancient Egyptian (Nubia) 90° 21′·2	Hindu (Bengal) 90° 18′·6	Fuegian 89° 5′·4
	Average Deviation from perpendicularity = 30′·4					
(η) "Mid-sagittal" Plane (ν.β.λ.) and Transverse ("Vertical") Plane	English (17th century) 92° 38′·7	Arab (Modern) 92° 10′·5	Hindu (Bengal) 90° 40′·8	Ancient Egyptian (Nubia) 90° 7′·2	Negro (Teita) 89° 37′·8	Fuegian 89° 24′·7
	Average Deviation from perpendicularity = 1° 5′·8					

Section (γ) indicates that the usual ($\nu.\beta.\lambda.$) Mid-sagittal Plane is even worse in the degree of perpendicularity to the Frankfurt Horizontal Plane, the average deviation amounting to 2° 27'·3 or more than four times that of the Plane of Maximum Symmetry.

Section (δ) shows us that the auricular axis (Bi-porionic line) is not perpendicular to the Plane of Maximum Symmetry, the average deviation for the six crania being slightly over 1°. From the standpoint of the present writer this is one measure of the asymmetrical location of the ears. Relative to a good approximation to a mirror plane the two ears are shifted slightly forward or backward, upward or downward.

Section (ϵ) gives the like angle between the auricular axis and the perpendicular to the $\nu.\beta.\lambda.$ Plane. The average angle for this skull is 2° 49'·8 or more than $2\frac{1}{2}$ times as great as in the case of the Plane of Maximum Symmetry.

Section (ζ) gives the angle between the Transverse Vertical Plane and that of Maximum Symmetry, the average deviation is half a degree, or the customary transverse vertical plane is not perpendicular to the plane of closest fit to the mid-sagittal points, as it should be in the opinion of the present writer.

Section (η) indicates, however, that the Plane of Maximum Symmetry is more than twice as fit as the $\nu.\beta.\lambda.$ Plane to represent the mid-sagittal section. If we start from the Frankfurt Plane and the Transverse Vertical Plane passing through the auricular axis as standard rectangular planes of the skull, then the Plane of Maximum Symmetry is more nearly perpendicular to both of these than the usual mid-sagittal section.

8. *Mean Square Deviation of the Mid-sagittal Points from various Planes.*

We can now look at another aspect of the relationship of the mid-sagittal points to the various planes which may be suggested for the mid-sagittal section. We may determine the Mean Square Distance, or the so-called Mean Square Residual, of the mid-sagittal points from the various planes. These values are given in Table VI. The five planes which may be considered as possible mid-sagittal planes are: (*a*) Our standard plane, the Plane of Maximum Symmetry: this must, of course, have the minimum Mean Square Distance. (*b*) The plane perpendicular to the auricular axis at the mean elevation, $\bar{z}$, of the mid-sagittal points. We may term this the "$\bar{z}$ Plane"; its Mean Square Distance is the σ_z^2 of our p. 231. (*c*) The plane perpendicular to the auricular axis and bisecting it. This may be termed the mid-porion sagittal plane. (*d*) The mid-sagittal plane through Nasion, Bregma and Lambda (the $\nu.\beta.\lambda.$ Plane). (*e*) Martin's Plane through Nasion, Inion and Basion.

We have already (see p. 228) discussed the last two planes, (*d*) and (*e*) of the Table. As far as the present skulls are concerned neither is comparable with (*a*), (*b*) and (*c*). If any craniologist finds the solving of a cubic equation too severe a mathematical labour, the coordinatograph will rapidly give him $\bar{z}$ and the Mean Elevation Mid-sagittal or z-Plane. The Plane of Maximum Symmetry is twice as good

as this plane, but the latter is very much better than either of the planes fixed by three mid-sagittal points only.

Clearly the mean square residual for any skull is a rough test of how far the mid-sagittal points lie in one plane, i.e. how far there is a true mid-sagittal plane, on the basis of which we could test the mirror symmetry of other points of the skull. A little thought, however, shows that this is only a rough approximation; the skulls are of different absolute sizes, and if we increased the linear dimensions of a skull by 5 °/ₒ we should increase the mean square residual by more than 10 °/ₒ. We need accordingly an index from which we have eliminated the absolute size of the skull. We cannot therefore assert that the relative mid-sagittal asymmetry of the above six skulls is measured by the numbers in the (*a*) column of Table VI.

TABLE VI.

Mean Square Distance of the Mid-sagittal Points from five Planes which may be treated as Mid-sagittal.

Skull	(*a*) Plane of Maximum Symmetry as Standard Mid-sagittal Plane	(*b*) Plane perpendicular to Auricular Axis at Mean Elevation of Mid-sagittal Points	(*c*) Plane perpendicular to Auricular Axis through Mid-porion	(*d*) Usual Mid-sagittal Plane, or Plane through Nasion, Bregma and Lambda	(*e*) Martin's Mid-sagittal Plane, or Plane through Nasion, Inion and Basion
Negro (Teita) ...	1·1451	2·7498	2·7593	5·7284	32·0419
Ancient Egyptian (Nubia)	1·2949	1·5245	2·4443	1·5167	1·5909
Fuegian	1·6768	2·8727	4·8327	8·5585	62·4137
Arab (Modern) ...	1·7280	9·5393	11·1308	4·0284	2·5998
Hindu (Bengal) ...	2·2452	2·4083	2·8062	31·1950	35·5764
English (17th century) ...	2·9204	3·6529	4·5923	27·3563	4·0257
Mean for six skulls	1·8351	3·7912⁵	4·7609	13·0055	23·0414

To allow for the absolute size of the skull I have taken three lengths of the skull in directions at right angles, choosing these lengths from the plan and elevation drawings rather than from calliper measurements of the skull. I have taken:

(*a*) The Bi-porionic Distance; this is a length on the axis of z.

(*b*) The perpendicular from the Apex on the auricular axis. This is the x coordinate of the Apex.

(*c*) The length of the projection of the line joining Nasion to Kappa onto the axis of y, i.e. onto the line joining the plan of Kappa to the plan of the Poria.

I have then squared the cube root of the product of (*a*), (*b*) and (*c*), and thus obtained a quantity depending on the squared linear dimensions of the skull, by which it seemed that the mean square residuals might be divided so as to obtain a reasonable index independent of the dimensions of the skull. The resulting

number has been multiplied by 100. In the case of our six skulls the following Table gives the resulting indices.

TABLE VII.

Indices of Mesial Asymmetry.

Skull	(α) Minimum Mean Square Residual	(β) Product of the Cranial Rectangular Lengths $(a) \times (b) \times (c)$	(γ) $\{(a) \times (b) \times (c)\}^{\frac{2}{3}}$	(δ) $\frac{10^6 (\alpha)}{(\gamma)}$ = Index
Negro (Teita)	1·1451	2,075,509·080	16271·1	70·38
Ancient Egyptian (Nubia)	1·2949	2,513,082·171	18487·3	70·04
Fuegian	1·6768	2,664,900·480	19220·5	87·24
Arab (Modern)	1·7280	2,245,151·580	17146·0	100·78
Hindu (Bengal)	2·2452	2,050,821·568	16141·8	139·09
English (17th century) ...	2·9204	2,694,690·504	19364·5	149·88

Only one change has been made in the order of the crania by reducing the minimum mean square residuals to an index independent of absolute size. It would be foolish to draw any conclusion as to the relative position of the races to which these six skulls belong from the order of either column (α) or (δ) of this Table, but the Table may suggest interesting problems, which might be followed up, if considerable racial groups were worked out* I have contented myself here by showing the manner in which numerical results may be obtained.

9. *The remaining Standard Planes.*

If we accept the view that the Plane of Maximum Symmetry is the most suitable plane to take as a mid-sagittal plane, and we term it our First Standard Plane, the question immediately follows: How are we to select our remaining standard planes? These will correspond respectively to the Horizontal Plane and Transverse Vertical Plane of the skull, but they must be chosen so as to be at right angles to one another and to the Plane of Maximum Symmetry.

Now the process described in my paper in the *Philosophical Magazine* leads to three mutual rectangular planes defined by Σ_1^2 and by Σ_2^2, and Σ_3^2 the other two roots of the cubic. Σ_2^2 and Σ_3^2 are easily found from a quadratic equation and are maxima values of the Mean Square Deviations. It might at first be supposed that these are the very planes we need. But we must remember that the first standard plane has been obtained on the basis of its approximation to the mirror plane, and only the mid-sagittal points have been used in its determination. It is accordingly somewhat one-sided to use solely these points in determining planes, which we need only limit as planes necessarily perpendicular to the Plane of Maximum Symmetry. Unfortunately when we come to the Horizontal Plane, and

* For example: Does asymmetry increase as we pass from more primitive to more highly civilised groups?

the Transverse Vertical Plane, we have no long series of points like the mid-sagittal points from which to determine approximate planes. The Apex and the Kappa are not natural points of the skull, but artificial points resulting from the definition of the Frankfurt Horizontal Plane. The Poria may be considered more nearly natural points although the Frankfurt Plane enters into their determination. The Orbitalia may be considered again as only in part natural points, as they depend upon the Poria, and the *lowest* points on the orbital rims are really meaningless, without the conception of the Horizontal Plane. But there is a greater difficulty arising here. When the skull is adjusted on the craniophor to the Frankfurt Horizontal Plane it will be found that in some cases several millimetres of the lower borders of the orbits may practically be parallel to that plane. This may not modify seriously the plans of the Orbitalia, but it renders their elevations above the drawing-board when the auricular axis of the skull is made perpendicular to the board, occasionally difficult of accurate determination.

Thus although it is perfectly easy to develop the mathematical solution for determining a plane perpendicular to a given plane, and fitting closely by a minimum square residual to a selected system of points—assumed in the case of a skull truly symmetrical to lie in that plane—yet in practice it is not easy to select such a system of points in the case either of the Horizontal Plane, or of the Transverse Vertical Plane.

I will not give here the mathematical theory by which a plane is constructed perpendicular to a given plane and of closest fit to a selected series of points, but merely indicate what resulted in the case of the Fuegian skull. The points selected were:

the two Poria (0, 0, 253·8) and (0, 0, 126·2),

and the two Orbitalia (13·3, 82·4, 219·0) and (5·1, 82·4, 145·8).

The equation to the plane perpendicular to the Plane of Maximum Symmetry of this skull and of closest fit to the above four points is*

$$\cdot 967{,}9125\, x - \cdot 250{,}8948\, y + \cdot 014{,}0420\, z + 13\cdot 946{,}272 = 0.$$

This plane makes an angle of 14° 33′·2 with the plane of $x = 0$, and is accordingly not very close to the Frankfurt Horizontal Plane. Clearly the plane perpendicular to our First Standard Plane and fitting closely to the Poria and Orbitalia is not, for this skull at any rate, at all approximated to by the Frankfurt Horizontal Plane.

The above plane is the plane of Least Square Residual from the Poria and Orbitalia which is perpendicular to the First Standard Plane. The plane of maximum mean square deviation subject also to the perpendicularity condition is

$$\cdot 251{,}0725\, x + \cdot 967{,}8841\, y - \cdot 012{,}7715\, z - 15\cdot 889{,}862 = 0.$$

This corresponds to the Transverse Vertical Plane, but it makes an angle of 14° 33′·6 with it.

* The plane determined in this way passes through the centroid of the Poria and Orbitalia.

These results are not satisfactory, if we wish our second and third standard planes to approximate fairly closely to the Frankfurt Horizontal Plane and the customary Transverse Vertical Plane. Accordingly, it seemed worth while investigating whether the other two roots of the fundamental cubic would give a system of rectangular planes with adequate approximation to the Frankfurt Horizontal Plane and to the customary Transverse Vertical Plane.

In the case of the Fuegian skull the three roots of the cubic are

$$\Sigma_1^2 = 1{\cdot}676{,}8435,$$
$$\Sigma_2^2 = 5121{\cdot}492{,}8812,$$
$$\Sigma_3^2 = 2848{\cdot}293{,}3502.$$

Σ_2^2 with the greater maximal value of the Mean Square Residual led to the plane

$$\cdot 4312{,}1286\,x + \cdot 9021{,}9390\,y - \cdot 0100{,}8176\,z = 12{\cdot}8930{,}3076.$$

This plane is closest to the Transverse Vertical Plane, or the reference plane $y = 0$, but it makes an angle of $25^\circ\ 33'{\cdot}1$ with it.

The other maximum Σ_3^2 provides the plane

$$-\cdot 9021{,}9942\,x + \cdot 4310{,}4409\,y - \cdot 0154{,}0415\,z = 22{\cdot}6879{,}0655,$$

which corresponds to the Frankfurt Horizontal Plane, but makes an angle with it of $25^\circ\ 33'{\cdot}1$, equal to that which the previous plane makes with the Transverse Vertical Plane. This did not seem very hopeful, but the like planes were worked out for the Negro skull from the Teita Hills. The three roots of the fundamental cubic (given on p. 237) are

$$\Sigma_1^2 = 1{\cdot}1451{,}0147,\quad \Sigma_2^2 = 4918{\cdot}7879{,}4908,\quad \Sigma_3^2 = 2430{\cdot}9817{,}4235.$$

Σ_2^2 gives us the equation

$$\cdot 0872{,}1082\,x + \cdot 9961{,}3765\,y + \cdot 0102{,}0028\,z = 9{\cdot}548{,}7522.$$

This plane is closest to the Transverse Vertical Plane, or $y = 0$; it makes an angle with it of $5^\circ\ 3'{\cdot}2$.

Σ_3^2 gives us

$$-\cdot 9959{,}4649\,x + \cdot 0874{,}1127\,y - \cdot 0212{,}0066\,z = 9{\cdot}397{,}5540,$$

which makes an angle of $-5^\circ\ 9'{\cdot}6$ with the plane of $x = 0$, or the Frankfurt Horizontal Plane. These results for the Negro skull are better than those for the Fuegian, but are not satisfactory, if we desire our Second and Third Standard Planes to approach fairly closely to the Frankfurt Horizontal Plane and the customary Transverse Vertical Plane.

No doubt the three roots of the fundamental cubic would provide three standard mutually rectangular planes possessing certain very definite physical relations with regard to any skull, but they would fail to provide close approximations to the Frankfurt Horizontal Plane. Accordingly, I started on a different route to find my Second Standard Plane. I sought a plane which should be at right angles to the First Standard Plane, i.e. the Plane of Maximum Symmetry, and should have a maximum cosine, i.e. a minimum angle with the Frankfurt Horizontal Plane. This should

form the Second Standard Plane. The Third Standard Plane must be perpendicular to the First and Second Standard Planes, and this will fully determine its direction cosines.

Mathematically: let λ, μ, ν be the direction cosines of the Second Standard Plane; l, m, n those of the First Standard Plane, and λ', μ', ν' of the Third Standard Plane. Let L, M, N be the direction cosines of the plane with which λ, μ, ν is to make a minimum angle θ or a maximum cosine $u = \cos\theta$. Then

$$u = L\lambda + M\mu + N\nu,$$
$$0 = l\lambda + m\mu + n\nu,$$
$$\lambda^2 + \mu^2 + \nu^2 = 1.$$

Hence for a maximum of u, if A and B be indeterminate multipliers

$$l + AL + B\lambda = 0, \quad m + AM + B\mu = 0, \quad n + AN + B\nu = 0.$$

Whence $$1 + A(Ll + Mm + Nn) = 0, \quad Au_0 + B = 0.$$

Accordingly, $$L(1 - l^2) - Mlm - Nnl = \lambda u_0$$
$$-Lml + M(1 - m^2) - Nmn = \mu u_0$$
$$-Lnl - Mnm + N(1 - n^2) = \nu u_0.$$

Whence

$$\frac{\lambda}{L(1 - l^2) - Mlm - Nln} = \frac{\mu}{-Lml + M(1 - m^2) - Nmn} = \frac{\nu}{-Lnl - Mnm + N(1 - n^2)} = \frac{1}{u_0} \quad \text{.........(xiii)},$$

which solve the general problem.

But for our special case: $L = 1$, $M = 0$, $N = 0$, and thus

$$\frac{\lambda}{1 - l^2} = \frac{\mu}{-ml} = \frac{\nu}{-nl} = \frac{1}{u_0} = \frac{1}{\sqrt{1 - l^2}}.$$

Thus $$\lambda = \sqrt{1 - l^2}, \quad \mu = -\frac{lm}{\sqrt{1 - l^2}}, \quad \nu = -\frac{ln}{\sqrt{1 - l^2}} \quad \text{..............(xiv)},$$

these determine the direction cosines of the Second Standard Plane.

For the Third Standard Plane we have

$$l\lambda' + m\mu' + n\nu' = 0$$

and $$\lambda\lambda' + \mu\mu' + \nu\nu' = 0,$$

or $$\frac{\lambda'}{m\nu - n\mu} = \frac{\mu'}{n\lambda - l\nu} = \frac{\nu'}{l\mu - m\lambda}$$

Or again substituting for λ, μ, ν,

$$\frac{\lambda'}{0} = \frac{\mu'}{n\sqrt{1 - l^2} + \dfrac{l^2 n}{\sqrt{1 - l^2}}} = \frac{\nu'}{\dfrac{-l^2 m}{\sqrt{1 - l^2}} - m\sqrt{1 - l^2}}.$$

Thus λ' must $= 0$, and

$$\frac{\mu'}{n} = -\frac{\nu'}{m} = \frac{1}{\sqrt{1 - l^2}},$$

Accordingly, we have finally:

$$\lambda' = 0, \quad \mu' = \frac{n}{\sqrt{1-l^2}}, \quad \nu' = -\frac{m}{\sqrt{1-l^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{xv}).$$

We have thus reached on our hypothesis fully determined directions of the three standard planes.

The reader will observe that our Third Standard Plane is invariably at right angles to the plane of $x=0$, since $\lambda'=0$, that is to the Frankfurt Horizontal Plane. We may now test how satisfactorily this arrangement works on our six illustrative crania.

(*A*) *Fuegian Skull.*

$l = -\cdot0095{,}7040, \quad m = +\cdot0157{,}3352, \quad n = +\cdot9998{,}3036.$

Direction Cosines of Second Standard Plane:

$\lambda = \cdot9999{,}5420, \quad \mu = +\cdot0001{,}5060, \quad \nu = +\cdot0095{,}6921.$

The Second Standard Plane makes an angle of 0° 32′·9 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$\lambda' = 0, \quad \mu' = +\cdot9998{,}7615, \quad \nu' = -\cdot015{,}3424.$

The Third Standard Plane makes an angle of + 0° 54′·1 with the customary Transverse Vertical Plane.

(*B*) *Ancient Egyptian Skull from Nubia.*

$l = +\cdot005{,}4027, \quad m = -\cdot006{,}2292, \quad n = +\cdot999{,}9660.$

Direction Cosines of Second Standard Plane:

$\lambda = \cdot9999{,}8540, \quad \mu = +\cdot0000{,}3365, \quad \nu = -\cdot0054{,}0260.$

The Second Standard Plane makes an angle of 0° 11′·6 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$\lambda' = 0, \quad \mu' = +\cdot9999{,}8060, \quad \nu' = +\cdot0062{,}2929.$

The Third Standard Plane makes an angle of + 0° 21′·4 with the customary Transverse Vertical Plane.

(*C*) *Arab Skull.*

$l = -\cdot0040{,}6975, \quad m = -\cdot0423{,}9970, \quad n = +\cdot9990{,}9244.$

Direction Cosines of Second Standard Plane:

$\lambda = \cdot9999{,}9127, \quad \mu = -\cdot0001{,}7256, \quad \nu = +\cdot0040{,}6609.$

The Second Standard Plane makes an angle of 0° 14′·3 with the Frankfurt Horizontal Plane.

Direction Cosines of the Third Standard Plane:

$\lambda' = 0, \quad \mu' = +\cdot9991{,}0116, \quad \nu' = +\cdot0424{,}0007.$

The Third Standard Plane makes an angle of 2° 25′·8 with the customary Transverse Vertical Plane.

(*D*) *Teita Negro Skull.*

$$l = -\cdot 0220{,}1936, \quad m = -\cdot 0083{,}0922, \quad n = +\cdot 9997{,}2301.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot 9997{,}5754, \quad \mu = -\cdot 0001{,}8301, \quad \nu = -\cdot 0220{,}1860.$$

The Second Standard Plane makes an angle of 1° 15′·7 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot 9999{,}6546, \quad \nu' = +\cdot 0083{,}1123^5.$$

The Third Standard Plane makes an angle of + 0° 28′·5 with the customary Transverse Vertical Plane.

(*E*) 17*th Century English Skull.*

$$l = +\cdot 0081{,}2136, \quad m = -\cdot 0109{,}6533, \quad n = +\cdot 9999{,}0690.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot 9999{,}6702, \quad \mu = +\cdot 0000{,}8906, \quad \nu = -\cdot 0081{,}2087.$$

The Second Standard Plane makes an angle of 0° 27′·9 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot 9999{,}3988, \quad \nu' = +\cdot 0109{,}6569.$$

The Third Standard Plane makes an angle of 0° 37′·7 with the customary Transverse Vertical Plane.

(*F*) *Hindu Skull.*

$$l = -\cdot 0017{,}4154, \quad m = -\cdot 0054{,}1230, \quad n = +\cdot 9999{,}8384.$$

Direction Cosines of Second Standard Plane:

$$\lambda = \cdot 9999{,}9848, \quad \mu = -\cdot 0000{,}0943, \quad \nu = +\cdot 0017{,}4151.$$

The Second Standard Plane makes an angle of 0° 6′·0 with the Frankfurt Horizontal Plane.

Direction Cosines of Third Standard Plane:

$$\lambda' = 0, \quad \mu' = +\cdot 9999{,}8536, \quad \nu' = +\cdot 0054{,}1231.$$

The Third Standard Plane makes an angle of 0° 18′·6 with the customary Transverse Vertical Plane.

As a result we see that our Second Standard Plane as defined above makes a mean angle (28′·1) for the six skulls with the Frankfurt Horizontal Plane of less than half a degree, while the Third Standard Plane makes a mean angle (51′·0), less than a degree, with the customary Transverse Vertical Plane. Thus with the above definitions of the Second and Third Standard Planes we reach planes which make respectively only small angles with such very familiar planes as the Frankfurt Horizontal Plane and the Transverse Vertical Plane.

One point alone remains unsettled, namely: We have determined by (xiv) and (xv) the direction cosines of these Standard Planes as functions solely of the direction cosines of the Plane of Maximum Symmetry. But we have not determined the meet of our three Standard Planes. This remains to be selected. After dealing with several points I came to the conclusion that the most suitable point to choose as origin of the Standard Planes was the point in which the First Standard Plane or Plane of Maximum Symmetry meets the auricular axis. The elevations of these points are respectively Fuegian, 189·054,163; Egyptian from Nubia, 188·671,058; Arab, 193·357,334; Teita Negro, 186·435,955; English, 196·758,098; Hindu, 186·608,610.

Paying attention to which Porion was uppermost (as noted on pp. 236—238) we find that in no case is the intersection more than 1·5 mm. from the Mid-porion, the maximum being 1·4427 in the case of the Arab skull. Four intersections deviate from the Mid-porion towards the Left Porion and two towards the Right Porion, the average of the deviations of the six intersections is only ·0136 mm. towards the Right Porion. The point chosen seems therefore a reasonable one, and enables us to write down very readily the equations to the Standard Planes. For our six skulls they are as follows:

(*A*) *Fuegian Skull.*

1st Standard Plane:

$$-\cdot 009{,}5704\, x + \cdot 015{,}7335\, y + \cdot 999{,}8304\, z = 189\cdot 022{,}082,$$

2nd Standard Plane:

$$\cdot 999{,}9542\, x + \cdot 000{,}1506\, y + \cdot 009{,}5692\, z = 1\cdot 809{,}0990,$$

3rd Standard Plane:

$$-\cdot 999{,}8762\, y + \cdot 015{,}3424\, z = 2\cdot 900{,}5446.$$

(*B*) *Egyptian Skull* (from Nubia).

1st Standard Plane:

$$\cdot 005{,}4027\, x - \cdot 006{,}2292\, y + \cdot 999{,}9660\, z = 188\cdot 664{,}644,$$

2nd Standard Plane:

$$-\cdot 999{,}9854\, x - \cdot 000{,}0337\, y + \cdot 005{,}4026\, z = 1\cdot 019{,}3143,$$

3rd Standard Plane:

$$\cdot 999{,}9806\, y + \cdot 006{,}2298\, z = 1\cdot 175{,}2867.$$

(*C*) *Arab Skull.*

1st Standard Plane:

$$-\cdot 004{,}0698\, x - \cdot 042{,}3997\, y + \cdot 999{,}0924\, z = 193\cdot 181{,}851,$$

2nd Standard Plane:

$$\cdot 999{,}9913\, x - \cdot 000{,}1726\, y + \cdot 004{,}0661\, z = \cdot 786{,}2103,$$

3rd Standard Plane:

$$\cdot 999{,}1012\, y + \cdot 042{,}4001\, z = 8\cdot 198{,}3145.$$

(*D*) *Teita Negro Skull.*

1st Standard Plane:

$$-\cdot 022{,}0194\, x - \cdot 008{,}3092\, y + \cdot 999{,}7230\, z = 186\cdot 384{,}314,$$

2nd Standard Plane:

$$-\cdot 999{,}7575\, x + \cdot 000{,}1830\, y + \cdot 022{,}0186\, z = 4\cdot 105{,}0587,$$

3rd Standard Plane:

$$\cdot 999{,}9655\, y + \cdot 008{,}3112\, z = 1\cdot 549{,}5065.$$

(*E*) *English Skull.*

1st Standard Plane:

$$\cdot 008{,}1214\, x - \cdot 010{,}9653\, y + \cdot 999{,}9069\, z = 196\cdot 739{,}780,$$

2nd Standard Plane:

$$-\cdot 999{,}9670\, x - \cdot 000{,}0891\, y + \cdot 008{,}1209\, z = 1\cdot 597{,}8469,$$

3rd Standard Plane:

$$\cdot 999{,}9399\, y + \cdot 010{,}9657\, z = 2\cdot 157{,}5883.$$

(*F*) *Hindu Skull.*

1st Standard Plane:

$$-\cdot 001{,}7415\, x - \cdot 005{,}4123\, y + \cdot 999{,}9838\, z = 186\cdot 605{,}594,$$

2nd Standard Plane:

$$\cdot 999{,}9985\, x - \cdot 000{,}0094\, y + \cdot 001{,}7415\, z = \cdot 324{,}9808,$$

3rd Standard Plane:

$$\cdot 999{,}9854\, y + \cdot 005{,}4123\, z = 1\cdot 009{,}9836.$$

The reader must bear in mind what these three Standard Planes signify:

The First Standard Plane is the Plane of Maximum Symmetry, or our nearest approach to a mirror plane, the plane which deviates least from the mid-sagittal points as judged by the Mean Square Residual.

The Second Standard Plane passes through the point where the first meets the auricular axis, is perpendicular to the first and makes the minimum angle with the Frankfurt Horizontal Plane, which fails in the condition of being perpendicular to the First Standard Plane. Thus our Second Standard Plane may be looked upon as an improved "Horizontal Plane."

The Third Standard Plane passes through the point where the first two meet the auricular axis, and is perpendicular to both of them. It may therefore be looked upon as an improved Transverse Vertical Plane.

The Second and Third Standard Planes will not as a rule pass through the auricular axis and neither will generally contain the Poria. The second will not usually pass through the Kappa, nor the third through the Apex. It may be not without interest to measure the mean departures of the Second Plane for our six skulls from the Poria and the Kappa, and of our Third Plane from the Poria and the Apex *.

* In the equations to the planes, as on pp. 236—238, it is needful to go to 6 or 7 decimals in order to get values of the direction cosines which will provide angles to a decimal of a minute. The values, being worked from those equations, are written down to the same number of decimals, but to use two or three decimal places in the distances is of course ample.

(*A*) *Fuegian Skull.*

2nd, or Horizontal Standard Plane.

The κ is above this plane with the perpendicular upon it = − ·000,4302. The R. Porion is above the plane − ·619,5640 and the L. Porion below it + ·601,4660. Thus, as judged by the Poria, the right ear is about ·6 above, and the left ear about ·6 mm. below the Standard Horizontal Plane.

3rd, or Transverse Vertical Standard Plane.

The Apex is + ·008,5022 in front of this plane which like the deviation of the κ is not sensible on the skull. The R. Porion is behind this Transverse Vertical Plane − ·993,3565 and the L. Porion in front of it + ·964,3337. Thus the left ear comes horizontally forwards, and the right ear retreats.

(*B*) *Nubian Egyptian Skull.*

2nd, or Horizontal Plane.

The κ is above this plane at − ·000,0522 again an insensible difference. The R. Porion is above it at − ·315,1279, and the L. Porion is below it at + ·302,3893. Again the right ear is raised above the Horizontal Plane, and the left lowered below it although the differences are only about ·3 mm.

3rd, or Transverse Vertical Standard Plane.

The Apex is at − ·006,4115 from this plane; the R. Porion is − ·363,3504, i.e. behind, and the L. Porion about + ·348,6586, i.e. in front of it, or the Poria are about one-third of the distance from this plane of the Poria in the case of the Fuegian skull.

(*C*) *Arab Skull.*

2nd, or Horizontal Standard Plane.

The κ is here + ·002,1754, practically at an insensible distance below the Horizontal Plane. The R. Porion is below at + ·241,7595 and the L. Porion is above the Horizontal Plane at − ·253,4915.

3rd, or Transverse Vertical Standard Plane.

The Apex is behind this plane at a distance − ·116,3451. The R. Porion is + 2·520,9411, i.e. behind, and the L. Porion − 2·643,3911, i.e. in front of the Transverse Vertical Plane. These are the most considerable deviations we have found for any of the six skulls.

(*D*) *Teita Negro Skull.*

2nd, or Horizontal Standard Plane.

The κ here is at a distance + ·047,0995 below this plane. The R. Porion is + 1·229,4295, i.e. below the Standard Horizontal, and the L. Porion is − 1·219,0388, i.e. above it.

3rd, or Transverse Vertical Standard Plane.

The Apex is in front of this plane at a distance + ·017,7523. The R. Porion is behind the Standard Transverse Vertical at + ·464,0638, and the L. Porion in front of it at − ·460,1417.

(*E*) *English* 17*th Century Skull.*

2nd, or Horizontal Standard Plane.

The κ is here $-\cdot 017,8341$ above the Horizontal Plane. The R. Porion is at $-\cdot 503,0299$, i.e. above, and the L. Porion at $+\cdot 515,3309$, i.e. below it.

3rd, or Transverse Vertical Standard Plane.

The Apex is behind this plane at a distance of $-\cdot 032,2620$. The R. Porion is at $-\cdot 679,2383$, i.e. in front, and the L. Porion at $+\cdot 695,8605$, i.e. behind the Transverse Vertical Plane.

(*F*) *Hindu Skull.*

2nd, or Horizontal Standard Plane.

The κ is $+\cdot 001,3340$ below this plane. The R. Porion is at $+\cdot 097,8892$, i.e. below, and the L. Porion at $-\cdot 095,7656$, i.e. above the Horizontal Plane.

3rd, or Transverse Vertical Standard Plane.

The Apex is behind this plane at $-\cdot 004,2814$ distance. The R. Porion is at $+\cdot 304,2197$, i.e. in front, and the L. Porion is at $-\cdot 297,6281$, i.e. behind the Transverse Vertical Plane.

Reviewing these results as a whole, we conclude as follows:

Our Standard Horizontal and Transverse Vertical Planes, which we believe to be more reasonable than the usual Frankfurt Horizontal and Transverse Planes because they are at right angles to the Plane of Maximum Symmetry, do not pass through the auricular axis, but the first passes very nearly through the Kappa, the average distance in the cases of our six skulls from it being only ·0115 mm. regardless of sign. We have previously seen that our Standard Horizontal Plane passes, with a mean deviation of ·0136 mm. only, close to the Mid-porion (p. 248). The Frankfurt Horizontal Plane *also passes through these two points.* If now the Frankfurt Horizontal Plane be turned round the line joining these two points as axis, it will pass to the position of the Standard Horizontal Plane, one Porion approaching it and one receding from it. This provides the difference in vertical height of the two Poria, or speaking popularly of the two ears.

The Second Standard Plane passes just as closely to the Mid-porion and very close to the Apex, i.e. at an average distance of ·0309 mm. from it. It is also perpendicular to the Frankfurt Plane. Accordingly if the usual Transverse Vertical Plane be rotated about the line joining Apex to Mid-porion through a small angle it will come nearly into the position of our Standard Transverse Vertical Plane. The rotation is, to judge from our six skulls only, somewhat greater than the rotation required to change the Frankfurt Horizontal into the Standard Horizontal Plane. The result of this rotation is to cause one Porion to retreat behind the Standard Transverse Plane and the other to advance in front of it. Popularly, one ear may be said with reference to the Plane of Maximum Symmetry to be farther back on the head than the other ear.

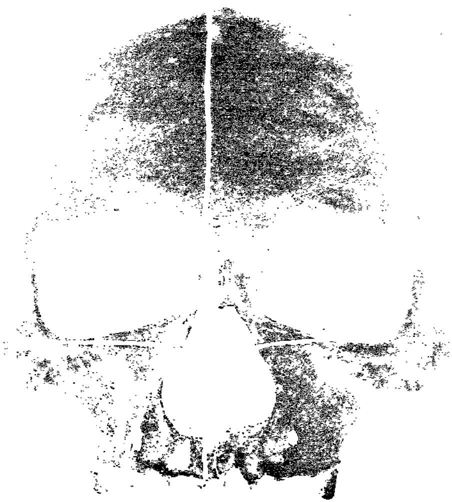
This more detailed numerical investigation indicates how the ears are displaced vertically and horizontally with regard to homologous positions as judged from a good representative of the mirror plane. Of course this solely emphasises in a different manner the point we had already reached, namely that the auricular axis is not perpendicular to the Plane of Maximum Symmetry.

The auricular axis of the skull is invaluable as an aid to the determination of cranial planes of reference, but it is far from an essential base line when we are investigating the symmetry of the skull. It cannot be legitimately used in determining standard horizontal and vertical planes, for in doing so we are assuming that it is necessarily perpendicular to a well-chosen Median Sagittal Plane, and this it is not.

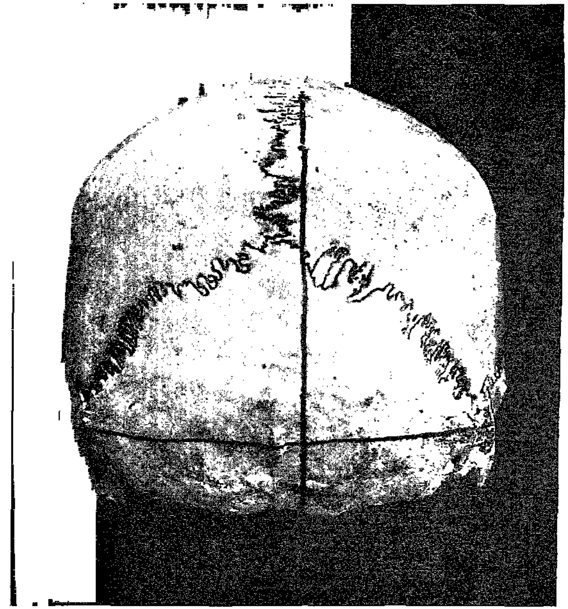
Summary.

By aid of the Cranial Coordinatograph it has been possible to obtain the coordinates of any point on the skull referred to three rectangular planes of reference. Thence we obtain the analytical equations of solid geometry to any lines or planes of the skull. Discussing the "mirror plane" of an absolutely symmetrical skull, we were led to find as our First Standard Plane the plane of the Mininum Square Deviation from the "mid-sagittal" points. This plane is that of maximum mesial symmetry. The Second and Third Standard Planes of the skull must be at right angles to each other and to the Plane of Maximum Symmetry. To make as little change as feasible we took our Second Standard Plane to make a minimum angle with the Frankfurt Horizontal Plane. This second plane practically passes through the Kappa and the Mid-porion, but is tilted to the Frankfurt Horizontal Plane. It is a truer Standard Horizontal Plane than the Frankfurt in that it is perpendicular to the close fitting Mid-sagittal Plane. Our Third Standard Plane is perpendicular to the first two and is a truer Transverse Vertical Plane than that through the auricular axis. These two standard planes indicate well the actual shift of the ear-orifices forwards and backwards, upwards and downwards, being, as we might naturally anticipate, an effect of the general asymmetry of the skull. The ears, being at a maximum distance from the Mid-sagittal Plane, show more than any other part of the skull, or of the living head, a maximum displacement from a symmetrical position, and really preclude the use of the auricular axis for the determination of a true horizontal, or a true transverse vertical plane.

The reader may ask: How much labour is involved in reducing a skull to analytical solid geometry? Originally it took me about two hours to two hours and a half to determine the Poria, Kappa and Apex on a Ranke craniophor, to adjust the skull on the staddle and read off the coordinates of 15 to 20 points. But with practice I think two skulls could easily be done in a morning of less than three hours. The numerical work is very straightforward computing, but laborious. The solving of the cubic is not so lengthy as it may appear, as we can be fairly sure that Σ_1^2 will lie between 1 and 3 sq. mm. Because its value is so small and the other roots of the cubic so large it is needful to keep a considerable number of decimal

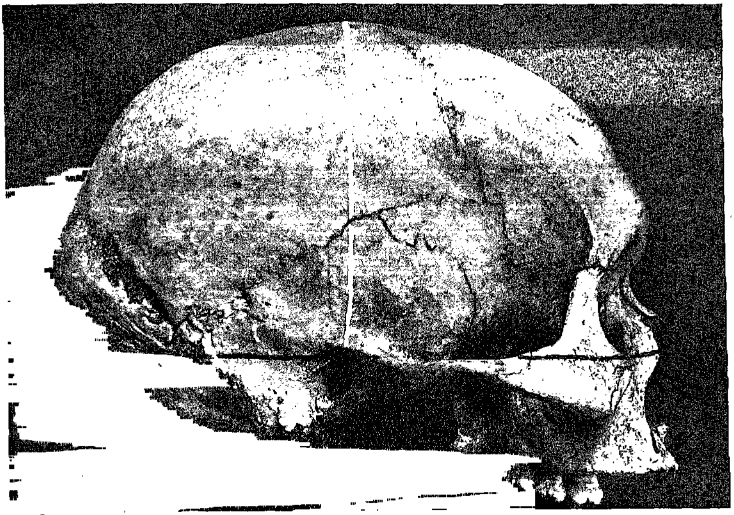


(*a*) Hindu Skull. *Norma facialis.*

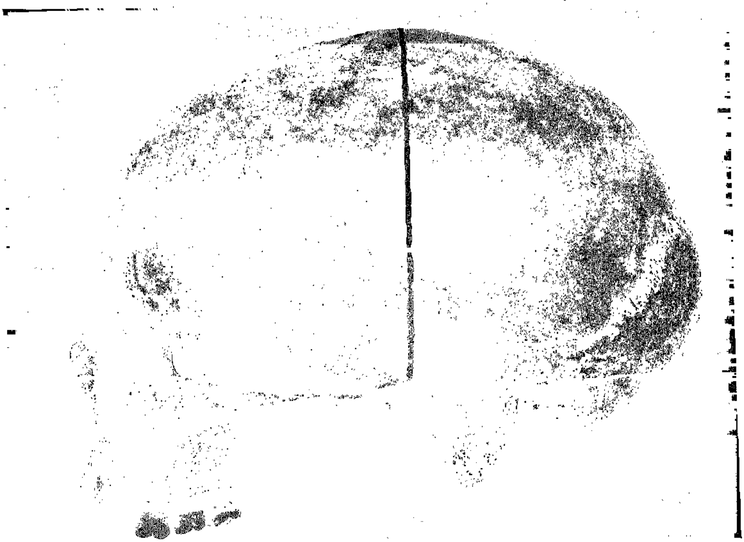


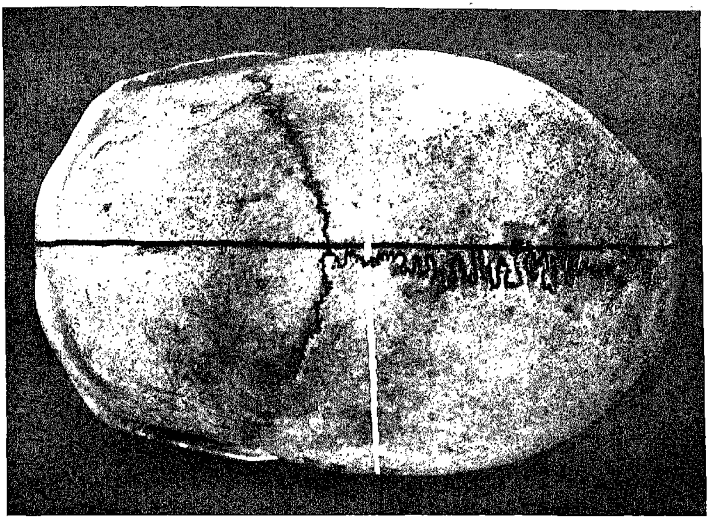
(*b*) Hindu Skull. *Norma occipitalis.*

This plate indicates how a plane perpendicular to the auricular axis and passing through the Mid-porion, and therefore perpendicular to the Frankfurt Horizontal Plane, fails to pass through any one of the "mid-sagittal points." The horizontal black curve is the trace of the Frankfurt Horizontal, and the vertical line is the trace of the plane through the Mid-porion perpendicular to the auricular axis.

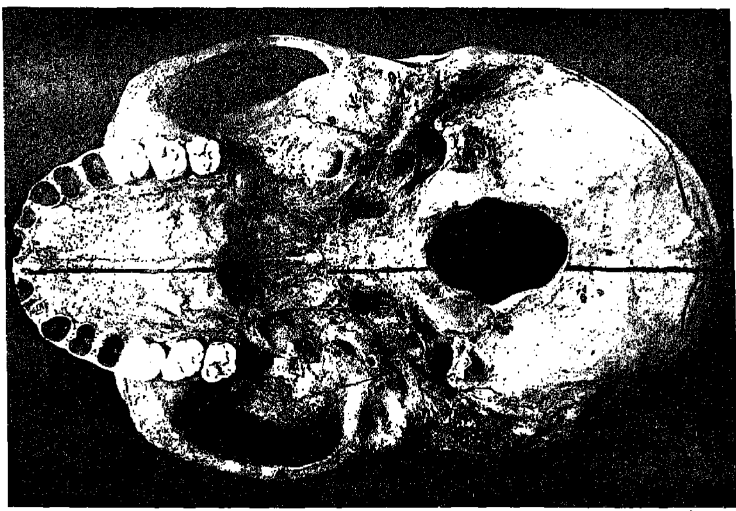


(c) Hindu Skull. *Norma lateralis* (Right Profile).

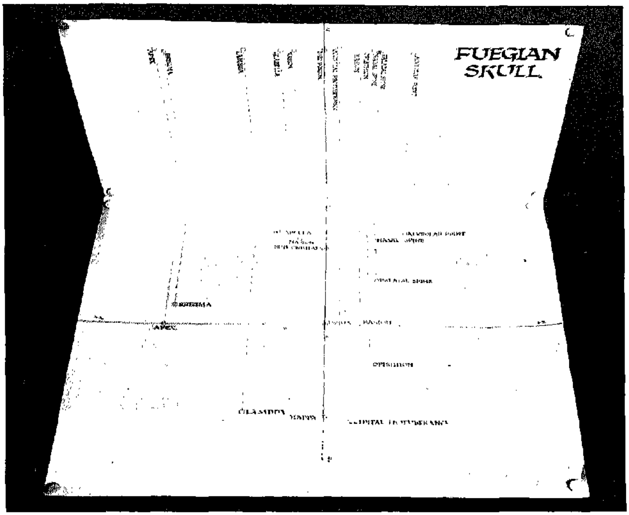




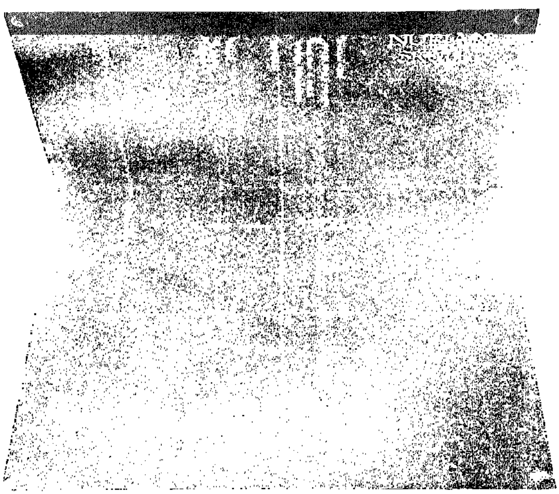
(e) Hindu Skull. *Norma verticalis.*



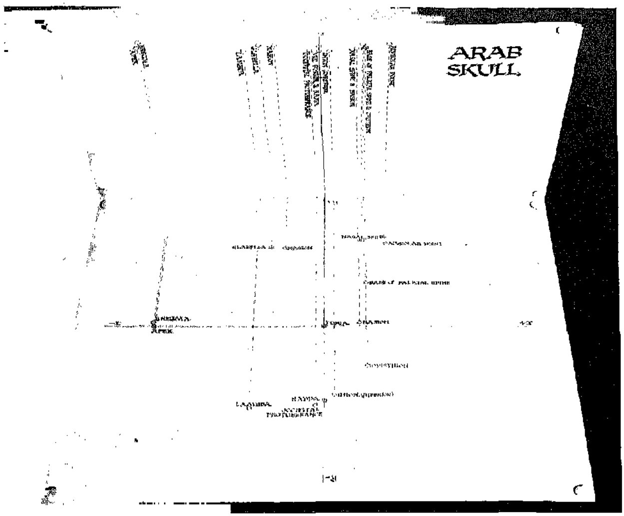
(*f*) Hindu Skull. *Norma basalis.*



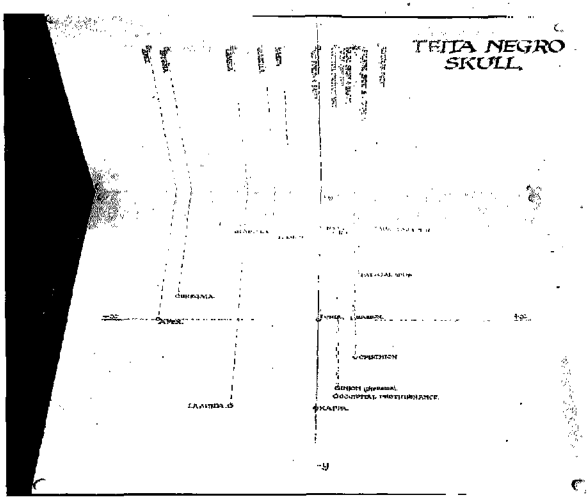
Plan and Elevation Model of a Fuegian Skull.



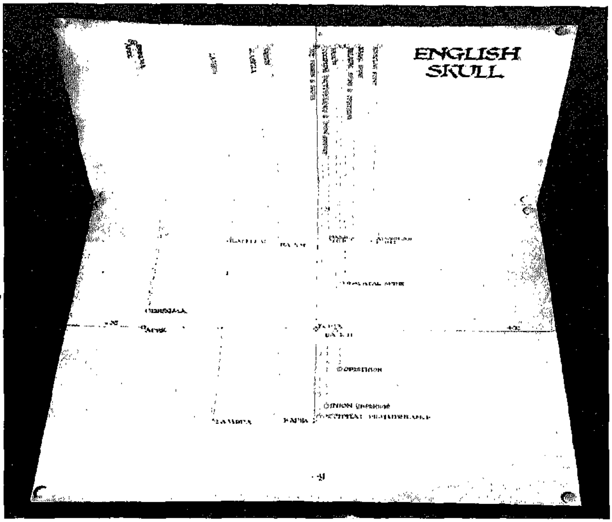
Plan and Elevation Model of an Egyptian Skull from Nubia.



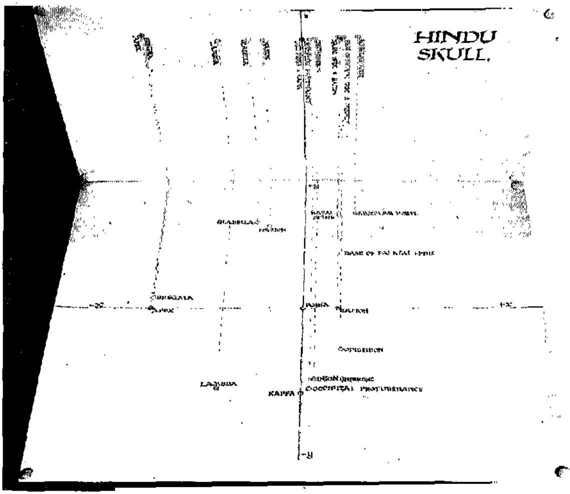
Plan and Elevation Model of an Arab Skull.



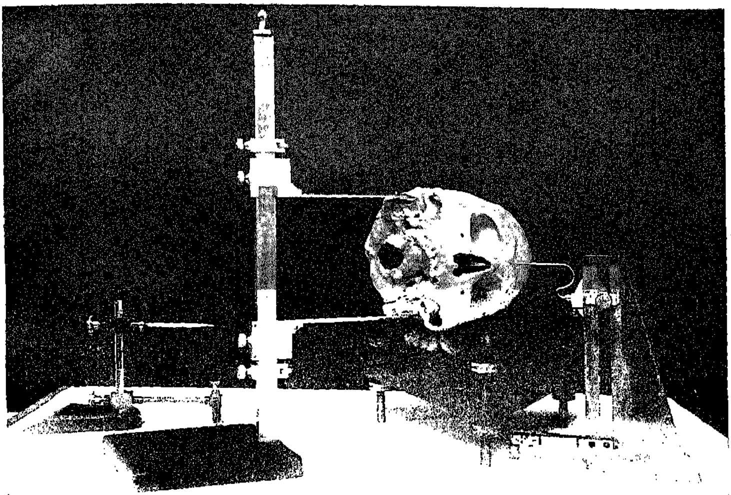
Plan and Elevation Model of a Negro Skull.



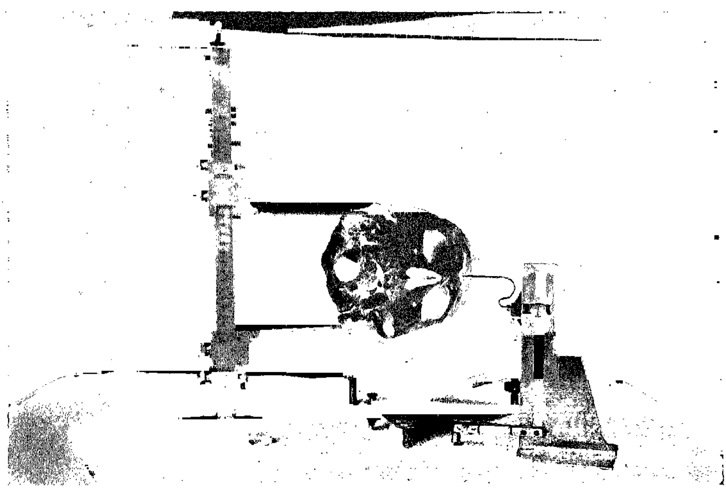
Plan and Elevation Model of an English Skull.



Plan and Elevation Model of a Hindu Skull.



(*a*) First form of Cranial Coordinatograph with Skull on Skull Staddle and auricular axis vertical. Klaatsch and Pearson independent projectors, the former set at glabella.



(*b*) Final form of Cranial Coordinatograph with a Pearson projector attached to its base, so that the plan is marked and elevation taken by the single instrument.

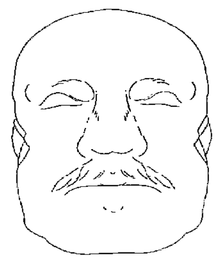
(*a*)
(i) Symmetricised Portrait of Right Side.

(*b*)
Drawing of Oliver Cromwell, after the bust in the Ashmolean Museum.

(*c*)
Symmetricised Portrait of Left Side.

(*a*)
(ii) Symmetricised Drawing of Right Side.

(*b*)
Drawing of Cromwell's Death Mask.



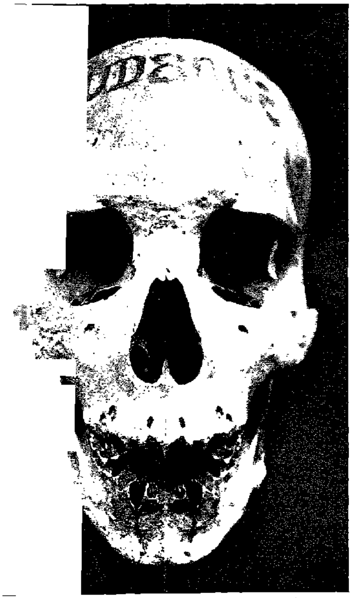
(*c*)
Symmetricised Drawing of Left Side.

(*a*) Symmetricised Portrait. Mirror Portrait of Right Side.

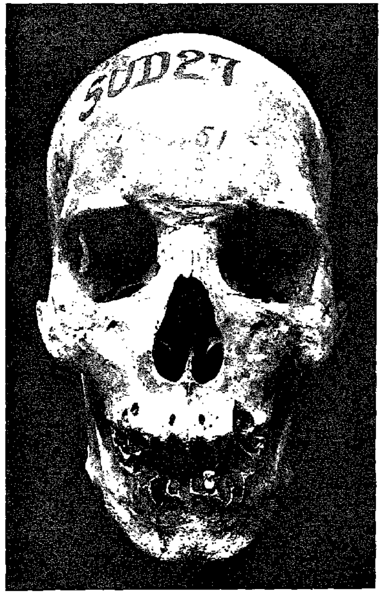
(*b*) Full Face Natural Portrait.

(*c*) Symmetricised Portrait. Mirror Portrait of Left Side.

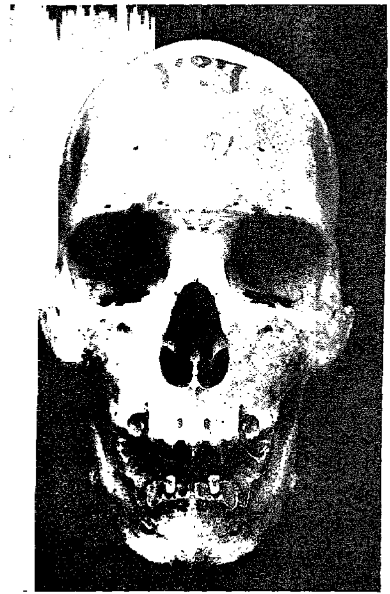
This plate indicates how much of personal character depends on asymmetry of the face.



(*a*)
Symmetricised *Norma facialis*.
Mirror Portrait of Right Side.



(*b*)
Natural *Norma facialis* of Ancient Egyptian Skull
from Nubia.



(*c*)
Symmetricised *Norma facialis*.
Mirror Portrait of Left Side.

places. I think that two skulls could be dealt with in 5 to 6 hours by a good computer, so that an investigator could make a daily output of two skulls, or 10 to 12 a week. A month to five weeks would complete a racial sample of some 50 crania. Thus 10 to 12 races could be studied in a year to eighteen months. This is less time than many students give to their thesis for a doctorate, and a most valuable study of the standard planes of the skull and its asymmetry would result.

The present paper only professes to be illustrative of what may be achieved by the use of a coordinatograph, and the application of solid geometry to craniometry.

I may be over-enthusiastic, but I unhesitatingly believe that there is a most promising field for the craniometricians who are the first to apply Cartesian geometry to the skull.

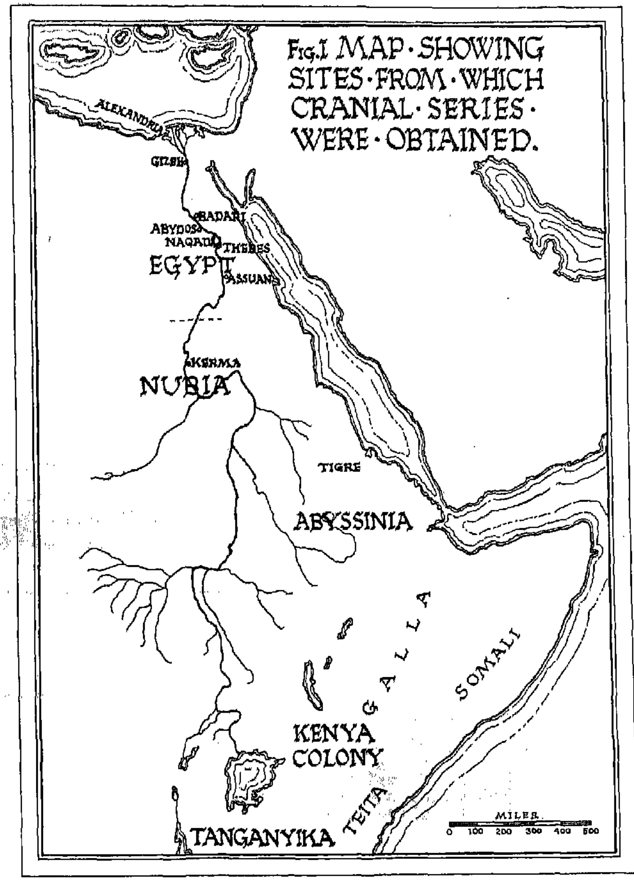
A STUDY OF TWELFTH AND THIRTEENTH DYNASTY SKULLS FROM KERMA (NUBIA).

By MARGOT COLLETT.
(Crewdson Benington Student.)

(1) *The Discovery of the Skulls at Kerma.* The skull series forming the subject of the present paper came from Kerma, a place lying on the east bank of the Nile between Argo and Tombos (see Fig. I). Kerma is 150 miles south of the boundary which divides the Anglo-Egyptian Sudan from Egypt and as the crow flies it is almost as far from Kerma to Thebes, a distance of nearly 450 miles, as it is from Thebes to Alexandria. The excavations were carried out by Dr George A. Reisner leading the joint expedition of Harvard University and the Boston Museum of Fine Arts in the seasons 1913–14 and 1915–16, and the following particulars are taken from his report*. No certain date for the occupation of Kerma is given earlier than the Twelfth Dynasty, but objects of earlier date have been found there and the settlement is believed to be one of great antiquity. The ruins of mud houses discovered under the foundations of buildings of Middle Dynastic date show clearly that there had been a settlement of considerable size in earlier times. Taking into consideration the local history of other parts of Nubia, it is most probable that during the Old Kingdom (1st—8th Dynasties) and the Early Middle Kingdom (9th—11th Dynasties) Kerma was one of the numerous Egyptian trading stations held by an agent and a few men, possibly local natives, at which the periodic expeditions from Egypt called to deliver and collect goods. These expeditions were sent out every two or three years carrying ointment, honey, faience and cloth, which were taken in exchange for resin, ivory, woods, oils, special grain, incense and leopard skins. The discovery of statuettes and pottery on which were inscribed the names of Pepy I and II, who were kings of the Sixth Dynasty, and of Amenemhat I of the Twelfth Dynasty, gave the first real evidence for purposes of dating. It is suggested that the settlement was slightly increased in the reign of Amenemhat I after the quelling of a native revolt. Dr Reisner gives 1970 B.C. as the date of this revolt. Later, in the reign of Sesostris I (also of the Twelfth Dynasty), and probably just after another native rising, a larger force was sent out from Egypt under the governorship of Hepzefa who was probably responsible for the erection of a fortress or fortified residence known as the Western Deffûfa. This building was the nucleus of a military settlement. It was of considerable size, containing a guard-room and several other rooms, some of which may have been used as storehouses. It is thought that the Deffûfa was built principally as a stronghold for the protection of goods brought from Upper Egypt for purposes of exchange,

* "Excavations at Kerma," Vol. I (Parts I—III) and Vol. II (Parts IV—V). *Harvard African Studies*, Vols. V and VI (1923).

and of the taxes collected from the local tribes. Owing to the lack of water nearby there would not have been a convenient harbourage for a large armed force in the building.



This governor (Hepzefa), besides being the builder of the Deffûfa, is believed to have been the founder of the Egyptian Cemetery in which we are chiefly interested. It remained in use for over 350 years. In the neighbourhood of the

Deffûfa were remains of only two other buildings, both being funerary chapels, one of which was attached to the earliest tumulus (K III) and the other to one of later date (K X). The pottery and grave furniture throughout this cemetery, the painting in the second chapel, the seals, and the numerous statuettes were all distinctly Egyptian in form and technique. On the other hand there were certain peculiarities quite unknown in the contemporary graves in Egypt proper. The principal among these peculiarities was the sacrificial, or so-called sâti, burial when in some cases as many as 320 people appear to have been buried alive with the body of their chief. This custom was prevalent in Egypt during Predynastic and First Dynastic times; it was also common in the earliest Nubian graves and it appears to have been the custom at Ur in early times. It was not only practised by the ruling classes at Kerma; the smaller independent graves of quite poor type also have one or possibly two human sacrifices. In the later graves, where the Nubian element became more marked, a ram or several rams seem to have been substituted for the human sacrifices in some cases, or there might be three rams and two human beings as opposed to one ram and four human beings in an earlier grave. The covering of the body with a cowhide shows Egyptian influence, but again the custom of bed burials was unknown in Upper Egypt at this time. In one of the later tumuli coffin burials were introduced contemporaneously with bed burials, another sign of Egyptian influence. The pottery became noticeably coarser in the later graves and, although developing along the same lines as in Egypt proper, a certain degeneration is evident.

The cemeteries at Kerma can be divided into three groups: the earliest Egyptian Cemetery founded by Hepzefa, the Nubian Cemetery which follows a transition period of Egyptian-Nubian graves and the much more recent Third and Fourth Century A.D. Meroitic Cemetery. With the exception of a few skulls from the Meroitic Cemetery which have not been measured all our material comes from the earliest Egyptian Cemetery which was in use during the Twelfth and Thirteenth Dynasties. The following table gives the numbers of skulls of which measurements were taken belonging to different graves or tumuli, including the Meroitic skulls:

No. of Tumulus or Grave	No. of Skulls		Dynasty	Nature of Interment
	Main	Subsidiary		
III	11	22	12th	Tumulus
IV	13	42	"	"
X	48	56	13th	"
XVI	4	16	"	"
XVIII	11	6	"	"
XXIX	1	—	"	Minor tumulus
XX	2	—	"	" "
XXIII	10	—	"	Tumulus
XXXIV	1	—	"	Minor tumulus
XXXV	1	—	"	" "
B	9	—	12th	Graves in open
Meroitic	4	—	3rd and 4th cent. A.D.	" "

The tumuli throughout the cemeteries were all constructed on the same plan. They consisted of one or two main chambers centrally placed, one of which contained the chief body, a sacrificial corridor dividing the tumulus in half, and built at right angles to this corridor a series of parallel walls gradually decreasing in height towards the outside. The subsidiary graves were placed between these walls and they are thought to have held the bodies of the officials belonging to the court of the chief. The height of the walls where they met the walls of the corridor was about 3 ft. and it decreased to about 1 ft. on the outside edge. The whole erection was covered with a mound of earth and surrounded by a ring of ox skulls which were possibly the remains of a funeral feast, and also by a ring of black stones for which we have no explanation. In addition to these tumuli, which differed in size considerably, there were also several independent graves consisting of oval or rectangular pits and containing one, two or three bodies each. Both primary and sacrificial burials were made in each form of funerary chamber or grave and it is not possible from the records available to distinguish the two varieties in all cases. It is probable, however, that considerably more than half of the skulls dealt with were those of sacrificial victims. The tumuli were numbered in chronological order as far as this was possible. The actual skeletons from the central corridor, representing the chief body and the sacrificial bodies, were lettered, and the skeletons from the subsidiary graves numbered. Of the total 310 skulls available 55 were excluded from the series measured for various reasons and this accounts for breaks in the serial numbering which was done in the Biometric Laboratory. The present paper deals only with the skulls, though many of them have mandibles and considerable numbers of the other bones of the skeleton have been preserved.

The archaeological evidence was apparently only capable of giving a very incomplete history of the settlement at Kerma, and the account of the excavations throws little or no light on many important points. We may conclude, however, that the original trading colony, which was founded at an early but unknown date, almost certainly consisted of pure Egyptians, although there was an early non-Egyptian village of considerable size and also a fairly large Nubian village in the vicinity. The trading settlement Inebuw Amenemhat (Kerma) was known to be in existence in the Twelfth Dynasty and evidence of a settlement much earlier than this is given. Reisner says that "the colony at Inebuw Amenemhat was... on the site of an old trading post of the 6th Dynasty." After the advent of Hepzefa the number of colonists was not substantially, if at all, increased by the traders from Egypt as far as we know, although there is no definite evidence on this point. Supposing there was no increase by immigration, the occupants must have interbred and remained in possession after the Twelfth Dynasty. The tenure was a military one and it is unlikely that there was much intermarriage with native peoples of the locality. The culture, as we have seen, remained essentially Egyptian in type until a much later period than that with which we are concerned, and the fact that a few peculiarities were observed such as the common practice of sacrifical burial and the use of a bed in burial cannot be accepted as evidence of blood

admixture with any foreign element. It would appear that all these peculiarities were merely survivals of elements of an earlier Egyptian culture. The bulk of our material belongs to a comparatively short period, probably not exceeding 250 years. Reisner says: "For the period from the death of Hepzefa (K III) to that of the person buried in K XX I estimate an interval of about 200 years."

It is fortunate that our knowledge of the craniology of Upper Egypt from Early Predynastic to Roman times is quite extensive, since the original colonists almost certainly came from there. It is known that the physical type in the north was undergoing slight but continual modification, and we may hope to give a definite answer to the questions whether the community which was apparently isolated in Nubia was, during the period for which we have acquaintance with it, of the same physical type as the contemporary population of Upper Egypt, or whether it was of the same type as the Upper Egyptian population of another period, or whether it had acquired distinctive characters which would differentiate it from all known populations of Egypt.

(2) *The Nature of the Kerma Series.* There are 310 skulls from Kerma preserved in the Biometric Laboratory. Fifty-five of these were excluded from the series, and no measurements of them were taken, for various reasons: 31 are posthumously distorted, eight are too fragmentary to be measured, one is scaphocephalic, seven are immature, four are of Meroitic date, two found in tumuli are so well preserved that they can only be supposed to represent modern intrusive burials and two others possess manifest negro characters. The remaining skulls, which are the only ones considered in later sections of this paper, are all in a similar state of preservation, which resembles that of most other Predynastic and Dynastic Egyptian series, and most of them are complete or nearly complete. With two exceptions, they all bear grave numbers, which assign them to the Twelfth and Thirteenth Dynasties, and these are given with the corresponding serial numbers in the appended tables of individual measurements. The specimens which were not marked by the excavators (Nos. 224 and 272) are presumed to belong to the same period as the others. When the skulls were set out on a table so that all could be seen together, few striking differences in type were observed except in the case of the two with clear negro characters which were excluded from the series on this account. It was evident that if any division of the remainder was to be made by merely examining the skulls, such would have to depend on the fact that some possess more negroid characters—viz. a flatter nasal bridge, a higher nasal index and a greater degree of prognathism—than others. But an attempt to divide the whole into two contrasted groups showed that it was quite impossible to distinguish the negroid and non-negroid specimens with any degree of exactness. Hence it was concluded that the safest procedure was to treat the total series as if it represented a single racial type which would obviously be one possessing negroid characters. The series did not appear to be more heterogeneous than many which have been dealt with in this way.

The sexing of the 255 adult skulls forming the series was done by Professor

Karl Pearson and Dr G. M. Morant. They distinguished 141 males and 114 females. This slight excess of the representatives of the former sex over those of the latter has frequently been found for a collection of excavated crania, and it has been supposed due to the fact that the weaker female specimens would be broken more easily and hence less likely to be saved by the excavators. In the present case many of the skulls are known to be those of sacrificial victims and this may also account in part for the presumed disparity between the sexes. Sex ratios, i.e. male means divided by female means, have been given by Kitson* for two European, two negro and the long Egyptian E series of 26th—30th Dynasty skulls. The Kerma values are closest to those for the Egyptians and Bantu negroes (Teita) from Kenya Colony and comparisons with these are made in our Table I. The correspondence of the sex ratios for these three series is extremely close and we may conclude that they were all sexed in a uniform way.

TABLE I.

Sex Ratios for the Kerma and other Series.

Character	Kerma	Egyptian E	Teita
L	1·045	1·047	1·046
B	1·029	1·025	1·024
LB	1·054	1·053	1·051
B'	1·040	1·027	1·039
H	1·058	1·038	1·044
S	1·039	1·034	1·037
U	1·044	1·038	1·037

A rough estimate of the age constitution of the cemetery population represented can be obtained by considering the state of closure of the coronal, sagittal and lambdoid sutures. The results for the Kerma series are summarised below, those three calvarial sutures being the only ones referred to.

Sex	All sutures open	Sutures beginning to close or partly closed	All sutures closed	Totals
♂	16 (11·3 %)	83 (58·9 %)	42 (29·8 %)	141
♀	40 (35·1 %)	51 (44·7 %)	23 (20·2 %)	114

The percentages may be compared with those given† for six other series examined by Dr Morant. For every one of these marked differences were found between the corresponding male and female frequencies, confirming the fact that the sutures close at a later age for women than for men. The Kerma male percentage of skulls showing all three sutures open is lower than any previously given, and the percentage with all sutures closed is only exceeded by one other. The females show a smaller proportion with all sutures open and a larger with all sutures closed than has been found for any other series examined. The present

* *Biometrika*, Vol. XXIII (1931), p. 275. † *Biometrika*, Vol. XXIV (1932), p. 170.

sample must hence be supposed to represent an older cemetery population than any of the others, although the contrary would have been expected owing to the fact that many of the people buried at Kerma met unnatural deaths as they were sacrificial victims.

We may enquire next whether there is any evidence of a change in the racial constitution of the population of Kerma during the Twelfth and Thirteenth Dynasties. The male means were computed for each of these Dynasties and the crude coefficient of racial likeness of $0{\cdot}92 \pm {\cdot}17$ for 31 characters was found between them. No significant differences between the means were found. The coefficient differs significantly from zero, but it may be too high owing to the fact that the Egyptian *E* standard deviations were used in computing it in place of the standard deviations for each of the samples compared. In any case it is of such a low order that it was thought safest to conclude that there is no evidence of a change in the racial nature of the Egyptian settlers at Kerma during the period considered. The female coefficient was not calculated, but a few of the more important means for the two Dynasties are compared in the table below.

Character	Male		Female	
	Twelfth Dynasty	Thirteenth Dynasty	Twelfth Dynasty	Thirteenth Dynasty
$100\,B/L$	71·8 (59)	72·5 (74)	73·1 (37)	73·4 (73)
$100\,B/H'$	101·0 (41)	100·2 (50)	101·8 (27)	101·9 (59)
$100\,NB/NH, R$	51·5 (48)	51·6 (63)	53·0 (28)	53·4 (55)
$N\angle$	64°·3 (45)	65°·4 (50)	67°·8 (31)	67°·4 (42)

The differences between these means in the case of each sex are quite insignificant. All the Twelfth and Thirteenth Dynasty skulls from Kerma accordingly were pooled, and the means, standard deviations and coefficients of variation with their probable errors for the total sample are given in Table II for all the characters measured and for the indices and angles*. I was able to make use of a considerable

* [Definitions of the measurements, which are denoted by the usual index letters, will be found in *Biometrika*, Vol. xx^B (1928), pp. 362—364. A single change was made in the way the nasal height *NH'* was measured. In recent craniometric studies published in *Biometrika* this has been taken, in addition to the Frankfurt nasal heights, from the nasion to the "base" of the anterior nasal spine with the object of providing a measurement which was supposed the same as the one defined by Broca and Martin and again in the Monaco scheme. While the definitions provided in these techniques are not perfectly clear, a re-examination suggested that a closer approach to the nasal height actually taken by those who have followed them could be obtained in the following way. The lower margins of the pyriform aperture are first drawn as pencil lines and the Frankfurt heights are taken to these curved lines on either side. A third pencil line, which will be horizontal on a symmetrical specimen, is then drawn so that it appears, when the skull is viewed in *norma facialis*, to be a straight line joining the lowest points of the margins on either side. The point where this horizontal line meets the boundary of the anterior nasal spine—itself a vaguely defined line—on the left-hand side is supposed to be Martin's nasospinale. The *NH'* given for the Kerma skulls is the chord from this point to the nasion, and in all cases it was found to be very close to the Frankfurt *NH, L*. The *NH'*'s given by Miss Kitson for the Naga skulls in the present volume of *Biometrika* were taken in the same way, but in earlier studies in the Biometric Laboratory the *NH'* was to the "base" of the anterior nasal spine.—G. M. M.]

number of measurements which had previously been taken by Mr G. C. Dunning and the individual measurements are given in Tables VIII, IX. The nature of the sample can be estimated from the constants in Table II. It may be asked first

TABLE II.

Constants of the Male and Female Kerma Series.

Character	Means		Standard Deviations		Coefficients of Variation	
	Male	Female	Male	Female	Male	Female
C	1383·3 ± 8·2 (72)	1229·9 ± 7·5 (47)	103·5 ± 5·8	86·8 ± 5·3	7·48 ± ·30	7·06 ± ·42
F	183·3 ± ·38 (138)	177·0 ± ·36 (114)	6·63 ± ·27	5·79 ± ·27	3·62 ± ·15	3·27 ± ·15
L	185·2 ± ·36 (138)	177·2 ± ·35 (114)	6·26 ± ·25	5·08 ± ·25	3·38 ± ·14	3·20 ± ·14
B	133·6 ± ·26 (136)	129·8 ± ·28 (112)	4·52 ± ·18	4·37 ± ·20	3·38 ± ·14	3·37 ± ·16
B'	92·8 ± ·23 (139)	89·2 ± ·27 (111)	4·08 ± ·16	4·20 ± ·19	4·40 ± ·19	4·71 ± ·21
B''	113·1 ± ·34 (68)	107·5 ± ·42 (60)	4·13 ± ·24	4·85 ± ·30	3·65 ± ·21	4·51 ± ·28
Biasterionic B	104·9 ± ·29 (113)	100·2 ± ·24 (102)	4·51 ± ·20	3·60 ± ·17	4·26 ± ·19	3·59 ± ·17
H'	133·4 ± ·32 (93)	127·8 ± ·31 (88)	4·65 ± ·23	4·38 ± ·22	3·48 ± ·17	3·43 ± ·17
H	135·7 ± ·34 (100)	128·2 ± ·35 (77)	5·01 ± ·24	4·56 ± ·25	3·69 ± ·18	3·56 ± ·19
OH	110·7 ± ·26 (125)	107·4 ± ·25 (91)	4·25 ± ·18	3·48 ± ·17	3·84 ± ·16	3·24 ± ·16
LB	101·8 ± ·26 (117)	96·6 ± ·27 (89)	4·16 ± ·18	3·75 ± ·19	4·09 ± ·22	3·88 ± ·19
S_1'	112·1 ± ·30 (120)	107·4 ± ·29 (109)	4·96 ± ·21	4·53 ± ·21	4·42 ± ·19	4·22 ± ·19
S_2'	115·1 ± ·34 (128)	111·0 ± ·35 (106)	5·78 ± ·24	5·26 ± ·24	5·02 ± ·22	4·74 ± ·22
S_3'	97·2 ± ·32 (124)	94·3 ± ·31 (97)	5·34 ± ·23	4·58 ± ·22	5·49 ± ·24	4·86 ± ·23
S_1	128·1 ± ·38 (126)	123·3 ± ·39 (109)	6·28 ± ·27	6·05 ± ·28	4·90 ± ·21	4·91 ± ·22
S_2	128·7 ± ·44 (128)	124·0 ± ·46 (108)	7·46 ± ·31	7·06 ± ·33	5·80 ± ·25	5·69 ± ·23
S_3	116·8 ± ·46 (124)	112·9 ± ·46 (97)	7·67 ± ·33	6·71 ± ·32	6·57 ± ·28	5·94 ± ·29
S	374·7 ± ·77 (133)	360·3 ± ·80 (105)	13·15 ± ·54	12·13 ± ·56	3·51 ± ·15	3·37 ± ·16
U	510·2 ± ·81 (133)	488·9 ± ·83 (107)	13·89 ± ·57	12·78 ± ·54	2·72 ± ·09	2·61 ± ·12
Q'	305·0 ± ·54 (123)	291·8 ± ·60 (89)	8·98 ± ·37	8·33 ± ·42	2·94 ± ·13	2·85 ± ·14
fml	36·4 ± ·20 (112)	34·6 ± ·15 (92)	3·13 ± ·14	2·14 ± ·11	8·60 ± ·39	6·18 ± ·30
fmb	30·1 ± ·16 (110)	28·4 ± ·16 (83)	2·42 ± ·11	2·12 ± ·11	8·04 ± ·37	7·40 ± ·39
$G'H$	69·6 ± ·29 (111)	65·7 ± ·25 (83)	4·48 ± ·20	3·33 ± ·17	6·45 ± ·29	5·07 ± ·27
GL	96·5 ± ·32 (97)	93·7 ± ·38 (68)	4·75 ± ·23	4·63 ± ·27	4·92 ± ·24	4·94 ± ·29
GB	95·3 ± ·28 (111)	90·9 ± ·30 (87)	4·64 ± ·21	4·20 ± ·21	4·87 ± ·22	4·62 ± ·24
J	127·5 ± ·38 (83)	118·9 ± ·39 (52)	5·16 ± ·27	4·16 ± ·28	4·05 ± ·24	3·50 ± ·23
NH, R	50·0 ± ·18 (112)	47·2 ± ·20 (86)	2·82 ± ·14	2·79 ± ·14	5·64 ± ·25	5·91 ± ·30
NH, L	50·1 ± ·19 (113)	47·3 ± ·20 (86)	3·05 ± ·14	2·70 ± ·14	6·09 ± ·28	5·71 ± ·30
NH'	50·0 ± ·19 (105)	47·1 ± ·19 (90)	2·95 ± ·15	2·73 ± ·14	5·90 ± ·27	5·80 ± ·29
NB	25·8 ± ·11 (114)	25·0 ± ·15 (89)	1·79 ± ·08	2·13 ± ·11	6·94 ± ·31	8·52 ± ·43
DS	11·6 (32)	11·1 (22)			—	—
DC	22·7 (33)	22·2 (23)			—	—
DA	34·5 (32)	33·0 (22)			—	—
SS	4·0 ± ·09 (95)	3·1 ± ·09 (61)	1·38 ± ·07	1·12 ± ·06	34·50 ± 1·88	36·13 ± 2·33
SC	10·8 ± ·13 (95)	10·2 ± ·17 (72)	1·99 ± ·10	2·15 ± ·12	18·43 ± ·93	21·08 ± 1·28
G_1	52·0 ± ·28 (87)	49·7 ± ·26 (59)	3·82 ± ·14	3·01 ± ·19	7·35 ± ·38	6·06 ± ·38
G_1'	47·3 ± ·21 (88)	45·7 ± ·24 (62)	2·94 ± ·15	2·84 ± ·17	6·22 ± ·32	6·21 ± ·38
G_2	39·4 ± ·22 (82)	37·8 ± ·23 (62)	2·90 ± ·15	2·71 ± ·16	7·36 ± ·39	7·17 ± ·44
EH	13·2 ± ·18 (55)	12·3 ± ·18 (47)	2·03 ± ·13	1·85 ± ·13	15·38 ± 1·01	15·04 ± 1·07
O_1, R	41·6 ± ·11 (116)	39·7 ± ·12 (80)	1·77 ± ·08	1·60 ± ·08	4·25 ± ·19	4·03 ± ·21
O_1, L	41·5 ± ·11 (105)	39·7 ± ·11 (84)	1·73 ± ·08	1·44 ± ·07	4·17 ± ·19	3·63 ± ·19
O_1', R	39·4 ± ·19 (45)	37·7 ± ·14 (47)	1·91 ± ·13	1·47 ± ·10	4·85 ± ·35	3·90 ± ·27
Lacrymal O_1, R	38·2 (28)	36·6 (40)	—	—	—	—
O_2, R	32·7 ± ·13 (114)	32·5 ± ·13 (83)	2·06 ± ·08	1·89 ± ·09	6·30 ± ·28	5·81 ± ·30
O_2, L	32·7 ± ·14 (103)	32·5 ± ·14 (85)	2·06 ± ·10	1·90 ± ·09	6·30 ± ·30	5·84 ± ·30

TABLE II (*continued*).

Character	Means		Standard Deviations	
	Male	Female	Male	Female
100 B/L	72·2 ± ·20 (133)	73·3 ± ·18 (112)	3·38 ± ·14	3·01 ± ·14
100 H'/L	72·4 ± ·19 (91)	72·4 ± ·17 (88)	2·67 ± ·13	2·37 ± ·12
100 H/L	73·2 ± ·18 (100)	72·9 ± ·18 (77)	2·61 ± ·12	2·33 ± ·13
100 B/H'	100·6 ± ·31 (91)	101·9 ± ·32 (88)	4·34 ± ·22	4·30 ± ·23
100 B/H	99·0 ± ·30 (98)	101·6 ± ·34 (77)	4·42 ± ·21	4·44 ± ·24
100 $(B-H')/L$	0·5 ± ·23 (89)	1·3 ± ·24 (88)	3·05 ± ·15	3·29 ± ·17
100 $G'H/GB$	72·7 ± ·33 (104)	72·2 ± ·33 (79)	5·00 ± ·23	4·32 ± ·23
100 NB/NH, R	51·6 ± ·25 (111)	53·2 ± ·35 (84)	3·94 ± ·18	4·72 ± ·25
100 NB/NH, L	51·6 ± ·25 (111)	53·3 ± ·34 (85)	3·95 ± ·18	4·78 ± ·24
100 NB/NH'	51·8 ± ·27 (105)	53·4 ± ·30 (85)	4·16 ± ·19	4·87 ± ·25
100 SS/SC	37·3 ± ·78 (95)	30·5 ± ·72 (89)	11·22 ± ·55	8·85 ± ·51
100 DS/DC	51·3 (32)	50·7 (22)	—	—
100 G_2/G_1	75·7 ± ·53 (67)	75·5 ± ·65 (42)	6·39 ± ·37	6·26 ± ·46
100 G_2/G_1'	83·6 ± ·58 (67)	82·5 ± ·71 (44)	7·04 ± ·41	7·00 ± ·50
100 GH/G_2	33·3 ± ·53 (55)	32·8 ± ·52 (47)	5·81 ± ·37	5·20 ± ·37
100 O_2/O_1, R	78·6 ± ·32 (113)	81·6 ± ·33 (83)	5·06 ± ·23	4·49 ± ·24
100 O_2/O_1, L	78·9 ± ·32 (104)	81·8 ± ·33 (84)	4·85 ± ·23	4·44 ± ·23
100 O_2/O_1', R	81·5 ± ·52 (44)	84·9 ± ·47 (46)	5·14 ± ·37	4·74 ± ·33
100 O_2/Lacr. O_1, R	84·0 (28)	87·7 ± ·52 (40)	—	4·84 ± ·36
100 fmb/fml	83·2 ± ·41 (106)	82·0 ± ·39 (81)	6·23 ± ·29	5·02 ± ·31
$Oc.$ $I.$	60·4 ± ·15 (124)	60·6 ± ·16 (97)	2·50 ± ·11	2·37 ± ·11
$N\angle$	64°·9 ± ·21 (95)	67°·6 ± ·30 (63)	3°·08 ± ·15	3°·52 ± ·21
$A\angle$	74°·1 ± ·24 (95)	72°·1 ± ·27 (63)	3°·52 ± ·17	3°·22 ± ·19
$B\angle$	41°·0 ± ·20 (95)	40°·3 ± ·18 (63)	2°·85 ± ·14	2°·15 ± ·13
Alveolar $P\angle$	83°·8 ± ·27 (69)	83°·6 ± ·45 (54)	3°·38 ± ·19	4°·90 ± ·32
Prosthion $P\angle$	81°·8 ± ·27 (88)	81°·8 ± ·40 (59)	3°·73 ± ·19	5°·21 ± ·32
$\theta_1\angle$	31°·0 ± ·22 (62)	28°·1 ± ·32 (46)	2°·62 ± ·16	3°·16 ± ·27
$\theta_2\angle$	10°·4 ± ·27 (62)	12°·3 ± ·53 (45)	3°·11 ± ·19	3°·25 ± ·37

whether the male and female series represent the same racial type. The differences of the corresponding mean indices and angles for the two sexes are found to exceed three times their probable errors in the case of 100 B/L (Δ/p.e. $\Delta = 4·0$), 100 SS/SC (6·4), 100 O_2/O_1, R (6·5), 100 O_2/O_1, L (6·3), 100 O_2/O_1', R (4·8), 100 B/H (5·7), 100 NB/NH, R (3·7), 100 NB/NH, L (4·0), 100 NB/NH' (3·6), $N\angle$ (7·4), $A\angle$ (5·5), $\theta_1\angle$ (5·1) and $\theta_2\angle$ (3·2). Sexual differences of the same sign as those now found are to be expected in the case of the first five of these indices. The difference in the case of 100 B/H may be due to chance causes as that for the very similar index 100 B/H' is insignificant. The other measurements selected in this way are the nasal indices, which all measure the same character in slightly different ways, and four angles which are necessarily related. They simply show that the female type has a significantly higher nasal index and a greater degree of prognathism (judging by the nasal but not by the profile angles) than the male*. This might suggest that there was more negro blood among the women than among the men

* These relations are still true if the Twelfth and Thirteenth Dynasties are considered separately, as is shown by the table on p. 260. But the sexual differences between the corresponding means there are probably all insignificant.

of Kerma, but it would be rash to accept such a conclusion without further evidence, and comparisons of a different kind made below (p. 272) fail entirely to substantiate it. The fact that the majority of the measurements of shape show either insignificant differences, or differences which would have been expected, must be taken to indicate that the male and female samples really represent the same racial type.

Sexual differences in variability may be considered next and comparisons were made between the coefficients of variation of the absolute measurements and the standard deviations of the indices and angles. For these constants 48 male values exceed the corresponding female values and for the remaining 19 the female are in excess. For the long Egyptian *E* series the male variabilities are the greater in 46 cases and the female in the other five, but a proportion more similar to that found for the present series has been observed for most other long cranial series available. The only significant difference found for the Kerma male and female coefficients of variation is for *fml* (Δ/p.e. $\Delta = 4·9$) and for the standard deviations of the indices and angles the only significant differences (Δ/p.e. $\Delta > 3$) are in the case of the *Alveolar* $P\angle$ (4·2), *Prosthion* $P\angle$ (4·0), $\theta_1\angle$ (6·0) and $\theta_2\angle$ (4·7). Finally we may compare the Kerma variabilities with those given for other series and it will suffice to consider only the Egyptian *E* series in this connection. It is known to be more homogeneous than almost all others that have been measured. Still comparing the coefficients of variation of absolute measurements and the standard deviations of indices and angles, it is found that 31 of the male Kerma variabilities are greater than the Egyptian *E* and 15 less, while four of the differences exceed three times their probable errors; for the females the Kerma variabilities are the greater in 26 cases, the lesser in 19 and there is equality in one case, while 10 of the differences may be considered significant. The Kerma series is thus rather more variable than the Egyptian *E*, but it is as homogeneous as many which have to be accepted as representing single and indivisible racial types.

(3) *Remarks on the Condition and Anomalies of the Kerma Skulls.* The crania from Kerma were examined for anomalies after the method customarily employed in the Biometric Laboratory in recent years. There are totals of 141 male and 114 female [illegible] * [illegible] owing to the incomplete nature of several of the skulls the total [illegible] been affected are recorded separately for several of the more [illegible]

(a) *Sutures.* [illegible] the condition of the coronal, sagittal and lambdoid sutures [illegible] on in the appended tables of individual measurements. [illegible] it may be assumed that any one, or all, of these sutures are [illegible] ire lengths. The approximate estimate of the age constitution [illegible] ch can be deduced from these data has been considered [illegible] of interest is the order of closure of the three

* An adult [illegible] is not included in this total and it was not measured since it appears to be [illegible] ttal suture is completely obliterated, the coronal is closing and the lambdoid [illegible] length.

principal sutures since a clear racial difference is supposed to be shown in this respect. The following table gives the frequencies found for the different orders of closure:

Sex	Sagittal closing before coronal and lambdoid	Sagittal and coronal closing together before lambdoid	Sagittal and lambdoid closing together before coronal	Coronal closing before sagittal and lambdoid	Lambdoid closing before sagittal and lambdoid	Totals	Coronal closing before lambdoid	Lambdoid closing before coronal
♂	75	20	3	18	1	117	79	20
♀	34	18	2	14	2	70	48	10

It is apparent from these figures that the sagittal shows a clear tendency to close before the other two sutures, while the coronal also tends to close before the lambdoid suture. The order of closure found with the greatest frequency in the case of both sexes is sagittal—coronal—lambdoid. Three other series have previously been examined in the Biometric Laboratory in exactly the same way and it was found that for negroes (Teita) the coronal suture showed a slight tendency to close before the sagittal in the case of the males, and a definite tendency in the case of the females, though the numbers (37 males and 19 females) on which these estimates were based are small. Both the coronal and sagittal sutures showed a marked tendency to close before the lambdoid. In the Hythe and Spitalfields English series the sagittal suture was found almost invariably to be the first to close, or to begin closing, followed by the coronal and lambdoid sutures which began closing at approximately the same time. Thus the Kerma type occupies an intermediate position between the Teita—a negro series—on the one hand and the Hythe and Spitalfields European series on the other. For the negroes the coronal suture closes before the sagittal and well before the lambdoid; for the Nubians the coronal suture closes after the sagittal but before the lambdoid again, and for the Europeans the coronal suture closes after the sagittal but almost at the same time as the lambdoid. These are only average results, and the numbers on which they are based are unfortunately small, but in confirmation of Gratiolet's Law they appear to indicate a definite racial difference in the order of sutural closing. The figures for the series examined in the Biometric Laboratory suggest that the tendency for the coronal suture to close before the lambdoid is rather more marked in the case of females than in the case of males for any particular racial series. As the numbers for the Kerma series are fairly adequate, it was considered worth while to examine whether the order of closure of the coronal and lambdoid sutures is associated with the magnitude of breadth measurements of the calvaria. The following table giving measurements for the two groups of male skulls from Kerma shows that the differences between the corresponding means are clearly insignificant:

Order of suture-closing	*B'*	*B*	Biasterionic *B*	*L*
Coronal suture closing before lambdoid	90·8 (72)	133·6 (67)	105·4 (57)	185·7 (67)
Lambdoid suture closing before coronal	93·8 (18)	134·8 (19)	105·7 (13)	185·7 (18)

Complete cases of metopism were found in four male out of a possible 89 skulls, and six female out of a possible 97 skulls, only small traces of a suture above the nasion being found in other specimens. All the longer European series to be examined have shown a percentage of about 10 for metopic sutures and in the case of negro series the frequency is about 1 per cent., among the 122 Teita adult skulls only one metopic suture being found. The Kerma skulls, therefore, with a percentage of 4·5 for the males and 6·2 for the females, again occupy an intermediate position between the negro and white races. Contact between the frontal and parietal bones in the case of a metopic suture being present is indicated by $LF+RP$ or $RF+LP$ and the measurement given is the length of the common suture. The males affected are Nos. 107 (contact?), 111 (at bregma), 113 (at bregma), and 126 (? $RF+LP$); and the females Nos. 163 ($LF+RP$, 3·1 mm.), 165 ($RF+LP$, 3·2 mm.), 166 ($LF+RP$, 8·2 mm.), 202 ($LF+RP$, 5·7 mm.), 219 ($LF+RP$, 2·9 mm.) and 224 ($LF+RP$, 7·7 mm.). In accordance with what has been previously found, contact between the left frontal and right parietal bones is the most frequent occurrence. It has been stated that when the metopic suture persists to an adult stage it closes at about the same age as the sagittal. Five out of the ten metopic Kerma skulls have both frontal and sagittal sutures open; in two other cases both are closed and in the remaining three the sagittal suture is open but the frontal is closed. One female specimen (No. 202) has the sagittal suture open but the frontal entirely closed and partly obliterated. Only when the squamous process made actual contact with the frontal bone were the cases of fronto-temporal articulation noted, these being in the males Nos. 62, 69, 136, 138 and 147 (on both sides), 70 and 124 (on the left but not on the right side), 3 and 51 (on the left while the right side was too defective to permit examination); and in the females Nos. 152, 194, 195, 199, 255 and 271 (on both sides), and 160, 173, 203 and 230 (on the right but not on the left side). For the males there were 89 cases on the right side where the sutures at the pterion were sufficiently open to permit examination for cases of fronto-temporal articulation and of these five were affected giving a percentage of 5·6, and on the left nine were affected out of a possible 101 (8·9 %); for the females, on the right side 10 were affected out of a total of 83 (12·0 %), and on the left six out of a possible 92 (6·5 %). These percentages are unusually high, as for European series in general the frequency is of the order of 1·6 per cent., but for negroes an average of 12 per cent. has been found*. The Kerma series it will be noticed once again occupies an intermediate position. No complete case of a horizontal suture across the malar bone occurs in the series, two female skulls however (Nos. 197 and 218) show slight traces of it. No traces of sutures between the ex- and supra-occipitals were observed.

(b) *Supernumerary Bones.* In the male series four cases of true interparietal bones were found out of a possible 111 skulls (3·6 %) and for the females only one out of a possible 104 cases (0·9 %) was noted, thus giving a percentage of 2·3 for

* Cf. le Double, *Traité des Variations des Os du Crâne de l'Homme* (1903), pp. 302—303.

the whole series. Owing to the irregularity of the percentages found for both European and negro series in this respect, it is not known whether any racial significance can be attached to the Kerma figures. Of the male specimens, No. 36 shows the *os pentagonale* and right *os triangulare* separate but with no suture between them; the other three skulls (Nos. 98, 121 and 130) have the *os pentagonale* only separate. The female specimen (No. 198) also has the *os pentagonale* only separate. An *os épactal** was noted in one male skull (No. 41): it is divided by a vertical suture which is to the right of the sagittal suture. There are no examples of an ossicle of bregma and only small Wormian bones were found in the coronal and sagittal sutures. The Wormian bones in the lambdoid suture appeared to be smaller and less frequent than in European series. Two unusual anomalies were noted; a male specimen (No. 107) has a large Wormian bone between the temporal squama and parietal on the left side and a female specimen (No. 153) has two large and two small Wormian bones in the same position on the left while the right side is normal (see Plate IV *b*). Several cases of an ossicle of lambda were observed. Records were made of the epipteric bones found, only those with a maximum diameter greater than 3 mm. being counted. In compiling the total numbers which might have possessed these supernumerary bones all those skulls were excluded on which the superior border of the greater wing of the sphenoid was indefinite on the side considered owing to synostosis. More males than females were excluded on this account. The following frequencies were found for the group having the region of the pterion intact on both sides and with the sutures there visible:

Sex	No epipteric bones	Epipteric bones (one or more) on both sides	One or more epipteric bones on the right but none on the left	One or more epipteric bones on the left but none on the right	Totals
♂	40	4	4	5	53
♀	43	9	7	9	68

A considerable difference is shown between the percentages for the two sexes, the males having 13 skulls with one or more epipteric bones out of a possible 53 cases (24·5 °/$_\circ$), and among the females there are 25 cases out of a possible 68 specimens (36·8 °/$_\circ$). According to le Double† it is not known for which sex the anomaly occurs with the greater frequency and comparison with the Hythe and Teita series does not encourage the assumption that there are any sexual or racial differences. The percentage for the Teita male series is 22·2 (or 8 out of a possible 36 cases) and for the female 19·5 (or 8 out of a possible 41 cases). For the Hythe skulls the male percentage is 32·0 and the female 25·8.

(*c*) *Teeth*. The state of preservation in which the teeth of the Kerma skulls were found was remarkably good, and there was a very high percentage of palates

* Cf. le Double, *Traité des Variations des Os du Crâne de l'Homme* (1903), p. 60.

† *Op. cit.* p. 306.

with no teeth lost before death. There are 111 complete male palates of which 71 have no teeth lost before death (64·0 %), and for the females the percentage is 69·5, 57 skulls from a total of 82 complete palates not having lost any teeth before death. Excluding a few cases, where it could not be determined whether the third molars had erupted or not owing to the loss of one or more molars before death, an investigation with regard to the presence or absence of third molars was made with the following results:

Sex	Region of third molars complete and undeformed on both sides	No third molar on either side	Third molar on right but not on left side	Third molar on left but not on right side	Percentage having one or both third molars missing
♂	96	9	3	1	13·5
♀	73	2	0	5	9·6

Considering the sides separately, in which case a few additional specimens with the region of the third molar complete on one side but defective on the other can be included, it is found for the males that on the right side there is an absence of the third molar in 12 cases out of a possible 102 (11·8 %) and on the left there are 12 showing an absence out of a possible 99 (12·1 %). For the females there is absence in seven cases out of a possible 79 (8·9 %) on the right and four out of a possible 82 (4·9 %) on the left. Several dental anomalies were observed among the male skulls: No. 79 has only a single premolar on the right, No. 92 (Plate VI *a*) apparently lacked two teeth other than molars, No. 123 has no canine on the right, while No. 128 lacks a canine on the left side. The second right incisor in skull No. 101 has erupted behind its normal position. A curious condition is present in No. 107 (Plate VI *d*) which apparently had only three incisors of which one was placed centrally so that it appears to have erupted between the premaxillary bones. The socket for the third incisor present on the left in skull No. 4 is the only case of an extra tooth among the male specimens. The female specimen No. 155 (Plate VI *c*) has no canine present on the left side and no third molar on the right. An absence of lateral incisors was noted in skull No. 216 (Plate VI *b*), and No. 266 has only one incisor on the left and no canine or third molar on the right. Several examples were observed of crowding of the teeth and of deformations of the alveolar margin due to disease (see Plate VI *a* showing an opening in the region of the first and second left molars).

(*d*) *Other Anomalies*. The relative sizes of the jugular foramina were compared with the following results:

Sex	*JR*	*J* =	*JL*
♂	74 (65 %)	19 (17 %)	20 (17 %)
♀	48 (62 %)	9 (12 %)	20 (26 %)

These results are quite in accordance with those for all series previously examined in this way. Precondyles, both single median and double, were noted in several cases, a female skull (No. 194) with a single median precondyle having the largest. There are no marked cases of a *fossa pharyngea*. A female specimen (No. 180) has a large para-mastoid process on the left side with an articular surface, and one case of a basi-occipital incisure was observed on the left side of another female skull (No. 193). Plate V *b* illustrates a curious condition present in a male skull (No. 86); this is a constriction of the inferior part of the basi-occipital where it unites with the sphenoid. The numbers of tympanic perforations were recorded but no details are given as no racial or sexual differences, or evidence that they were more numerous on one side than on the other, could be deduced. What appears to be an infantile condition of some interest is present in a female skull (No. 216: Plate V *d*) which shows a complete failure to unite on the superior and inferior margins of the tympanic plate on the left side and a union only just made on the right*. None but very small exostoses were found. Three healed wounds were observed among the female skulls from a total of 114, but, as is to be expected, the wounds on the male skulls were more numerous, totalling 16 for the 141 skulls. Those of greatest interest are on male specimens and photographs of two of these are given: No. 74 (Plate IV *a*) has a depressed wound on the right parietal and temporal squama and a healed fracture of the left malar bone; No. 64 (Plate V *c*) has what appears to be a wound on the right maxillary bone below the orbit. No. 7 shows a fracture and subsequent rejoining of the left zygomatic arch. One case of a wound on the nasal bones is noted in skull No. 66. Traces of roughening of the glenoid fossa due to arthritis were found on several skulls of which the males Nos. 46, 119 and 146, and the females Nos. 151 and 236 were the most marked. The only other sign of disease was observed on the female skull No. 274 (Plate V *a*); the diseased parts are two symmetrically placed areas on either side of the lambda, the lambdoid suture being prematurely closed where it lies within the diseased areas. This may be a case of periostitis. Holes similar to those found by Prof. Elliot Smith on Nubian skulls† and supposed due to insects were evident on several of the Kerma skulls (see Plate III *b* below). The holes are mostly small and not nearly so numerous as on the skull he figures. No signs of healing are shown on our specimens and the excavations were almost certainly made after death.

(4) *Comparisons between the Kerma and other Racial Series.* In order to gain a knowledge of the racial relationships of the Kerma series, the Coefficients of Racial Likeness were calculated between its means and those available for a number of other series. With the usual notation, the form of the crude coefficient used is

$$\frac{1}{m}\Sigma\frac{(M_s-M_{s'})^2}{\sigma_s^2}\times\frac{n_s n_{s'}}{n_s+n_{s'}}-1\pm{\cdot}67449\sqrt{\frac{2}{m}}=\frac{1}{m}\Sigma(\alpha)-1\pm{\cdot}67449\sqrt{\frac{2}{m}}.$$

* The adolescent *Sinanthropus* skull shows a similar fissure of the tympanic plate, and the condition is normal for infants to-day.

† *The Archaeological Survey of Nubia, Report for* 1907—1908, Cairo (1910), Vol. II, p. 290, and *Plates Accompanying Vol. II*, Plate xxx, Fig. 2. See also "The alleged Discovery of Syphilis in Prehistoric Egyptians," *The Lancet*, August 22 (1908), p. 521.

The reduced coefficient is defined to be

$$50 \times \frac{\bar{n}_s + \bar{n}_{s'}}{\bar{n}_s \bar{n}_{s'}} \left[\frac{1}{m} \Sigma(\alpha) - 1 \pm \cdot 67449 \sqrt{\frac{2}{m}}\right],$$

where $\bar{n}_s$ and $\bar{n}_{s'}$ are the mean numbers of skulls available for the characters used in computing the coefficient for the first and second series in the comparison, respectively. The reduced coefficient is supposed to give the best measure of the absolute divergencies between the types which it is possible to find at present. The standard deviations of the long Egyptian E series of Twenty-sixth to Thirtieth Dynasty skulls were used in computing the coefficient.

The male and female Kerma series are of approximately equal length, but this is not the case for the majority of cranial series for which measurements have been provided. In general the male series are longer than the corresponding female ones and there are far more of them available. Our estimates of racial affinity are thus based principally on the comparisons of the male series. There are 20 of these series of sufficient length representing different sections of the population of Egypt from Early Predynastic to Roman times. Of the sites represented Gizeh and Deshasheh with Medum are in Lower Egypt, but all the others are in Middle or Upper Egypt. Table III gives all the coefficients of racial likeness between these Egyptian and the Kerma series and the reduced male values range from 1·11 to 19·56. Working on data provided by Morant, Woo has given* the lowest reduced coefficients of racial likeness found for each of 23 male Egyptian series, and they range approximately from $-0{\cdot}5$ to $+4{\cdot}5$. Three of the values now found with the Kerma series are less than 4·5 and two are as low as any which it has generally been possible to find for these types. We are thus led to conclude that the population of Kerma in the Twelfth and Thirteenth Dynasties was of typical Egyptian type. It has been shown by Morant† that in Upper Egypt the character of the population was changing slowly from Early Predynastic to Roman times, the type having gradually lost its original negroid characters and at the same time increased its calvarial breadth and hence its cephalic index. This general tendency was illustrated by the majority of the series available, and the exceptions to it which were found could be supposed due to the peculiarities of local types or possibly to a local and restricted admixture of part of the population with alien immigrants. The question to what period the Upper Egyptian series which most closely resemble the Kerma belong is one of particular interest. Its closest connections are found with the Naqada series which is probably of Middle Predynastic data and with a Late Predynastic series measured by Thomson and MacIver‡. A First Dynasty series is next in order and then the Kerma type is

* *Biometrika*, Vol. XXII (1930), p. 75. † *Ibid.* Vol. XVII (1925), pp. 1—52.

‡ For the series measured by Thomson and MacIver only 14 of the complement of 31 characters used in computing the coefficient of racial likeness are given. For these 14 characters the male crude coefficient between the Kerma and Naqada A and Q series is $1{\cdot}21 \pm {\cdot}25$ and the reduced value is $1{\cdot}49 \pm {\cdot}31$. These only differ slightly from the values in Table III given for the total 31 characters, and it may be assumed that the 14 characters provided by Thomson and MacIver lead to close approximations to the coefficients which would be obtained if all 31 characters could be used in comparisons with their series. It may be concluded that the Kerma series is equally related to the Naqada and to the Late Predynastic series of Thomson and MacIver.

TABLE III.

Coefficients of Racial Likeness between the Kerma (12th and 13th Dynasties) and other Series.*

Locality or Race	Period	Measured by	$\bar{n}$ ♂	$\bar{n}$ ♀	Kerma $\bar{n}$ ♂	Kerma $\bar{n}$ ♀	Crude Coefficients ♂	Crude Coefficients ♀	Reduced Coefficients ♂	Reduced Coefficients ♀
Naqada (*A* and *Q* series)† ...	Predynastic	Fawcett	65·8	—	107·3	—	0·91 ± ·17 (31)	—	1·11 ± ·21	—
El-Amrah and Hou† ...	Late Predynastic	Thomson and MacIver	105·6	119·9	107·6	86·6	1·54 ± ·25 (14)	3·39 ± ·35 (14)	1·15 ± ·19	3·33 ± ·26
Abydos, El-Amrah and Hou†	1st Dyn. (Private Tombs)	"	33·6	54·3	107·6	86·6	1·97 ± ·25 (14)	1·59 ± ·35 (14)	3·84 ± ·50	2·37 ± ·38
Abyssinia (Tigre District)‡	Modern	S. Sergi	62·0	—	105·2	—	3·91 ± ·18 (28)	—	3·01 ± ·23	—
Abydos, El-Amrah and Hou†	Early Predynastic	Thomson and MacIver	40·9	—	107·6	—	3·35 ± ·25 (14)	—	5·65 ± ·43	—
Hou and Abydos†	12th—15th Dyns.	"	65·9	87·3	107·6	86·6	4·72 ± ·25 (14)	8·35 ± ·35 (14)	5·78 ± ·31	9·60 ± ·29
Denderah†	6th—12th Dyns.	"	168·6	140·4	107·6	86·6	7·68 ± ·25 (14)	7·09 ± ·25 (14)	5·65 ± ·19	6·61 ± ·24
El-Kubanieh North‡ ...	Middle Dyns.	Toldt	32·5	—	108·5	—	3·16 ± ·18 (28)	—	6·27 ± ·36	—
Denderah†	Ptolemaic	Thomson and MacIver	76·6	—	107·6	—	6·24 ± ·25 (14)	—	7·09 ± ·29	—
Shekh Ali†	18th Dyn.	"	40·0	—	107·6	—	4·27 ± ·25 (14)	—	7·33 ± ·44	—
El-Kubanieh South† ...	Early and Middle Dyns.	Toldt	63·6	—	108·5	—	5·89 ± ·18 (26)	—	7·35 ± ·22	—
Badari§	Early Predynastic	Stoessiger	33·8	20·3	107·3	81·7	3·99 ± ·17 (31)	2·13 ± ·17 (31)	7·77 ± ·33	6·55 ± ·52
Thebes†	18th—21st Dyns.	E. Schmidt	161·9	—	101·0	—	9·70 ± ·17 (30)	—	7·77 ± ·14	—
Galla and Somali‖	Modern	Pooled	26·2	—	105·6	—	3·26 ± ·19 (24)	—	7·77 ± ·46	—
Thebes‡	18th—20th Dyns.	Stahr	53·6	—	104·0	—	5·72 ± ·18 (29)	—	8·09 ± ·25	—
Abydos†	18th Dyn.	Thomson and MacIver	49·9	—	107·6	—	6·94 ± ·25 (14)	—	10·17 ± ·37	—
Negroes from Egypt‖ ...	Modern	E. Schmidt	34·1	19·5	102·1	80·5	8·57 ± ·18 (29)	1·17 ± ·18 (29)	12·85 ± ·35	3·73 ± ·56
Denderah†	Roman	Thomson and MacIver	49·3	—	107·6	—	9·34 ± ·25 (14)	—	13·62 ± ·38	—
Gizeh†	26th—30th Dyns.	Pearson and Davin	854·3	—	105·0	—	20·35 ± ·17 (31)	—	15·25 ± ·09	—
Abydos†	1st Dyn.	Motley	32·3	—	107·3	—	7·71 ± ·17 (31)	—	15·52 ± ·34	—
Teita (Kenya Colony)‖ ...	Modern	Kitson	33·8	33·8	107·3	81·7	9·12 ± ·17 (31)	10·07 ± ·17 (31)	15·71 ± ·29	14·89 ± ·25
Sedment‡	9th Dyn.	Woo	37·1	—	107·3	—	8·01 ± ·17 (31)	—	16·14 ± ·31	—
Tamils¶	Modern	Harrower	35·0	—	102·3	—	8·10 ± ·17 (31)	—	17·24 ± ·33	—
Nepalese¶	"	Morant	45·7	—	107·3	—	11·36 ± ·17 (31)	—	17·76 ± ·27	—
Abydos†	18th and 19th Dyns.	E. Schmidt	29·9	—	102·1	—	8·13 ± ·18 (29)	—	18·66 ± ·38	—
Deshasheh and Medum† ...	4th and 5th Dyns.	Thomson and MacIver	39·9	—	107·6	—	11·38 ± ·25 (14)	—	19·56 ± ·44	—
Dravidians¶	Modern	Turner	30·9	—	103·0	—	10·21 ± ·21 (20)	—	21·46 ± ·45	—
Tanganyika‖	"	Reid	37·4	18·3	110·6	87·2	12·12 ± ·20 (22)	6·36 ± ·20 (32)	21·50 ± ·36	21·05 ± ·67
Hottentots‖	"	Pooled	24·2	—	109·1	—	9·98 ± ·19 (24)	—	25·19 ± ·49	—
Hindus¶	"	"	74·5	—	104·8	—	26·28 ± ·20 (23)	—	30·15 ± ·23	—
Veddahs¶	"	"	33·8	—	105·3	—	17·36 ± ·20 (23)	—	33·91 ± ·39	—

* The $\bar{n}$'s are the mean numbers of skulls available for the characters used in computing the coefficients. The numbers in round brackets following the coefficients are the numbers of characters on which they are based.

† References and male means for these series are given by Morant in *Biometrika*, Vol. XVII (1925), pp. 14—43. The female means of Thomson and MacIver's 6 [illegible]—12th and 12th—15th Dynasty series are in *Biometrika*, Vol. XXII (1930), p. 70, and the other female means of their series used have not been publis[illegible].

‡ [illegible] and r[illegible] for these series are given by Woo in *Biometrika*, Vol. XXII (1930), pp. 69 and 74, and in Appendix I to his paper.

§ [illegible] Vol. [illegible] 1927), Table II, facing p. 117. [illegible]

‖ [illegible]

found to be equally removed from those of Early Predynastic and Early Dynastic populations. It is also approximately equally, but more distantly, removed from the earliest Predynastic type (Badari) and from several found in Upper Egypt about the Eighteenth Dynasty. The relationships of the Kerma to most of the Later Dynastic series, including the Roman, are more distant still. Exceptions to any orderly arrangement suggested by these facts must, of course, be expected. They are found, for example, in the rather close connection between the Kerma series and the Ptolemaic and in the dissimilarity of the former to the First Dynasty series from Abydos measured by Motley, to the Ninth Dynasty series from Sedment and to the Fourth and Fifth Dynasty series from Deshasheh and Medum, though it must be remembered that the last series comes from Lower Egypt. In these cases the other series, and not the Kerma, must be supposed peculiar, as has been pointed out in earlier papers. The evidence provides abundant justification for considering that our series represents the typical population found in Upper Egypt in Late Predynastic times, and it can only just be distinguished from other series representing that population. But actually our series is of Twelfth and Thirteenth Dynasty date, though it is clearly differentiated from the contemporary population of Upper Egypt. Obvious conclusions are suggested by these facts. It may be supposed that the colony at Kerma was founded in Late Predynastic times by a body of emigrants from Upper Egypt who were racially typical of the population there and that this type persisted at Kerma unchanged by admixture with any non-Egyptian elements, or by the factors which were modifying the parent population, until the Twelfth and Thirteenth Dynasties. It appears to be extremely probable that these hypotheses are the correct ones, but they could only be submitted to direct proof if we had skeletal evidence of the racial constitution of the population at Kerma between Early Predynastic and Twelfth Dynasty times

It is certainly unlikely that any coefficients of racial likeness of the same order as the lowest in Table III could be found between the Kerma and any non-Egyptian series. There is a close connection shown with Abyssinians from the Tigre district, but all the closest connections as yet found for this series have been with Dynastic Egyptian types and it is supposed to represent a survival until modern times of part of that population. It has been shown by Stoessiger* that there is a moderately close resemblance between the Predynastic Egyptian series and modern Indian ones, and by Kitson† that the former also resemble various negroid and Bantu negro East African types. It is among these two groups of races that we should expect to find those outside Egypt which will bear the closest resemblance to the Kerma type. Male coefficients of racial likeness were therefore calculated between the Kerma and all the best series available for the two groups and these are given in Table III. Of all the alien types compared, the Galla and Somali shows the lowest reduced coefficient, though it is higher than eleven of the values found with the Egyptian (including the Abyssinian) series. A moderately close connection with that "Hamitic" series was to be expected. Neither is it surprising to find that the

* *Biometrika*, Vol. XIX (1927), pp. 125—185. † *Ibid.* Vol. XXIII (1931), pp. 285—300.

next closest connection is with a modern series from Egypt of which the origin is unknown and which is only judged to be one of negroes on account of the cranial measurements. The Teita from Kenya Colony come next in order and they are undoubtedly Bantu negroes, while the Tamils are a little further removed. The resemblances in these cases, and those between the Kerma and the other East African and Indian types, are very much more distant than the majority of those between the Kerma and the Egyptian types. The fact that a negro series shows a lower reduced coefficient than three Egyptian series may be attributed to the fact that the latter are not of pure Egyptian origin. It has been shown by Woo, for example, that his Sedment series and that of the Fourth and Fifth Dynasties from Deshasheh and Medum are more closely related to modern Egyptians and modern Cretans than are the other Dynastic Egyptian series. There is again no suggestion that the Kerma population was of any but pure Egyptian origin.

From the 31 series for which male coefficients of racial likeness were calculated, eight were selected for the purpose of comparing the female means with the Kerma values. There are no adequate female means available for most of the other series. If the male and female means for each of two series compared really represent the same population, then we should expect to find that the reduced coefficients for the two sexes do not differ significantly. This is actually the case in five out of the eight comparisons made in Table III, the differences for these being all less than 2·5 times their probable errors. For the other three—viz. the Late Predynastic series from El-Amrah and Hou, the Twelfth to Fifteenth Dynasty from Hou and Abydos, and the modern series of negroes from Egypt—the sexual differences between the coefficients are markedly significant. The means for the last series are based on such small numbers that no stress need be placed on the discordance. In the case of each of the other two a direct comparison of the male and female means suggests that the samples may not represent exactly the same population. It has been shown in section (2) above that the Kerma male and female indices and angles only differ clearly in an unexpected way in the case of the nasal indices and of the angles of the fundamental triangle, the female type having the higher index and the greater nasal angle, indicating a greater degree of prognathism. This might suggest that the female Kerma sample was more negroid than the male, but the coefficients with the Teita and Tanganyika negro series fail entirely to substantiate this view. The Kerma sexual differences for these characters, though of an order which must be considered significant, may only have been due to chance causes.

A detailed comparison of measurements considered singly need not be made. If only the 31 characters used in computing the coefficients of racial likeness are considered, the α's found show whether the differences between the means compared should be considered significant or not. We will suppose, as usual, that a significant difference is indicated if the α is greater than ten. The proportions of significant to non-significant α's are found to differ markedly for different characters as has been observed in all previous comparisons of this kind. The coefficients (given in Table III) were computed between the Kerma male series, on the one hand, and

21 series of the ancient Egyptian type (including the modern Abyssinians), on the other. There are eleven comparisons between median sagittal arcs (S) from nasion to opisthion—the measurement not being available for the other ten series—and not a single one of these is significant. For the basio-bregmatic height (H') only one significant difference out of 21 is found, and for the glabella-occipital length there are only three α's greater than ten in 21 comparisons. Such characters will clearly be of no value if a classification of the material on the basis of single measurements, or groups of a few measurements, is attempted. The only measurements which are likely to be of any value for such purposes are those for which more than 30 per cent., say, of the possible comparisons show significant differences. In the present case these are: 100 B/L (61·9 °/ₒ), 100 NB/NH (52·4 °/ₒ), B (42·9 °/ₒ), 100 G_2/G_1 (42·9 °/ₒ), NH (38·1 °/ₒ), B' (36·4 °/ₒ) and Q' (33·3 °/ₒ). The percentage for 100 G_2/G_1 is only based on a total of seven comparisons and no stress can be laid on it, while the other characters selected in this way really only indicate that two primary factors are concerned. The differences between the maximum calvarial breadths (B) are necessarily associated with those between the transverse arcs (Q') and cephalic indices—the calvarial length and height remaining practically constant—and with those between the minimum frontal diameter (B'); while the differences between the nasal heights are necessarily associated with differences between the nasal index—the nasal breadth remaining practically constant. In comparing the Egyptian series Morant found that the essential differences were, in general, those between the calvarial breadths and the characters dependent on that character, and also those between the nasal indices. The Kerma series thus differs most markedly from the Egyptian series in precisely the same characters as they, on the whole, differ most markedly from one another, or, in other words, the Kerma type must be considered to be a perfectly typical representative of the Egyptian stock, as we have previously concluded from a comparison of the coefficients of racial likeness. When comparison is made between the Kerma and the non-Egyptian series the incidence of characters which differ most essentially alters as we pass from series to series and a detailed description of these differences would not be profitable.

(5) *Type Contours.* The individual contours were drawn and the type contours were constructed from them by following the usual methods employed in the Biometric Laboratory. The mean measurements are given in Tables IV—VI and the male and female types are shown in Figs. II—VII. Comparison of the type contour with the mean calliper measurements in Table VII shows a very satisfactory agreement. The differences are all less than 1 mm. except in the case of the male and female S_1'—the chord from nasion to bregma—and the female GL. The difference of 1·4 mm. in the last case is chiefly due to the fact that the two means are based on very different numbers and when found for the same 33 skulls it is reduced to 0·6. It is probable that the rather large differences found in the case of S_1' were caused by the fact that the nasion was not located in precisely the same position when drawing the contours as when taking the calliper length. Table VII shows clearly that the practice of raising or lowering the pointer of the tracer in order

TABLE IV.

Mean Measurements of Kerma Transverse Contours.

Sex	*MA*	*1R = 1L*	*2R*	*2L*	*3R*	*3L*	*4R*	*4L*	*5R*	*5L*	*6R*	*6L*	*7R*
♂	110·7 (124)	55·4 (124)	58·8 (124)	58·2 (124)	61·4 (120)	61·0 (121)	63·8 (116)	63·2 (121)	64·8 (118)	64·0 (122)	63·8 (121)	63·8 (124)	62·9 (121)
♀	107·7 (90)	52·8 (90)	56·1 (90)	56·5 (90)	58·9 (87)	59·4 (89)	61·8 (84)	61·6 (90)	63·3 (85)	63·0 (90)	63·1 (88)	61·8 (90)	61·6 (89)

Sex	*7L*	*8R*	*8L*	*9R*	*9L*	*10R*	*10L*	*A¼R*	*A¼L*	*ZR, R*		*ZR, L*	
										x	*y*	*x*	*y*
♂	62·4 (124)	59·0 (121)	58·4 (124)	50·6 (122)	49·8 (124)	35·9 (124)	35·1 (124)	17·4 (123)	17·1 (124)	3·4 (124)	58·7 (124)	3·2 (124)	58·9 (124)
♀	60·2 (90)	58·4 (89)	56·1 (90)	50·5 (90)	48·0 (90)	36·8 (90)	34·6 (90)	17·4 (90)	16·5 (90)	3·7 (89)	55·8 (89)	4·0 (89)	55·7 (89)

TABLE V.

Mean Measurements of Kerma Horizontal Contours.

Sex	*FO*	*F¼R*	*F¼L*	*F½R*	*F½L*	*2R*	*2L*	*2½R*	*2½L*	*3R*	*3L*
♂	185·1 (124)	23·1 (124)	21·6 (123)	35·4 (124)	35·0 (123)	47·2 (124)	46·9 (123)	46·1 (124)	46·1 (123)	47·6 (119)	47·4 (124)
♀	174·7 (90)	21·7 (90)	22·3 (90)	32·5 (89)	33·3 (90)	45·1 (90)	44·2 (90)	46·4 (88)	44·7 (89)	48·3 (87)	46·6 (90)

Sex	*4R*	*4L*	*5R*	*5L*	*6R*	*6L*	*7R*	*7L*	*8R*	*8L*
♂	54·2 (117)	53·5 (122)	61·0 (119)	59·6 (123)	65·3 (120)	63·9 (123)	66·3 (123)	66·2 (124)	63·8 (124)	63·1 (124)
♀	53·1 (86)	51·8 (88)	59·2 (87)	57·8 (88)	63·4 (87)	62·3 (90)	64·7 (90)	64·3 (90)	62·3 (90)	62·6 (90)

Sex	*9R*	9L	10R	*10L*	O¼R	O¼L	*TR*		*TL*	
							x	*y*	*x*	*y*
♂	57·5 (124)	57·1 (124)	45·0 (124)	44·3 (124)	25·9 (124)	25·0 (124)	20·5 (124)	48·9 (124)	19·9 (123)	48·4 (123)
♀	55·5 (90)	55·8 (90)	43·6 (90)	43·7 (90)	24·7 (90)	23·9 (90)	18·6 (89)	46·8 (89)	17·5 (90)	45·2 (90)

TABLE VI.

Mean Measurements of Kerma Median Sagittal Contours.

Sex	$N\gamma$	Ordinates above $N\gamma$												
		$O=N$	$N\frac{1}{2}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{8}$
♂	182·5 (98)	19·4 (98)	36·5 (98)	58·5 (98)	71·2 (98)	78·7 (98)	83·1 (98)	85·0 (98)	85·3 (98)	81·9 (98)	73·3 (98)	55·1 (98)	27·0 (98)	20·5 (98)
♀	174·2 (73)	17·0 (73)	37·5 (73)	56·7 (73)	68·7 (73)	75·7 (73)	80·1 (73)	81·7 (73)	81·7 (73)	78·4 (73)	69·9 (73)	51·9 (73)	26·3 (73)	19·4 (73)

Sex	Ordinates below $N\gamma$							Vertex		Bregma		Glabella	
	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{2}$	$\gamma\frac{1}{4}$	x from N	y	x from N	y	x from N	y
♂	61·5 (75)	53·4 (82)	52·1 (72)	49·0 (96)	38·7 (98)	28·3 (98)	20·2 (98)	100·6 (98)	86·0 (98)	76·8 (98)	83·6 (98)	1·7 (98)	12·1 (98)
♀	54·7 (64)	49·8 (66)	49·3 (58)	47·5 (73)	36·4 (73)	27·8 (73)	20·6 (73)	94·9 (73)	82·4 (73)	73·0 (73)	80·5 (73)	1·2 (73)	12·7 (73)

Sex	Occipital Pt.		λ		Sub-Orb.		Aur. Pt.		Opisthion		Inion		Basion	
	x from γ	y*	x from γ	y	x from N	y	x from γ	y	x from γ	y	x from γ	y	from γ	from N
♂	0·4 (98)	3·1 (98)	5·2 (98)	27·8 (98)	8·3 (91)	27·8 (91)	93·1 (98)	26·7 (98)	57·4 (92)	54·4 (92)	15·8 (85)	37·8 (85)	106·4 (84)	102·0 (84)
♀	0·3 (73)	1·5 (73)	4·7 (73)	26·9 (73)	6·5 (71)	25·8 (71)	90·2 (73)	26·3 (73)	55·7 (69)	52·9 (69)	17·0 (40)	36·0 (40)	101·5 (64)	96·5 (64)

Sex	Alv. Pt.		Nose					Palate				Frontal	
								P		P		Max. Sub. to $N\beta$	
	from N	from Bas.	(i)	(ii)	(iii)	$\angle LN\gamma$	NL	x from N	y	x from Alv.	y	x from N	y
♂	69·6 (50)	96·2 (46)	0·4 (80)	1·9 (73)	5·2 (48)	116°·9 (17)	19·4 (17)	47·4 (49)	50·6 (49)	37·0 (41)	16·4 (41)	49·7 (98)	26·4 (98)
♀	66·4 (37)	95·1 (33)	1·0 (63)	2·9 (59)	6·4 (40)	117°·7 (14)	19·4 (14)	43·4 (60)	47·6 (60)	37·3 (36)	15·3 (36)	47·0 (73)	26·2 (73)

Sex	Occipital Max. Sub. to λ Op.		Max. Sub. to $N\lambda$		Max. Sub. to GI		Sp.		Sub. from ½ Bas.—Sp. Chord		N.S.		Prosthion	
	x from λ	y	x from N	y	x from G	y	x from N	y	x from Bas.	y	x from N	y	x from N	y
♂	50·2 (93)	27·4 (93)	90·4 (98)	71·4 (98)	104·0 (85)	105·0 (85)	68·6 (82)	34·7 (82)	14·2 (81)	1·2 (81)	3·8 (45)	49·3 (45)	7·8 (52)	65·7 (52)
♀	45·8 (69)	26·3 (69)	84·3 (73)	68·8 (73)	95·1 (39)	98·3 (39)	64·3 (55)	33·1 (55)	12·3 (59)	1·5 (59)	7·7 (57)	45·2 (57)	10·6 (44)	60·8 (44)

* The occipital points are above the $N\gamma$ lines.

TABLE VII.

A Comparison of Calliper and Type Contour Measurements.*

Character	Male		Female	
	Contour	Calliper	Contour	Calliper
L	185·2 (98)	185·2 (138)	176·7 (73)	177·2 (114)
OH	110·7 (124)	110·7 (125)	107·7 (90)	107·4 (91)
S_1'	113·5 (98)	112·1 (126)†	108·7 (73)	107·4 (109)‡
S_2'	114·9 (98)	115·1 (128)	110·4 (73)	110·4 (106)
S_3'	97·4 (92)	97·2 (124)	94·7 (69)	94·3 (97)
fml	36·6 (84)	36·4 (112)	34·7 (64)	34·6 (92)
H'	134·3 (84)	133·4 (93)	128·1 (64)	127·8 (88)
$G'H$	69·6 (50)	69·6 (111)	66·4 (37)	65·7 (82)
GL	96·2 (46)	96·5 (97)	95·1 (33)	93·7 (68)§
LB	102·0 (84)	101·8 (117)	96·5 (64)	96·6 (89)

* The contour OH is the height (MA) of the transverse type contour, and all other contour measurements are taken from the sagittal type. The maximum length (L) is measured on that figure; $G'H$, GL and LB are used in its construction and all other measurements compared are found by calculation from lengths used in the construction of the type.

† For the 98 male skulls for which sagittal contours were drawn the mean calliper S_1' is 112·1.

‡ For the 73 female skulls for which sagittal contours were drawn the mean calliper S_1' is 107·3.

§ For the 33 female skulls for which the contour GL is given the mean calliper GL is 94·5.

to pass exactly through the "points" in drawing the sagittal figures does not introduce any serious discrepancies. For all practical purposes there is little danger in assuming that all the points shown on that section really lie in a single plane in the case of each individual skull*. If the series be a long one it might be more satisfactory to omit the specimens for which this is least true.

The male transverse type is almost exactly symmetrical, all differences between the right and left sides of the same parallel being less than 1 mm. Greater differences are found in the case of the female figure and the largest is 2·5 for the ninth parallel. The Kerma sections show no striking peculiarities. A number of indices, providing measurements of shape, have been derived from these figures for the purpose of making racial comparisons more exact, and for all these the Kerma values fall well within the ranges given by racial series previously studied. The male and female indices are closely similar as is generally found. By superposition, with aid of the tracings provided, the Kerma outlines are found to be extraordinarily close of those provided by the Teita negro series and rather further removed from the Badari types.

The horizontal type conteurs are very approximately symmetrical, the maximum difference between the right and left sides of the same parallel being 1·5 in the case of the male figure and 1·7 for the female. They are again found to have no

* [I do not agree with this view. The "practice" does not seem to me satisfactory, as the sagittal section of the individual skull ceases to be a section of that skull at all, and the diagraph thus used can lose the advantages it has over the dioptograph. Ed.]

unusual characters and the indices which have been used for comparative purposes are all, with one exception, within the ranges previously found. The most striking sexual difference in the case of the horizontal contours is seen in the shapes of the sections of the temporal fossae. Immediately behind the temporal lines (marked by the points TR and TL) the male outline is contracted to a greater extent than the female. This difference has been measured by expressing the total length of

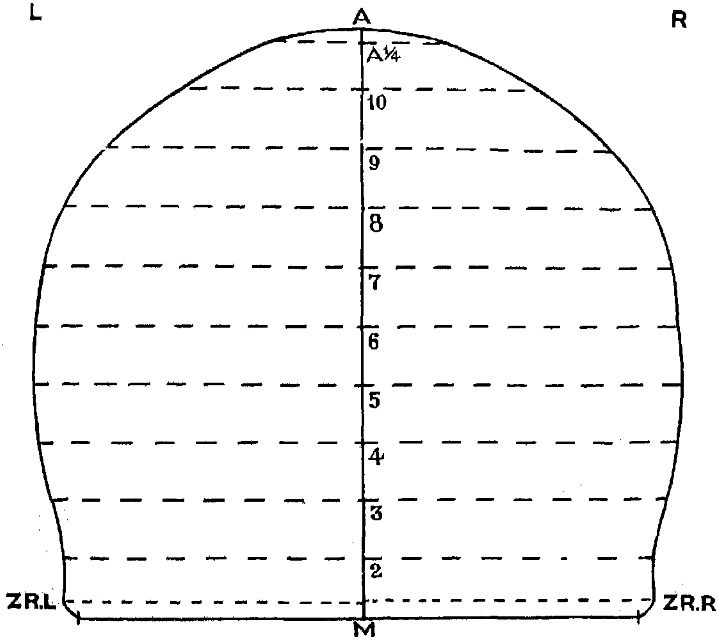


Fig. II Transverse Type Contour, based on 124 ♂ Kerma Skulls.

parallel 3 as a percentage of the width between the temporal lines, i.e. very approximately $TR(y) + TL(y)$. The male Kerma index is 97·7 and the female 103·1, and this difference is rather greater than any previously found between the male and female values for the same racial series. The male index is also lower, indicating more marked temporal fossae, than any previously found. The Kerma type contours are again found to be remarkably similar to the Teita, the only difference which appears to have any significance being dependent on the fact that

the sections of the temporal fossae are more accentuated on the Kerma male figure. The Kerma and Badari types differ more markedly.

The sagittal type sections possess no features which would have been unexpected in contours representing an Egyptian series, except, perhaps, a rather greater degree of prognathism. All the indices and angles which have been used to compare the most salient racial differences of the median sagittal section have values for the

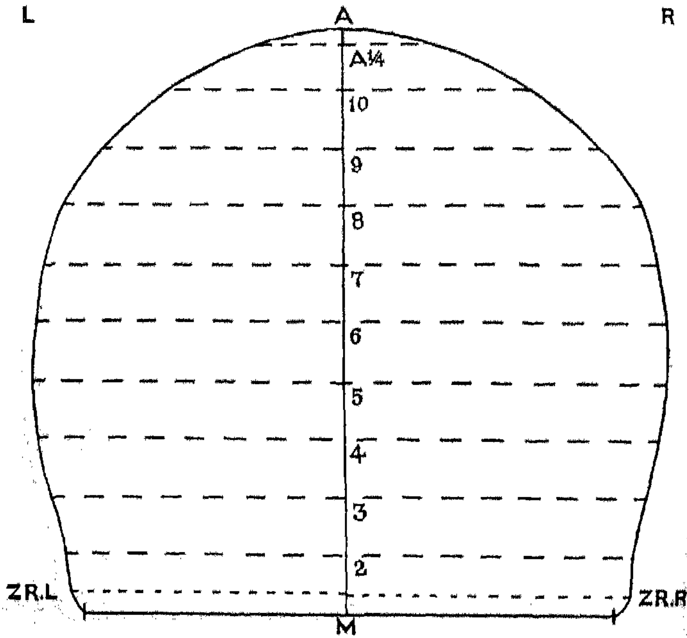


Fig. III Transverse Type Contour, based on 90 ♀ Kerma Skulls.

Kerma series which fall well within the ranges given by the types previously recorded. Mean sexual differences are generally found for some of these and those of the same sign and the same order are observed in the present series. For example, the index expressing the maximum subtense of the frontal arc as a percentage of the $N\beta$ chord is 23·2 for the males and 24·1 for the females, and the fact that the female frontal bone is, on the average, more vertical than the male can be illustrated by angular measurements. A comparison of the mean indices

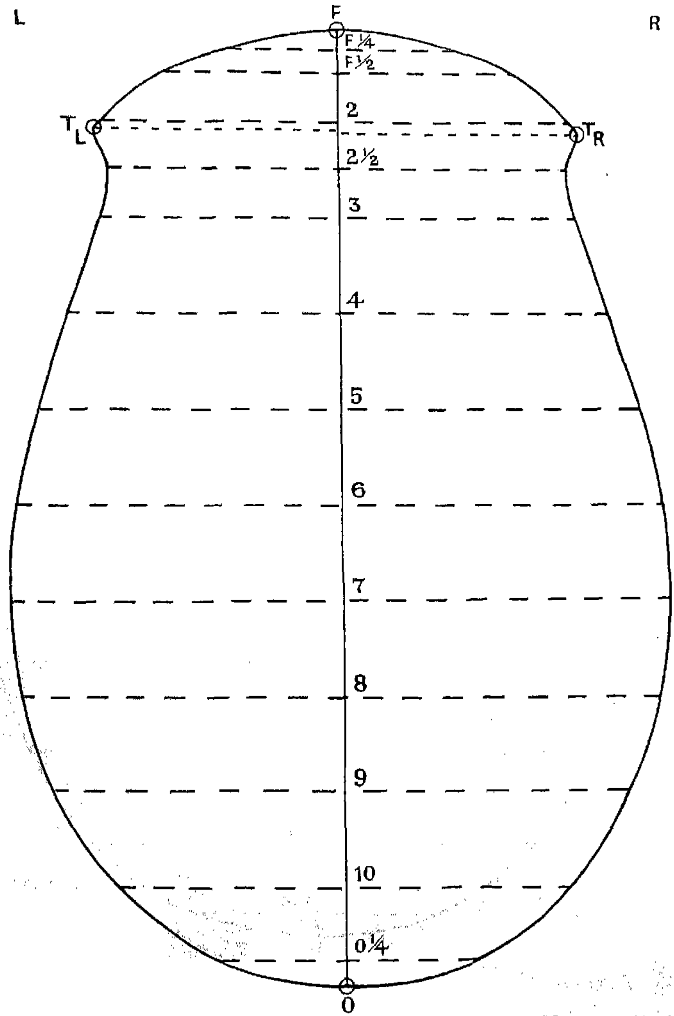


Fig. IV Horizontal Type Contour, based on 124 ♂ Kerma Skulls.

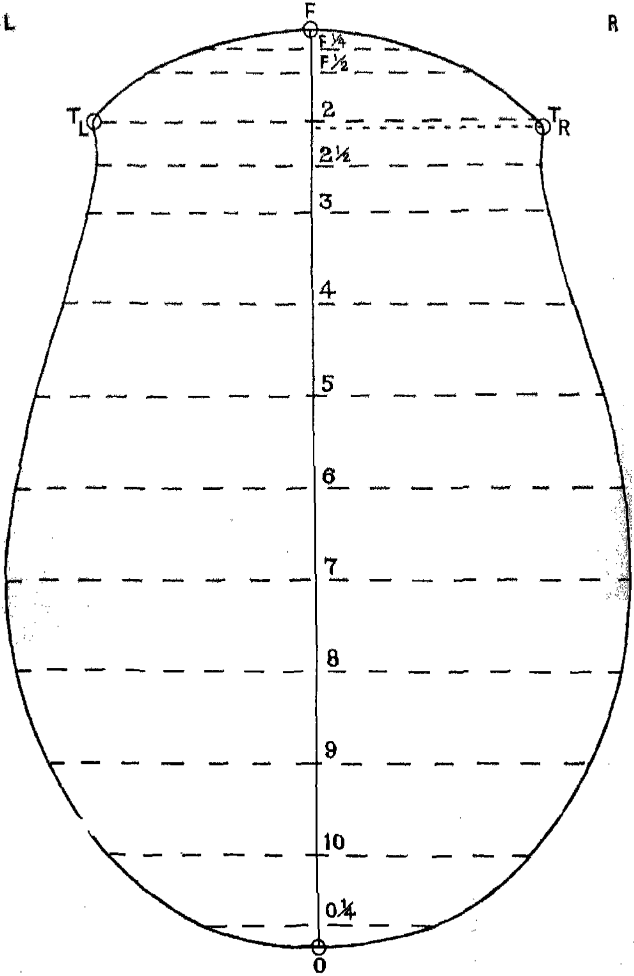


Fig. V Horizontal Type Contour, based on 90 ♀ Kerma Skulls.

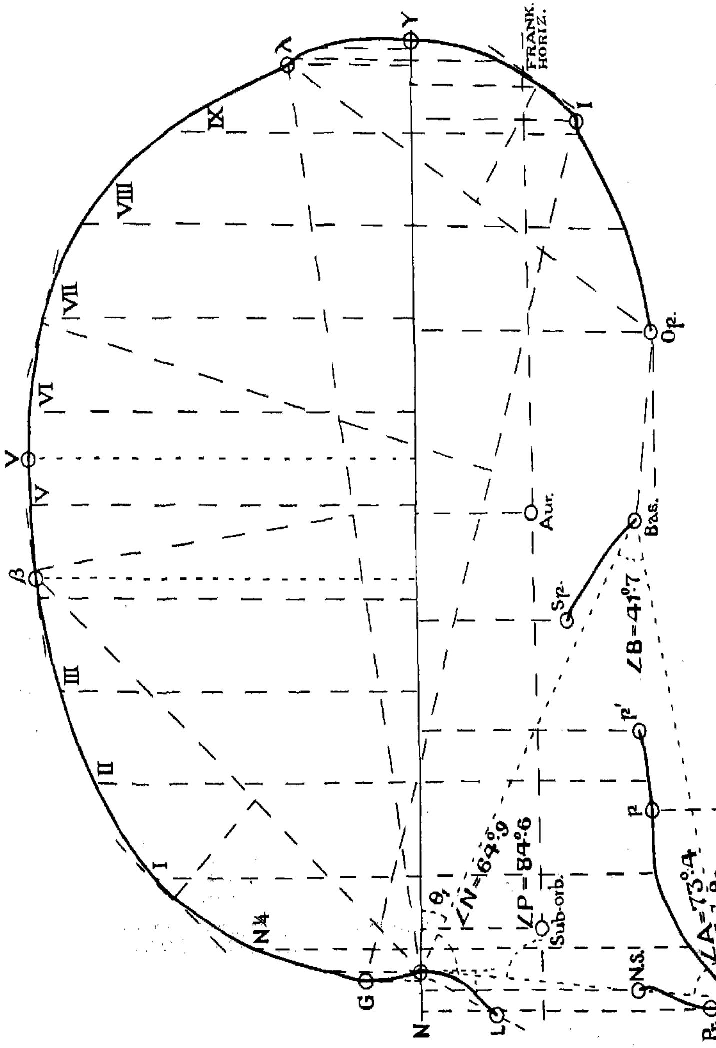


Fig. VII Sagittal Type Contour, based on 98 ♂ Kerma Skulls.

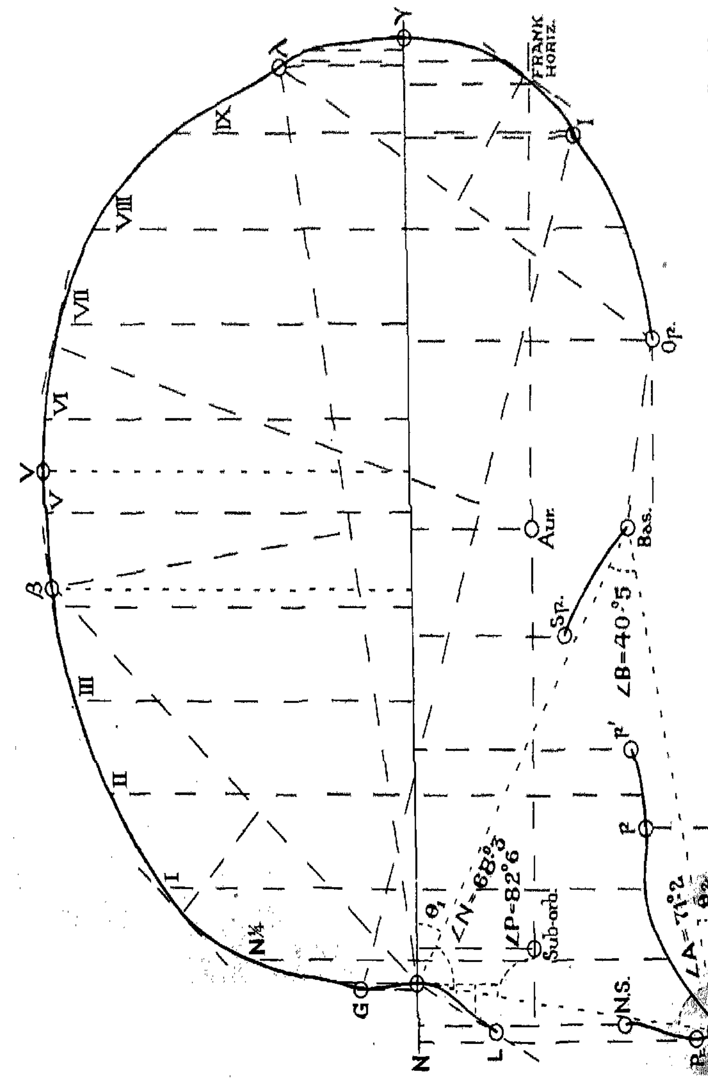


Fig. VII Sagittal Type Contour, based on 73 ♀ Kerma Skulls.

and angles derived from calliper measurements for the two series showed that the most marked differences in shape which would not have been anticipated were found for the angles of the fundamental triangle. Confirmation of the correctness of the means can be obtained from the sagittal type contours. The values are:

Sex	Method of Measurement	$N\angle$	$A\angle$	$B\angle$
♂	By callipers on skull Scaled on type contour	64°·9 (95) 64°·9 (46)	74°·1 (95) 73°·4 (46)	41°·0 (95) 41°·7 (46)
♀	By callipers on skull Scaled on type contour	67°·6 (63) 68°·3 (33)	72°·1 (63) 71°·2 (33)	40°·3 (63) 40°·5 (33)

By superposing the types it is found that the Kerma sagittal section bears a much closer resemblance to the Badari than to the Teita. Strangely enough, considering that one of these series represents a tribe of Bantu nègroes, the degrees of prognathism of these three are almost identical, but the section of the nasal bones projects considerably less in the Teita than in the Egyptian types.

(6) *Conclusions.* The series of skulls described in this paper represents the population of the Egyptian settlement at Kerma (Nubia) in the Twelfth and Thirteenth Dynasties. Only adults were dealt with and of these 141 appeared to be males and 114 females. No distinction can be made between the series representing the Twelfth and Thirteenth Dynasties respectively. The male and female series may be supposed to represent the same racial type. The male variabilities are rather greater than the female, as is generally found, and the population as a whole must be considered rather more variable than those in some Dynastic Eygyptian cemeteries, but still as homogeneous as most for which cranial samples are available. The type is distinctly negroid and the frequencies with which some anomalies are found assign it to an intermediate position between those of Bantu negroes on the one hand and European types on the other. The coefficients of racial likeness show that all the closest relationships of the Kerma series are with Upper Egyptian series of Predynastic and Early Dynastic date. The closest connections found are with Late Predynastic types and the Kerma sample may be supposed to represent the typical population of Upper Egypt at that time. It is distinctly further removed from the types contemporary with it found in Upper Egypt. Hence it is concluded that the settlement was probably founded in Late Predynastic times and that the racial type there persisted unchanged until the Thirteenth Dynasty although the parent population had been modified in the interval. Comparisons are also made with negro and Indian series, but no unexpectedly close connections are shown. The type contours are provided.

In conclusion I must thank Mr G. C. Dunning, some of whose measurements I have used, Miss M. Kirby, who drew the map and type contours, and Dr G. M. Morant, to whom I am greatly indebted for much help and encouragement.

DESCRIPTION OF PLATES OF KERMA SKULLS.

I. Typical male skull (No. 27), *Norma facialis* (0·9 natural size).

II. Typical male skull (No. 27), *Norma lateralis* (0·6 natural size).

III. (*a*) Typical male skull (No. 27), *Norma verticalis* (0·7 natural size).
This specimen has a cephalic index of 72·0, and the male mean for the series is 72·2.

(*b*) Male skull (No. 130; 0·7 natural size). This shows holes presumed to have been made by insects after death.

IV. (*a*) Male skull (No. 74; 0·7 natural size), with a depressed wound on the parietal and temporal bones and a healed fracture of the malar bone.

(*b*) Female skull (No. 153; 1·2 natural size), with large Wormian bones between the temporal squama and parietal bone.

V. (*a*) Female skull (No. 274; 0·7 natural size), showing diseased areas on either side of the lambda.

(*b*) Male skull (No. 36; 1·4 natural size), showing constriction of the basi-occipital where it unites with the sphenoid.

(*c*) Male skull (No. 64; 0·8 natural size), showing a wound below the right orbit.

(*d*) Female skull (No. 216; 1·3 natural size), showing failure of the tympanic elements to unite.

VI. (*a*) Male skull (No. 92; 1·2 natural size), with the sockets of two teeth other than molars lacking and a diseased opening on the left-hand side.

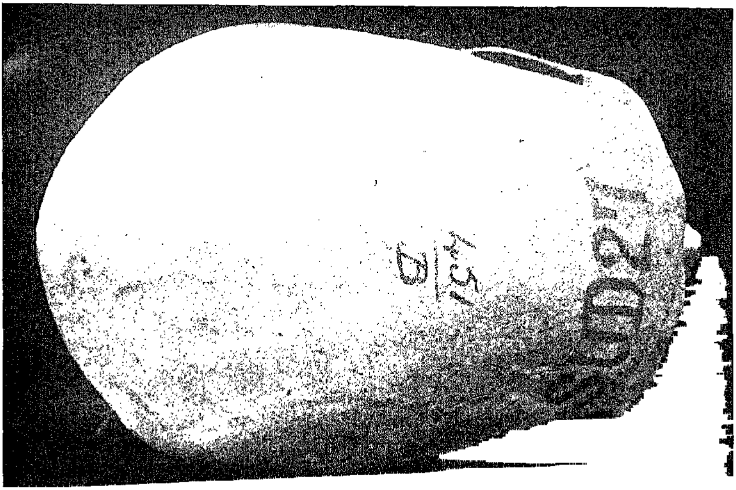
(*b*) Female skull (No. 216; 1·2 natural size), with no sockets for lateral incisors.

(*c*) Female skull (No. 155; 1·6 natural size), with no sockets for the left canine and right third molar.

(*d*) Male skull (No. 107; 1·5 natural size), with sockets for three incisors only, the socket for the central incisor being apparently between the pre-maxillary bones.

Serial No.	*GL*	*GB*	*J*	*NH'*	*NH, R*	*NH, L*	*NB*
1	—	94·7	130·5	51·7	52·6	51·6	24·4
2	96·4	97·8	134	52·0	51·8	51·5	24·2
3	—	—	—	—	—	—	—
4	—	90·0	—	52·3	52·0	51·9	23·4
5	92·9	100·2	—	47·0	47·1	46·8	25·6
6	98·0	—	128	50·8	52·2	51·4	25·2
7	86·6	98·4	—	48·0	46·6	47·3	22·8
8	100·1	94·0	135·5	45·7	47·8	45·4	27·0
9	—	87·3	—	—		—	—
10	—	95·4	128·5	49·0	48·8	47·9	25·8
11	—	94·0	127·5	50·6	52·8	50·9	25·8
13	105·5	101·9	135	50·2	49·6	50·2	27·7
14	95·9	91·0	123	45·1	45·4	45·9	24·3
15	98·1	93·9	—	50·2	50·3	50·1	22·5
16	94·5	96·5	129	46·6	45·8	46·6	24·4
17	—	—	—	—	—	—	—
18	—	—	—	—	—	—	—
19	—	—	—	—	—	—	—
20	103·4	105·5	133	48·7	50·0	49·4	27·2
21	92·5	94·7	122	49·0	49·5	49·0	26·8
22	—	—	—	—	—	—	—
23	104·3	98·4	133·5	56·2	56·0	56·7	24·7
24	—	—	—	—	—	—	—
25	99·2	101·9	—	54·6	54·3	55·0	28·0
26	92·0	92·0	129	46·0	47·5	47·5	27·5
27	96·5	103·1	131·5	54·2	54·0	54·3	26·5
28	95·0	92·0	—	55·5	56·5	56·8	24·0
29	100·0	100·7	129	52·9	51·8	52·8	24·9
30	97·0	96·0	132	49·6	49·2	50·0	24·5
31	—	90·5	—	—	—	—	25·2
32	—	—	121	—	—	—	—
33	104·6	—	—	—	49·0	49·0	—
34	99·3	90·5	128	52·2	51·8	52·4	26·4
35	101·4	100·5	134	50·0	49·2	50·0	27·8
36	89·9	103·1	133	48·7	48·7	48·4	24·5
37	104·3	94·0	—	46·9	46·7	46·6	26·1
38	102·3	103·3	—	55·2	54·4	54·2	28·2
39	91·5	88·5	124·5	49·5	48·6	49·5	27·9
40	—	—	—	—	—	—	—
41	90·0	94·0	133	59·9	59·7	60·0	28·0
43	96·2	90·5	130	50·5	51·8	52·4	26·1
44	92·0	94·0	—	—	—	—	—
45	94·3	90·9	131	52·6	53·4	53·4	25·8
46	100·5	97·7	117	50·3	49·0	50·1	26·9
47	101·4	98·2	120	52·5	51·0	52·0	26·0
48	102·5	98·0	127	51·4	50·2	51·4	27·4
49	90·5	94·7	126	52·4	54·2	52·8	28·1
50	106·8	106·2	135	53·1	52·5	52·4	27·1
51	—	—	—	—	—	—	—
52	—	93·0	128	—	—	—	—
53	100·3	96·1	—	48·5	49·2	49·5	25·9
54	97·6	98·6	133	52·5	52·5	51·9	25·6
55	99·8	94·0	129	50·0	49·9	49·6	26·1
56	93·7	92·5	124 ?	51·3	52·0	51·6	23·9
57	98·5	93·6	127·5	45·0	45·9	44·0	22·8
58	91·3	—	—	48·9	49·7	49·0	27·0
59	99·4	102·4	141·5	54·5	54·1	54·6	27·4
60	95·4	98·5	124	51·6	51·5	51·9	26·0
61	101·0	—	—	54·3	55·0	56·2	29·3
62	95·7	95·6	126	48·5	49·8	49·3	25·9
63	99·5	90·5	127·5	55·3	53·9	55·9	24·6
64	91·0	—	—	46·1	46·1	47·5	28·4
65	95·7	92·6	130	48·2	48·6	48·8	23·0
66	—	89·4	125·5	44·6	45·8	45·8	24·0
67	91·8	92·8	—	48·2	47·7	47·4	25·5
68	—	—	—	—	—	—	—
69	101·1	93·7	131	48·4	47·9	48·6	27·7
70	98·7	95·1	134	46·6	46·3	46·3	23·5
71	91·2	92·2	124	48·5	48·7	48·6	27·0
72	94·1	—	—	51·4	51·7	52·2	30·4
73	—	88·0	130	50·3	49·6	50·0	24·7

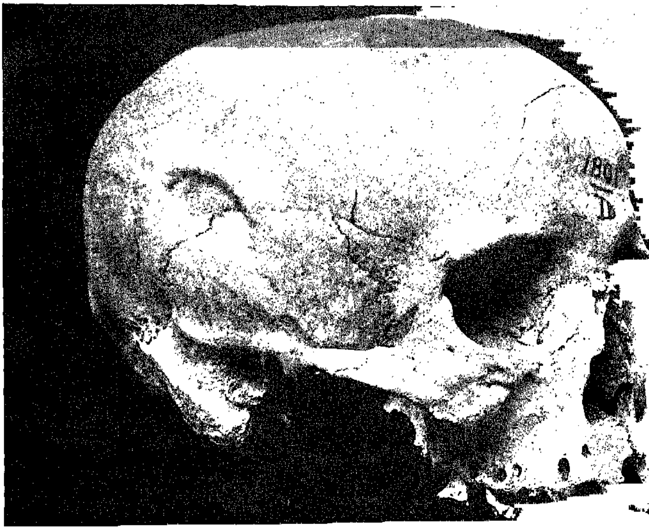
Typical Male Kerma Skull (No. 27).



(*a*) Typical male skull.

(*b*) Male skull showing holes presumed to have been made by insects after death.

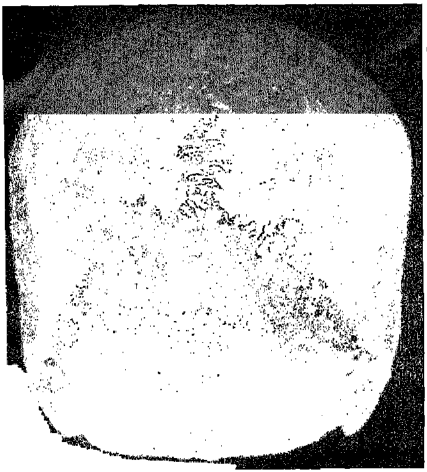
Male Kerma Skulls. *Normae verticales.*



(*a*) Male skull (No. 74), with wounds on the parietal and malar bones.

(*b*) Female skull (No. 153), with large Wormian bones between the temporal squama and parietal bone.

Anomalous Kerma Skulls



(*a*) Female skull (No. 274), showing diseased areas on either side of the lambda.

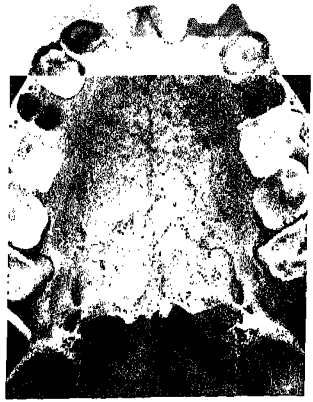
(*b*) Male skull (No. 36), showing constriction of the basi-occipital where it unites with the sphenoid.

(*c*) Male skull (No. 64), showing a wound below the orbit.

(*d*) Female skull (No. 216), showing failure of the tympanic elements to unit.

Anomalous Kerma Skulls.

(*a*) Male skull (No. 92), with the sockets of two teeth other than molars lacking.



(*b*) Female skull (No. 216), with no sockets for lateral incisors.

(c) Female skull (No. 155), with no sockets for the left canine and right third molar.

(*d*) Male skull (No. 107), with sockets for three incisors only.

Anomalous Palates of Kerma Skulls.

ON THE LIKELIHOOD THAT ONE UNKNOWN PROBABILITY EXCEEDS ANOTHER IN VIEW OF THE EVIDENCE OF TWO SAMPLES.

BY WILLIAM R. THOMPSON. From the Department of Pathology, Yale University.

Section 1.

IN elaborating the relations of the present communication interest was not centred upon the interpretation of particular data, but grew out of a general interest in problems of research planning. From this point of view there can be no objection to the use of data, however meagre, as a guide to action required before more can be collected; although serious objection can otherwise be raised to argument based upon a small number of observations. Indeed, the fact that such objection can never be eliminated entirely—no matter how great the number of observations—suggested the possible value of seeking other modes of operation than that of taking a large number of observations before analysis or any attempt to direct our course. This problem is more general than that treated in *Section* 2, and is directly concerned with any case where probability criteria may be established by means of which we judge whether one mode of operation is *better* than another in some given sense or not.

Thus, if, in this sense, P is the probability estimate that one *treatment* of a certain class of individuals is *better* than a second, as judged by data at present available, then we might take some monotone increasing function of P, say $f_{(P)}$, to fix the fraction of such individuals to be treated in the *first manner*, until more evidence may be utilised, where $0 \leqslant f_{(P)} \leqslant 1$; the remaining fraction of such individuals $(1 - f_{(P)})$ to be treated in the *second manner*; or we may establish a probability of treatment by the two methods of $f_{(P)}$ and $1 - f_{(P)}$, respectively. If such a discipline were adopted, even though it were not the best possible, it seems apparent that a considerable saving of individuals otherwise sacrificed to the inferior treatment might be effected. This would be important in cases where either the rate of accumulation of data is slow or the individuals treated are valuable, or both.

If we arbitrarily decide to eliminate the second treatment in favour of the first at this time, then the expectation of sacrifice to the inferior treatment would be $(1 - P)$ for *all* subsequently treated individuals; whereas, if, for example, we take $f_{(P)} = P$, the expectation of such sacrifice would be *temporarily*

$$P(1 - P) + (1 - P)P = 2PQ,$$

where $Q = 1 - P$. Obviously, $2PQ \leqslant \frac{1}{2}$ and, if $P \neq \frac{1}{2}$, then $2PQ < \frac{1}{2}$; whence a saving is made in contrast to the so-called *alternate case method*. In the *long run*, if a real preference exists between the two *treatments*, the expected saving by continued application of this method of apportionment rather than by making immediate final decision is sensibly $1 - P$ of individuals subsequently treated.

Obviously, if we are to operate in this manner, we need methods of evaluation of P for *small* as well as *large* numbers of observations. In the latter case many approximate methods are available in all fields although bounds to approximation have not been considered generally.

In *Section* 2 a sampling problem is treated, which is equivalent to a special case, where we are to judge between two rival treatments upon the basis of the probability of occurrence of a given critical event following such treatment. These probabilities are assumed unknown, but denoted by $\bar{p}_1$ and $\bar{p}_2$; and it is assumed that, independently for each of these, *a priori* $\bar{p}_i$ is equally likely to lie in either of any two equal intervals in its possible range, (0, 1). Our available experience consists solely of the data:

Of n_1 individuals treated by the first method, r_1 experienced the critical event and s_1 did not; and of n_2 treated by the second, r_2 and s_2 were the corresponding numbers with respect to the critical event.

In any given case it must be decided whether these requirements are met or not, and whether we may apply the well-known Principle of Bayes to convert the problem to the form of *Section* 2. Statistical criteria are often employed, however, in situations in which certain deviations from the conditions required in their development can be tolerated, when a better procedure is not available.

Section 2.

Consider the case of two infinite populations for which the unknown probabilities of occurrence of a given critical event are $\bar{p}_1$ and $\bar{p}_2$, and the probability of obtaining a sample containing r critical occurrences and s failures in $n = r + s$ trials is $\binom{n}{r} \cdot \bar{p}_i^r (1 - \bar{p}_i)^s$, where $i = 1, 2$, respectively. Furthermore, assume that one sample has been drawn at random from each population, the respective values of r and s being denoted by r_i and s_i (where $i = 1, 2$) and $n_i = r_i + s_i$; and that *independently* for $i = 1$ or 2 the probability that $\bar{p}_i$ lies in the interval $(p, p + dp)$ is $P^{(i)}_{p, p+dp}$, where

$$(1) \qquad P^{(i)}_{p,p+dp} = \frac{\int_p^{p+dp} \binom{n}{r} \cdot p^r \cdot q^s \cdot dp}{\int_0^1 \binom{n}{r} \cdot p^r \cdot q^s \cdot dp},$$

where $q = 1 - p$, $r = r_i$, $s = s_i$, and $n = r + s$. Then

$$(2) \qquad P^{(i)}_{p,1} = \frac{(n+1)!}{r! \cdot s!} \int_p^1 p^r \cdot q^s \cdot dp = \sum_{\alpha=0}^{r} \binom{n+1}{\alpha} \cdot p^\alpha \cdot q^{n+1-\alpha};$$

the last expression having been indicated by K. Pearson* in this relation. In the notation employed by him and by Müller† we may write

$$(3) \qquad 1 - P^{(i)}_{p,1} = P^{(i)}_{0,p} = I_p(u, v) = \frac{B_p(u, v)}{B_1(u, v)} = 1 - I_q(v, u),$$

where $u = r_i + 1$ and $v = s_i + 1$. The object of the present communication is to give a reduced‡ rational algebraic evaluation of the probability ($P_{\tilde{p}_2 > \tilde{p}_1}$) that for the postulated systems $\tilde{p}_2$ exceed $\tilde{p}_1$, and to indicate certain relations between its value (later designated by $\psi_{(r_1, s_1, r_2, s_2)}$) and the sum of the first $r_2 + 1$ terms of a *hypergeometric series* which has appeared in the work of K. Pearson§‖ as well as in the *Incomplete B-* and *I-functions**† of (3).

Obviously, we may write

$$(4) \qquad P_{\tilde{p}_2 > \tilde{p}_1} = \frac{(n_1+1)!}{r_1!\, s_1!} \int_0^1 p_1^{r_1} . q_1^{s_1} . \frac{(n_2+1)!}{r_2!\, s_2!} \int_{p_1}^1 p_2^{r_2} . q_2^{s_2} . dp_2 . dp_1 \qquad \text{(i)}$$

$$= \frac{(n_1+1)!}{r_1!\, s_1!} \int_0^1 \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} . p^{r_1+\alpha} . q^{s_1+n_2+1-\alpha} . dp \qquad \text{(ii)}$$

$$= \frac{(n_1+1)!}{r_1!\, s_1!} . \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} . \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{(n_1+n_2+2)!} \qquad \text{(iii)}$$

$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!} . \sum_{\alpha=0}^{r_2} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{r_1!\,\alpha!\,.\,s_1!\,(n_2+1-\alpha)!} \qquad \text{(iv)}$$

$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!} . \sum_{\alpha=0}^{r_2} \frac{(r_1+r_2-\alpha)!\,(s_1+s_2+1+\alpha)!}{r_1!\,(r_2-\alpha)!\,s_1!\,(s_2+1+\alpha)!}, \qquad \text{(v)}$$

whence we have

$$(5) \qquad P_{\tilde{p}_2 > \tilde{p}_1} = \frac{\sum_{\alpha=0}^{r_2} \binom{r_1+r_2-\alpha}{r_1} . \binom{s_1+s_2+1+\alpha}{s_1}}{\binom{n_1+n_2+2}{n_1+1}},$$

where, of course, $n_i = r_i + s_i$.

Now, it is obvious that $P_{\tilde{p}_2 > \tilde{p}_1} = P_{\tilde{p}_1 < \tilde{p}_2} = P_{\tilde{q}_1 > \tilde{q}_2} = P_{\tilde{q}_2 < \tilde{q}_1}$, where $\tilde{q}_i = 1 - \tilde{p}_i$, and thus in similar manner we have

$$(6) \qquad P_{\tilde{p}_2 > \tilde{p}_1} = \frac{\sum_{\alpha=0}^{s_1} \binom{s_1+s_2-\alpha}{s_2} . \binom{r_1+r_2+1+\alpha}{r_2}}{\binom{n_1+n_2+2}{n_1+1}}.$$

Furthermore, $P_{\tilde{p}_2 > \tilde{p}_1} = 1 - P_{\tilde{p}_1 > \tilde{p}_2}$, as the probability that $\tilde{p}_1$ is exactly equal to $\tilde{p}_2$ is zero by hypothesis. Hence we have two other similar sums which may be used with this difference relation to evaluate the probability under consideration.

* Pearson, Karl: *Biometrika*, Vol. XVI (1924), pp. 202—203.

† Müller, J. H.: *Biometrika*, Vol. XXII (1930—31), pp. 284—297.

‡ The earliest work directed to this end is discussed by Todhunter. Cf. *A History of the Mathematical Theory of Probabilty*. Cambridge and London (1865), pp. 419—420.

§ Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

‖ Pearson, Karl: *Biometrika*, Vol. XX^A (1928), pp. 149—174.

In actual evaluation we may make use of the well-known *pyramid form* of tabulation of the binomial coefficients,

$$
(7)\qquad
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

which is readily constructed by the property that each entry except those on the boundaries (which are always unity) is the sum of the two nearest entries of the row next above*. The corresponding factors of the successive terms in the sum to be evaluated may be found in order on given diagonals, ascending in the case of the first and descending in the case of the second factor. The bounding diagonals of units may be deleted in practice (as may all entries to the right of the middle column). Then the first factor of the first term of the sum in (5) is the r_1-th entry in the row whose first entry (at the left) is $r_1 + r_2$, and the corresponding factors of successive terms lie successively above on the diagonal through the first and parallel to the left-hand boundary. Similarly, the other factors are found, but proceeding in the opposite direction on the appropriate diagonal.

Now, for any positive number, x, let $Q_{(x)} = x^{x+\frac{1}{2}}$. Then, by Stirling's formula, we have

$$(8)\qquad m\,! = Q_{(m)} \cdot \sqrt{2\pi} \cdot e^{-m} \cdot e^{\frac{\theta}{12m}},$$

where $0 \leqslant \theta \leqslant 1$; whence (4)(v) gives

$$(9)\qquad P_{p_2>p_1} = \frac{Q_{(n_1+1)} \cdot e^{w}}{Q_{(n_1+n_2+2)} \cdot Q_{(r_1)} \cdot Q_{(s_1)}} \cdot \sum_{\alpha=0}^{r_2} \binom{n_2+1}{r_2-\alpha} Q_{(r_1+r_2-\alpha)} \cdot Q_{(s_1+s_2+1+\alpha)},$$

where
$$|12w| < \frac{1}{r_1} + \frac{1}{s_1} + \frac{1}{n_1+1};$$

which may be used to advantage in approximation of $P_{p_2>p_1}$ when r_1 and s_1 are *large* and r_2 *small*.

Now we may define $\psi_{(r_1, s_1, r_2, s_2)}$ for any four rational integers (not negative) as identical with the right member of (5), where $n_i = r_i + s_i > 0$. Then we have shown several equivalent expressions of the same function and that

$$(10)\qquad \psi_{(r_1, s_1, r_2, s_2)} = \psi_{(s_2, r_2, s_1, r_1)} = 1 - \psi_{(s_1, r_1, s_2, r_2)}.$$

* Glaisher, J. W. L.: "A Table of Binomial-Theorem Coefficients," *Messenger of Mathematics*, Vol. 47 (1917), pp. 97—107.

From the conditions stated it may be expected that if we set $p=\frac{r_1}{n_1}$ and $q=1-p$, then (provided $0<p<1$),

$$\text{(11)} \qquad \lim_{n_1\to\infty} \frac{\psi_{(r_1, s_1, r_2, s_2)}}{I_q(s_2+1, r_2+1)} = 1.$$

That this is true may be verified if we exclude the cases, $p=0, 1$. Further bounds to approximation of this limit by the ratio ($\bar{R}$) of these functions for given values of n_1 may be found as follows:

By (4)(iv) we may write

$$\text{(12)} \qquad \psi_{(r_1, s_1, r_2, s_2)} = \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} \cdot \frac{\dfrac{(r_1+\alpha)!}{r_1!} \cdot \dfrac{(s_1+n_2+1-\alpha)!}{s_1!}}{\dfrac{(n_1+n_2+2)!}{(n_1+1)!}},$$

and by (2) and (3), introducing the appropriate values of $r=r_2$ and $s=s_2$, we have

$$\text{(13)} \qquad I_q(s_2+1, r_2+1) = \sum_{\alpha=0}^{r_2} \binom{n_2+1}{\alpha} \cdot p^{\alpha} \cdot q^{n_2+1-\alpha},$$

where $p=\frac{r_1}{n_1}$ and $q=\frac{s_1}{n_1}$. Obviously, therefore, as all terms of both sums in (12) and (13) are positive, if we exclude the special cases where p or $q=0$, $\bar{R}$ is bounded by the greatest and least values attainable for the ratio of a term in the sum of (12) to the corresponding term in (13). Thus we may define

$$\text{(14)} \qquad \omega_1 = \text{Min.}\left[\left(\frac{r_1+1}{r_1}\right)^{\alpha}\left(\frac{s_1+1}{s_1}\right)^{n_2+1-\alpha}\left(\frac{n_1}{n_1+n_2+2}\right)^{n_2+1}\right]$$

and

$$\omega_2 = \text{Max.}\left[\left(\frac{r_1+\alpha}{r_1}\right)^{\alpha}\left(\frac{s_1+n_2+1-\alpha}{s_1}\right)^{n_2+1-\alpha}\left(\frac{n_1}{n_1+2}\right)^{n_2+1}\right]$$

for $0 \leqslant \alpha \leqslant r_2$; and, obviously, then

$$\text{(15)} \qquad \omega_1 < \bar{R} < \omega_2, \text{ and } \lim_{n_1\to\infty} [\bar{R}] = 1.$$

In the excluded cases it is also readily verified that

$$\text{(16)} \qquad \lim_{n_1\to\infty} [\psi_{(r_1, s_1, r_2, s_2)}] = I_q(s_2+1, r_2+1).$$

The relation of this function to the sum of a given number of consecutive terms of a *hypergeometric series* is particularly interesting in view of the occurrence of such series in the investigations of K. Pearson *†. By (4)(iv) we may write:

$$\text{(17)} \qquad \psi_{(r_1, s_1, r_2, s_2)} = \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!} \cdot \sum_{\alpha=0}^{r_2} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!},$$

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

† Pearson, Karl: *Biometrika*, Vol. xx[A] (1928), pp. 149—174.

which is the sum of the first r_2+1 terms of a *hypergeometric series* multiplied by a constant. Similarly, we may write

$$(18)\quad \psi_{(s_1, r_1, s_2, r_2)} = \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!} \cdot \sum_{\alpha=0}^{s_2} \frac{(s_1+\alpha)!\,(r_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}$$
$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!} \cdot \sum_{\alpha=0}^{s_2} \frac{(r_1+r_2+1+\alpha)!\,(s_1+s_2-\alpha)!}{(r_2+1+\alpha)!\,(s_2-\alpha)!}$$
$$= \frac{(n_1+1)!\,(n_2+1)!}{(n_1+n_2+2)!\,r_1!\,s_1!} \cdot \sum_{\alpha=r_2+1}^{n_2+1} \left[\frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}\right],$$

obviously (as $n_2 = r_2 + s_2$); and by previous demonstration

$$(19)\qquad \psi_{(r_1, s_1, r_2, s_2)} + \psi_{(s_1, r_1, s_2, r_2)} = 1,$$

whence

$$(20)\qquad \frac{(n_1+n_2+2)!\,r_1!\,s_1!}{(n_1+1)!\,(n_2+1)!} = \sum_{\alpha=0}^{n_2+1} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}.$$

The last relation, demonstrated above by independent proof, has been established previously by Pearson*† (with different notation). Thus we may regard $\psi_{(r_1, s_1, r_2, s_2)}$ as defined by the identity

$$(21)\qquad \psi_{(r_1, s_1, r_2, s_2)} \equiv \frac{\displaystyle\sum_{\alpha=0}^{r_2} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}}{\displaystyle\sum_{\alpha=0}^{n_2+1} \frac{(r_1+\alpha)!\,(s_1+n_2+1-\alpha)!}{\alpha!\,(n_2+1-\alpha)!}},$$

extending the domain of definition to include the value, $r_2 = n_2 + 1$; but retaining the restrictions, $n_i = r_i + s_i \geqslant 0$, and that -1 be the least value of r_1, s_1, r_2, and s_2 (only one of which shall be admitted to be negative). Then, by this extension, we have

$$(22)\qquad \psi_{(r_1, s_1, 0, 0)} \equiv \frac{s_1+1}{r_1+s_1+2}, \text{ and } \psi_{(r_1, s_1, n_2+1, -1)} \equiv 1,$$

which lie outside the domain of the initial discussion, and we extend to the new domain the relation of (10) *formally*; i.e.

$$(23)\qquad \psi_{(r, s, r', s')} \equiv \psi_{(s', r', s, r)} \equiv 1 - \psi_{(s, r, s', r')}.$$

K. Pearson† has considered the problem of likelihood of various values of R and S, the number of *marked* members and *unmarked* members, respectively, in a finite universe of aggregate number, $N = R + S$; assuming N fixed and all values of $R, S \geqslant 0$ equally likely *a priori* and that our sole experience from which judgment is to be made is that a *random sample* has been drawn containing exactly r marked and s unmarked members (R and S being used here in place of Pearson's p and q to avoid confusion). Then by (iii) and (iv) of the article† just mentioned, we have a means of evaluating the probability, $\bar{P}_R$, that the universe contains no more than R marked members by the relation,

$$(24)\qquad \bar{P}_R \equiv \psi_{(r, s, R-r, N-R-s-1)},$$

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

† Pearson, Karl: *Biometrika*, Vol. xx^A (1928), pp. 149—174.

which may be verified readily. Similarly, in the case of the problem considered earlier by Pearson*—having drawn one random sample from a certain infinite population, the sample containing exactly r' *marked* and s' *unmarked* members, we are required to find the probability (under the given conditions) that if we draw another random sample of n'' individuals from the same population it will contain no more than r'' marked members—the required value is given for $r'' \leqslant n''$ by

$$\psi_{(r', s', r'', n''-r''-1)}.$$

In the tabulation of values of $\psi_{(r,s,r',s')}$ for ascending values of the arguments the work may be greatly simplified by certain relations in addition to those given in (22) and (23) in much the same manner as the binomial coefficients may be tabulated† by mere summation of two values already given. To this end let us examine two functions defined by

$$N_{(r,s,r',s')} \equiv \sum_{\alpha=0}^{r'} \binom{r+r'-\alpha}{r} \binom{s+s'+1+\alpha}{s} \tag{25}$$

and $\quad D_{(r,s,r',s')} \equiv D_{(n,n')} \equiv \binom{n+n'+2}{n+1}$ for $n = r+s$, and $n' = r'+s'$.

Obviously, by the original definition of $\psi_{(r,s,r',s')}$, extended in (21), (22) and (23), then

$$\psi_{(r,s,r',s')} \equiv \frac{N_{(r,s,r',s')}}{D_{(r,s,r',s')}}, \tag{26}$$

where we extend the definition of (25) for $r, s, r', s' \geqslant 0$ by

$$N_{(r,s,-1,s')} \equiv 0, \text{ and } N_{(r,s,r',-1)} \equiv D_{(r+s,r'-1)}, \tag{27}$$

and by (26) and (23) we have

$$N_{(r,s,r',s')} \equiv N_{(s',r',s,r)} \equiv D_{(r+s,r'+s')} - N_{(s,r,s',r')}, \tag{28}$$

as it is obvious that $D_{(n,n')} \equiv D_{(n',n)}$. Furthermore, by the well-known relation,

$$\binom{a}{b} \equiv \binom{a-1}{b-1} + \binom{a-1}{b}, \tag{29}$$

$$D_{(n,n')} \equiv D_{(n,n'-1)} + D_{(n-1,n')},$$

we have for $s > 0$ in (25)

$$N_{(r,s,r',s')} \equiv \sum_{\alpha=0}^{r'} \binom{r+r'-\alpha}{r} \left[\binom{s+s'+\alpha}{s-1} + \binom{s+s'+\alpha}{s} \right], \tag{30}$$

whence we have in any case under its definition, obviously, by (20), (27) and (28), the identities

$$\begin{aligned} N_{(r,s,r',s')} &\equiv N_{(r,s-1,r',s')} + N_{(r,s,r',s'-1)} \\ &\equiv N_{(r-1,s,r',s')} + N_{(r,s,r'-1,s')}. \end{aligned} \tag{31}$$

By (25) and (29), obviously, the same relation holds for the D-function; and we may write

$$\psi_{(r,s,r',s')} \equiv \frac{N_{(r,s-1,r',s')} + N_{(r,s,r',s'-1)}}{D_{(r,s-1,r',s')} + D_{(r,s,r',s'-1)}}. \tag{32}$$

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

† Glaisher, J. W. L.: "A Table of Binomial-Theorem Coefficients," *Messenger of Mathematics*, Vol. 47 (1917), pp. 97—107.

By means of these relations it is evident that it suffices to tabulate the N and the D-functions by additions of corresponding pairs of values already listed or readily obtained by the relations of (28) if we proceed from the lowest values of the arguments upward; and we need list values only for the cases where $r+s \geqslant r'+s'$ and $r \geqslant s$ where all four variables may be restricted to positive values, as by (25) we have

$$N_{(r,s,0,s')} \equiv D_{(s-1,s')}. \tag{33}$$

By (25) we may write

$$N_{(r,s,r',s')} \equiv \sum_{a=0}^{r'} D_{(r-1,r'-1-a)} \cdot D_{(s-1,s'+a)}, \tag{34}$$

which is of value if $D_{(n,n')}$ is tabulated for $n \geqslant n'$ through ascending values of n and n'. This may be done rapidly by means of (29) and the relation $D_{(n,n')} \equiv D_{(n',n)}$ and is more convenient than the pyramid form of the corresponding binomial coefficients. In the tabulation we may restrict n and n' to the positive integers, employing the relation $D_{(n,0)} \equiv n+2$. A short table of the N and D functions is appended as an illustration, the required probability being the ratio of these corresponding values.

Section 3.

If a system of operation such as suggested in *Section* 1 were adopted extensively for the case considered in *Section* 2, reference to values of $\psi_{(r,s,r',s')}$ for small values of the arguments should be required frequently; and, accordingly, a simple method of formation of a table of these would be valuable. The method given in *Section* 2 seems to serve this purpose; and, in conjunction with the relations of (28), many values need not be listed. The short table is given merely in illustration. It really deals completely with all cases of $n, n' \leqslant 5$ although certain cases are not listed where the values are readily obtainable from those given and (28). The several general evaluations at the head of the table would permit deletion of many more, e.g. any instance where one of the four arguments is zero; but they have been retained for illustrative purposes.

The function, $D_{(n,n')} \equiv \binom{n+n'+2}{n+1}$, is readily tabulated in a convenient form for increasing values of $n \geqslant n' \geqslant 1$, as has been mentioned under (34) above by adding to $n+2$ successively the values of $D_{(n-1,n')}$, already listed, and taking a sub-total after each addition, and finally doubling the last sub-total. These sub-totals and the final double are the required successive values of $D_{(n,n')}$. The value of such a table extends far beyond that of the immediate problem; and, by means of it and relation (34), $\psi_{(r,s,r',s')}$ may be calculated rapidly or approximated with any required precision for considerably higher values of the arguments than it may be convenient to have tabulated $N_{(r,s,r',s')}$. In accord with some prescribed tolerance and limited extent, a table of approximate values of the D-function could be made with greater ease, but apparently not readily extensible within the same relative tolerance without revision. Just how far these tables should extend would depend upon demands for their use. All questions as to approximate methods should be decided

Short Table of $N_{(r,s,r',s')}$ and $D_{(n,n')}$ ($n = r + s$, and $n' = r' + s'$)

r	s	r'	s'	N	D
a	b	c	d	$\bar{N}$	$\bar{D}$
b	a	d	c	$\bar{D}-\bar{N}$	$\bar{D}$
d	c	b	a	$\bar{N}$	$\bar{D}$
r	s	-1	s'	0	
r	0	0	0	1	$r+2$
r	0	0	s'	1	
r	0	1	s'	$r+2$	
r	s	0	0	$s+1$	$n+2$
r	s	0	s'	$D_{(s-1,s')}$	
0	0	-1	0	0	1
0	0	0	0	1	2
1	0	0	0	1	3
0	1	0	0	2	
1	0	0	1	1	6
1	0	1	0	3	
2	0	0	1	1	10
2	0	1	0	4	
1	1	0	1	3	
2	0	0	2	1	20
2	0	1	1	4	
3	0	0	1	1	15
3	0	1	0	5	
2	1	0	1	3	
2	1	1	0	9	
3	0	0	2	1	35
3	0	1	1	5	
3	0	2	0	15	
2	1	0	2	4	
2	1	1	1	13	
2	1	2	0	25	
3	0	0	3	1	70
3	0	1	2	5	
3	0	2	1	15	
2	1	1	2	17	
4	0	0	1	1	21
4	0	1	0	6	
3	1	0	1	3	
3	1	1	0	11	
2	2	0	1	6	
4	0	0	2	1	56
4	0	1	1	6	
4	0	2	0	21	
3	1	0	2	4	
3	1	1	1	16	
3	1	2	0	36	
2	2	0	2	10	
2	2	1	1	28	
4	0	0	3	1	126
4	0	1	2	6	
4	0	2	1	21	
4	0	3	0	56	
3	1	0	3	5	
3	1	1	2	21	
3	1	2	1	51	
3	1	3	0	91	
2	2	0	3	15	
2	2	1	2	45	
4	0	0	4	1	252
4	0	1	3	6	
4	0	2	2	21	
4	0	3	1	56	
3	1	1	3	26	
3	1	2	2	66	
5	0	0	1	1	28
5	0	1	0	7	
4	1	0	1	3	
4	1	1	0	13	
3	2	0	1	6	
3	2	1	0	18	
5	0	0	2	1	84
5	0	1	1	7	
5	0	2	0	28	
4	1	0	2	4	
4	1	1	1	19	
4	1	2	0	49	
3	2	0	2	10	
3	2	1	1	34	
3	2	2	0	64	
5	0	0	3	1	210
5	0	1	2	7	
5	0	2	1	28	
5	0	3	0	84	
4	1	0	3	5	
4	1	1	2	25	
4	1	2	1	70	
4	1	3	0	140	
3	2	0	3	15	
3	2	1	2	55	
3	2	2	1	115	
3	2	3	0	175	
5	0	0	4	1	462
5	0	1	3	7	
5	0	2	2	28	
5	0	3	1	84	
5	0	4	0	210	
4	1	0	4	6	
4	1	1	3	31	
4	1	2	2	91	
4	1	3	1	196	
4	1	4	0	336	
3	2	0	4	21	
3	2	1	3	81	
3	2	2	2	181	
3	2	3	1	301	
3	2	4	0	406	
5	0	0	5	1	924
5	0	1	4	7	
5	0	2	3	28	
5	0	3	2	84	
5	0	4	1	210	
4	1	1	4	37	
4	1	2	3	112	
4	1	3	2	252	
3	2	2	3	262	

by several statisticians in consultation at a time when a definite programme for the use of these methods is formed.

In (24) and the paragraph in which it stands is given the relation between the hypergeometrical series studied by Pearson*† and $\psi_{(r,s,r',s')}$ by means of which it is obvious that any approximation methods valid for estimation of P_E in (24) are equally valid for the estimation of the corresponding ψ-function, several of which have been suggested by K. Pearson. The *I*-function of Pearson is related to the ψ-function by (11) to (15) also; and another approximation of $\psi_{(r_1,s_1,r_2,s_2)}$ is given in (9) with indicated domain of validity.

A further treatment of the ψ-function and the method of apportionment will be provided in a later paper.

* Pearson, Karl: *Philosophical Magazine*, Series 6, Vol. 13 (1907), pp. 365—378.

† Pearson, Karl: *Biometrika*, Vol. xx^A (1928), pp. 149—174.

ON ASYMPTOTIC FORMULAE FOR THE HYPERGEOMETRIC SERIES.

I. HYPERGEOMETRIC SERIES IN WHICH THE FOURTH ELEMENT, x, IS UNITY.

BY O. L. DAVIES, M.Sc.

The hypergeometric series $F(\alpha, \beta, \gamma, 1)$ arises frequently in problems of chance when samples are taken from a finite population. For instance, in a population of size M in which p individuals possess a certain character A, and $q\ (= M-p)$, not A, the chance of drawing a sample of size n in which there are r of A and $s\ (=n-r)$, not A, is clearly

$$\frac{n!}{r!\,s!}\,\frac{p}{M}\cdot\frac{p-1}{M-1}\cdots\frac{p-r+1}{M-r+1}\cdot\frac{q}{M-r}\cdot\frac{q-1}{M-r-1}\cdots\frac{q-s+1}{M-n+1}\quad\ldots\ldots(1).$$

Hence the distribution of r and s in repeated samples of size n is given by the terms of the series

$$N\cdot\frac{(M-n)!\,p!}{M!\,(p-n)!}\left[1+\frac{nq}{1!\,(p-n+1)}+\frac{n(n-1)\,q\,(q-1)}{2!\,(p-n+1)(p-n+2)}+\ldots\right]\ldots(2),$$

where N is the number of samples. If we write $n=-\alpha$, $q=-\beta$, $p+1-n=\gamma$, this series can be transformed into the usual hypergeometric form

$$N\frac{(\gamma-\beta-1)!\,(\gamma-\alpha-1)!}{(\gamma-\alpha-\beta-1)!\,(\gamma-1)!}\left[1+\frac{\alpha\beta}{1!\,\gamma}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}+\ldots\right]$$

$$=N\frac{(\gamma-\beta-1)!\,(\gamma-\alpha-1)!}{(\gamma-\alpha-\beta-1)!\,(\gamma-1)!}F(\alpha,\beta,\gamma,1)\ldots\ldots(3).$$

The chance of drawing in a sample of size n at least r individuals possessing character A is equal to the sum of the first $(r+1)$ terms of (2). This is what is meant by the *probability integral* of the series; its determination is of paramount importance.

It is well known that, when M is infinite, and if p and q now measure the chance of an individual being A or not A respectively, the distribution of r and s in repeated samples of size n is given by the terms of the expansion of the bino[mial] $N(p+q)^n$. Its probability integral has been expressed by an incomplete [β-] function*. When neither p nor q is very small and n fairly large, the distrib[ution] may be fitted very closely by a normal curve. If, however, either p or q is [very] small, the distribution may be represented by a Poisson series. This may also [be]

* Karl Pearson, *Biometrika*, Vol. XVI. pp. 202—3.

fitted quite closely by a normal curve if n be sufficiently large. The probability integrals of both these distributions, normal and Poisson, have been tabulated*.

Hypergeometric series of type $F(\alpha, \beta, \gamma, 1)$ may arise in at least two other ways†, namely:

(i) The proportional frequencies of drawing in successive samples of a fixed size n, r marked and $s\,(=n-r)$ unmarked individuals $(r=0, 1, 2, 3, \ldots, n)$, from a population having previously drawn a sample of size N with p marked and $q\,(=N-p)$ unmarked individuals, are given by the successive terms of the hypergeometric series

$$F(\alpha, \beta, \gamma, 1), \text{ where } \begin{cases} \alpha=-n, \ \beta=q+1, \\ \gamma=-(p+n), \ \epsilon=\gamma-\alpha-\beta-1=-(N+2). \end{cases}$$

(ii) A sample of size n with r marked and $s\,(=n-r)$ unmarked individuals is taken from a finite population of size N. The likelihoods of N having p marked $(p=0, 1, 2, \ldots, N)$ and $q\,(=N-p)$ unmarked individuals are given by the successive terms of the hypergeometric series

$$cF(\alpha, \beta, \gamma, 1),$$

where

$$\alpha=r+1, \ \beta=-(N-n), \ \gamma=-(N-r),$$
$$\epsilon=\gamma-\alpha-\beta-1=-(n+2),$$
$$F(\alpha, \beta, \gamma, 1)=c^{-1}=\frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)}.$$

An asymptotic expression for the remainder of the series $F(\alpha, \beta, \gamma, 1)$ after the $(s+1)$st term has been provided by M. J. M. Hill‡. It may be written in the form

$$R(\alpha, \beta, \gamma, s)=\frac{\Pi(s+1, \gamma-1)\,(s+1)^{\gamma+\beta-\gamma}f(\alpha, \beta, \gamma, s)}{\Pi(s+1, \alpha-1)\,\Pi(s+1, \beta-1)\,(\gamma-\alpha-\beta)},$$

where

$$f(\alpha, \beta, \gamma, s)=1+\frac{(\gamma-\alpha)(\gamma-\beta)}{(\gamma-\alpha-\beta+1)(\gamma+s+1)}$$
$$+\frac{(\gamma-\alpha)(\gamma-\alpha+1)(\gamma-\beta)(\gamma-\beta+1)}{(\gamma-\alpha-\beta+1)(\gamma-\alpha-\beta+2)(\gamma+s+1)(\gamma+s+2)}+\ldots.$$

I have applied this formula to a number of particular hypergeometric series and found that in most cases the series $f(\alpha, \beta, \gamma, s)$ did not converge with sufficient rapidity to be of practical use for finding the sum of a number of *significant* terms of the hypergeometric series. However, the formula was very useful for evaluating the extreme tail of the series.

Professor Burton H. Camp§ has introduced a method for the approximate determination of the tail of a frequency distribution, continuous or discrete. His

* *Tables for Statisticians and Biometricians*, Part I, Tables II and LI, LII.

† Karl Pearson, *Philosophical Magazine*, March, 1907, pp. 365—378; *Biometrika*, Vol. v. 1907, pp. 172—175, Vol. XIII. 1920, pp. 1—16, Vol. XX^A. 1928, pp. 149—174.

‡ *Proc. Lond. Math. Soc.* Ser. 2, Vol. 5.

§ *Biometrika*, Vol. XVI. p. 163.

formula has been applied* to the series $F(\alpha, \beta, \gamma, 1)$, but it fails to give good results when the "stump" lies within $\pm 2\sigma$ of the mode, σ being the standard deviation of the series whose discrete terms are treated as a frequency distribution.

In this paper we shall obtain approximations to the sum of a finite number of terms of the series $F(\alpha, \beta, \gamma, 1)$ by fitting to it a Pearson-type curve. The question will be investigated of how closely the probability integral of the series may be represented by the probability integral of a Pearson-type curve.

The series

$$F(\alpha, \beta, \gamma, 1) = 1 + \frac{\alpha\beta}{1!\,\gamma} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)} + \dots,$$

when infinite, is convergent as long as $\gamma > \alpha + \beta$. The sth moment of the terms of the series, represented by a histogram with grouping unit "c," about a point c before the midordinate of the first block is

$$\left(1 \cdot 1^s + \frac{\alpha\beta}{1!\,\gamma} 2^s + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)} 3^s + \dots\right) c^s.$$

Applying Raabe's test, the condition of convergence of this more general series is found to be $\gamma > \alpha + \beta + s$, or $\epsilon > s - 1$, where $\epsilon = \gamma - \alpha - \beta - 1$. We cannot, therefore, have an infinite hypergeometric series with all its moments finite.

Convenient expressions for the moment coefficients of the above series, finite or infinite, have been found by Professor Karl Pearson. They are

$$N = \text{sum of series} = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad \epsilon > 0,$$

$$= \frac{\Gamma(\alpha-\gamma+1)\,\Gamma(\beta-\gamma+1)}{\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(1-\gamma)} \qquad \epsilon < 0,$$

$$\mu_1' = c\frac{\alpha\beta}{\epsilon},$$

$$\mu_2 = \sigma^2 = \frac{c^2\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}{\epsilon^2(\epsilon-1)},$$

$$\mu_3 = \frac{c^3\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)(2\alpha+\epsilon)(2\beta+\epsilon)}{\epsilon^3(\epsilon-1)(\epsilon-2)},$$

$$\mu_4 = \frac{c^4\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}{\epsilon^4(\epsilon-1)(\epsilon-2)(\epsilon-3)}\left[\epsilon^3(1+\epsilon) + 6\epsilon^2\{\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)\} + 3(6+\epsilon)\alpha(\alpha+\epsilon)\beta(\beta+\epsilon)\right].$$

We will now test the goodness of fit of the Pearson curve $P(x)$ to $F(\alpha, \beta, \gamma, 1)$, where P and F have the same first four moments.

I*a*. *Infinite Hypergeometric Series.* α, β, γ *positive.*

Since we are to fit by the first four moments, we must initially assume $\epsilon > 3$ to ensure all expressions finite.

* *Biometrika*, Vol. XVII. p. 61.

The first two betas of a hypergeometric series in which x is unity, have the following form:

$$\beta_1 = \frac{\epsilon - 1}{(\epsilon - 2)^2} \frac{(2\alpha + \epsilon)^2 (2\beta + \epsilon)^2}{\alpha\beta (\alpha + \epsilon)(\beta + \epsilon)}$$

$$= \frac{\epsilon - 1}{(\epsilon - 2)^2}\left(4 + \frac{\epsilon^2}{\alpha(\alpha + \epsilon)}\right)\left(4 + \frac{\epsilon^2}{\beta(\beta + \epsilon)}\right),$$

$$\beta_2 = \frac{3(\epsilon - 1)(\epsilon + 6)}{(\epsilon - 2)(\epsilon - 3)} + \frac{\epsilon^2(\epsilon - 1)}{(\epsilon - 2)(\epsilon - 3)}\left[\frac{\epsilon(\epsilon + 1)}{\alpha(\alpha + \epsilon)\beta(\beta + \epsilon)} + 6\left\{\frac{1}{\alpha(\alpha + \epsilon)} + \frac{1}{\beta(\beta + \epsilon)}\right\}\right].$$

β_1 and β_2 are functions of the three parameters α, β and ϵ, being symmetrical with respect to α and β. As either α or β increases, both β_1 and β_2 decrease. When α and β are large with respect to ϵ, i.e., when γ approaches the value $(\alpha + \beta)$, β_1 and β_2 approach the simple expressions

$$\beta_1 = 16 \frac{\epsilon - 1}{(\epsilon - 2)^2},$$

$$\beta_2 = \frac{3(\epsilon - 1)(\epsilon + 6)}{(\epsilon - 2)(\epsilon - 3)}.$$

The criterion K which aids in the determination of the type of a curve, is

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1)}.$$

For the above values of β_1 and β_2, K is unity. Hence, these two betas trace out a line on the β_1, β_2 plane which coincides with the Type V line. When ϵ is fairly large, but still small compared with α and β, the distribution represented by the terms of the hypergeometric series tends to become normal.

Substituting the general values of β_1 and β_2 in the expression $2\beta_2 - 3\beta_1 - 6$ and simplifying, we have

$$\frac{\epsilon^2(\epsilon - 1)}{(\epsilon - 2)^2(\epsilon - 3)}\left[12\left\{\frac{1}{\epsilon - 1} + \frac{1}{\alpha(\alpha + \epsilon)} + \frac{1}{\beta(\beta + \epsilon)}\right\} - \frac{\epsilon(\epsilon^2 - 7\epsilon + 4)}{\alpha(\alpha + \epsilon)\beta(\beta + \epsilon)}\right]$$

$$= \frac{12\epsilon^2}{(\epsilon - 2)^2(\epsilon - 3)} \frac{1}{\alpha(\alpha + \epsilon)} \frac{1}{\beta(\beta + \epsilon)}\Big[\alpha(\alpha + \epsilon)\beta(\beta + \epsilon)$$
$$+ (\epsilon - 1)\{\alpha(\alpha + \epsilon) + \beta(\beta + \epsilon)\} - \frac{\epsilon(\epsilon - 1)(\epsilon^2 - 7\epsilon + 4)}{12}\Big]$$

The constants α, β and γ have been taken positive, hence the point (β_1, β_2) lies above, below, or on the Type III line $2\beta_2 - 3\beta_1 - 6 = 0$, according as the expression

$$X \equiv \alpha(\alpha + \epsilon)\beta(\beta + \epsilon) + (\epsilon - 1)\{\alpha(\alpha + \epsilon) + \beta(\beta + \epsilon)\} - \frac{\epsilon(\epsilon - 1)(\epsilon^2 - 7\epsilon + 4)}{12}$$

is negative, positive or zero.

The condition for a Type III curve is clearly $X = 0$. The algebra may be greatly simplified by writing

$$\frac{1}{\alpha(\alpha + \epsilon)} \frac{1}{\beta(\beta + \epsilon)} = \rho,$$

$$\frac{1}{\alpha(\alpha + \epsilon)} + \frac{1}{\beta(\beta + \epsilon)} = \sigma.$$

The betas then take up the comparatively simple forms

$$\beta_1=\frac{\epsilon-1}{(\epsilon-2)^2}\{16+4\epsilon^2\sigma+\epsilon^4\rho\},$$

$$\beta_2=\frac{3(\epsilon-1)(\epsilon+6)}{(\epsilon-2)(\epsilon-3)}+\frac{\epsilon^2(\epsilon-1)}{(\epsilon-2)(\epsilon-3)}\{\epsilon(\epsilon+1)\rho+6\sigma\}$$

and we find

$$2\beta_2-3\beta_1-6=\frac{12\epsilon^2}{(\epsilon-2)^2(\epsilon-3)}\left[1+(\epsilon-1)\sigma-\frac{\epsilon(\epsilon^2-7\epsilon+4)(\epsilon-1)\rho}{12}\right],$$

$$4\beta_2-3\beta_1=\frac{\epsilon^2(\epsilon-1)}{(\epsilon-2)^2(\epsilon-3)}[12+12(\epsilon-1)\sigma+\epsilon(\epsilon^2+5\epsilon-8)\rho],$$

$$\beta_2+3=[6+6(\epsilon-1)\sigma+\epsilon^2(\epsilon^2-1)\rho]\frac{\epsilon^2}{(\epsilon-2)(\epsilon-3)}.$$

Hence, substituting these values in the criterion

$$K=\frac{\beta_1(\beta_2+3)^2}{4(2\beta_2-3\beta_1-6)(4\beta_2-3\beta_1)},$$

we find

$$K=\tfrac{1}{4}\frac{\{16+4\epsilon^2\sigma+\epsilon^4\rho\}}{\{12+12(\epsilon-1)\sigma-\epsilon(\epsilon-1)(\epsilon^2-7\epsilon+4)\rho\}}\times\frac{\{6+6(\epsilon-1)\sigma+\epsilon(\epsilon^2-1)\rho\}^2}{\{12+12(\epsilon-1)\sigma+\epsilon(\epsilon^2+5\epsilon-8)\rho\}}.$$

α, β, γ and ϵ are, by hypothesis, positive quantities. Hence, σ, ρ, σ^2 and $\sigma\rho$ are positive quantities. The expression $\epsilon^2-7\epsilon+4$ is a minimum when $\epsilon=\frac{7}{2}$. Its value is then $-\frac{33}{4}>-9$. Hence

$$K>\tfrac{1}{4}\frac{(16+4\epsilon^2\sigma+\epsilon^4\rho)}{\{12+12(\epsilon-1)\sigma+9\epsilon(\epsilon-1)\rho\}}\times\frac{\{6+6(\epsilon-1)\sigma+\epsilon(\epsilon^2-1)\rho\}^2}{\{12+12(\epsilon-1)\sigma+\epsilon(\epsilon^2+5\epsilon-8)\rho\}}.$$

Comparing the coefficients of σ, ρ, $\sigma\rho$, σ^2, ρ^2 in the numerator and denominator of this expression, we find that K is always greater than unity. Hence the two betas of the hypergeometric series $F(\alpha, \beta, \gamma, 1)$ with positive constants, lie above the Type V line which divides the Type IV and Type VI regions on the β_1, β_2 plane.

By making α and β indefinitely large compared with ϵ (which is equivalent to making the quantities ρ and σ indefinitely small) we have already shown that the first two betas tend to lie on the Type V line. Therefore, *the Type V line forms the lower bound* to the β_1 and β_2 area of the above hypergeometric series. This is significant, because it means that when fitting by the first four moments, the Type IV curve never arises.

It now remains to find the upper boundary to the β_1 and β_2 area. This will be found as follows.

The hypergeometric series

$$\left\{1+\frac{\alpha\beta}{1!\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}+\dots\right\}y_0,$$

where y_0 is the inverse of the sum of the series, may be written in the form

$$\left[1+\frac{\frac{\alpha\beta}{\gamma}}{1!}+\frac{1}{2!}\left(\frac{\frac{\alpha\beta}{\gamma}\frac{(\alpha+1)(\beta+1)}{\gamma}}{1+\frac{1}{\gamma}}\right)+\dots\right]y_0.$$

Make α, β and γ tend to infinity in such a way that $\frac{\alpha\beta}{\gamma}$ remains constant and equal to m. In the limit the series takes the form

$$\left(1+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\dots\right)e^{-m},$$

which is the exponential or Poisson series. Hence, *the Poisson limit to the binomial is also a limit to the hypergeometric series.*

Now $$\epsilon=\gamma-\alpha-\beta-1,$$

therefore $$\frac{\epsilon}{\alpha\beta}=\frac{\gamma}{\alpha\beta}-\frac{1}{\beta}-\frac{1}{\alpha}-\frac{1}{\alpha\beta}.$$

In the limit, therefore, $$\frac{\epsilon}{\alpha\beta}=\frac{\gamma}{\alpha\beta}=\frac{1}{m}.$$

Hence we may obtain the same Poisson limit by making α, β and ϵ tend to infinity in such a way that $\frac{\alpha\beta}{\epsilon}$ remains equal to a constant quantity m.

We have

$$\beta_1=\frac{\epsilon-1}{(\epsilon-2)^2}\frac{(2\alpha+\epsilon)^2(2\beta+\epsilon)^2}{\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)}$$

$$=\frac{(\epsilon-1)\alpha\beta}{(\epsilon-2)^2}\frac{\left(\frac{2}{\alpha}+\frac{\epsilon}{\alpha^2}\right)^2\left(\frac{2}{\beta}+\frac{\epsilon}{\beta^2}\right)^2}{\left(\frac{1}{\alpha}+\frac{\epsilon}{\alpha^2}\right)\left(\frac{1}{\beta}+\frac{\epsilon}{\beta^2}\right)}$$

and $$\beta_2=\frac{3(\epsilon-1)(\epsilon+6)}{(\epsilon-2)(\epsilon-3)}+\frac{\epsilon^2(\epsilon-1)}{(\epsilon-2)(\epsilon-3)}\left[\frac{\frac{\epsilon}{\alpha\beta}(\epsilon+1)}{\alpha^2\beta^2\left(\frac{1}{\alpha}+\frac{\epsilon}{\alpha^2}\right)\left(\frac{1}{\beta}+\frac{\epsilon}{\beta^2}\right)}+6\left\{\frac{1}{\alpha(\alpha+\epsilon)}+\frac{1}{\beta(\beta+\epsilon)}\right\}\right].$$

Proceeding to the limit, β_1 and β_2 take the values

$$\left.\begin{aligned}\beta_1&=\frac{\epsilon}{\alpha\beta}=\frac{1}{m}\\(\beta_2-3)&=\frac{1}{m}\end{aligned}\right\}.$$

Eliminating m, we arrive at the Poisson line

$$\beta_2 - 3 = \beta_1.$$

Substitute now the general values of β_1 and β_2 in $\beta_2 - 3 - \beta_1$, we find

$$\beta_2 - \beta_1 - 3 = \frac{2}{(\epsilon-2)^2(\epsilon-3)}[(7\epsilon^2 - 16\epsilon + 12) + \epsilon^3(\epsilon-1)(\sigma + \overline{\epsilon-1}\,\rho)].$$

Since $\epsilon > 3$, this expression is always positive. Hence the points (β_1, β_2) never lie above the Poisson line. They never actually reach this line, which may be considered as a mathematical limit.

Hence, the limits to the β_1, β_2 area for the hypergeometric series in which α, β and γ are positive quantities and x unity, are

$$\begin{cases} \beta_2 - 3 = \beta_1 & \text{upper limit} \\ \beta_1(\beta_2+3)^2 = 4(2\beta_2 - 3\beta_1 - 6)(4\beta_2 - 3\beta_1) & \text{lower limit.} \end{cases}$$

These two lines arise from the Gaussian point $\left.\begin{matrix}\beta_1 = 0\\ \beta_2 = 3\end{matrix}\right\}$.

Examples.

I. $$\alpha = 10, \quad \beta = 30, \quad \gamma = 101,$$
$$\epsilon = \gamma - \alpha - \beta - 1 = 60, \quad c = 1.$$

Substituting these values in the formulae for the moments and betas, we find

$$\log N = 1{\cdot}639{,}8377,$$
$$\nu_2 = 8{\cdot}898{,}305{,}085,$$
$$\beta_1' = {\cdot}855{,}217{,}711,$$
$$\beta_2' = 4{\cdot}351{,}447{,}584.$$

Position of the mode measured from start of histogram is

$$\frac{(\alpha-1)(\beta-1)}{(\epsilon+2)} = 4{\cdot}209{,}677.$$

Mean of series measured from start of histogram is

$$\mu_1' = \frac{\alpha\beta}{\epsilon} + {\cdot}5 = 5{\cdot}5.$$

Corrected moments and betas, by Sheppard, are

$$\mu_2 = 8{\cdot}814{,}9718,$$
$$\beta_1 = {\cdot}879{,}70242,$$
$$\beta_2 = 4{\cdot}377{,}2277.$$

The point (β_1, β_2) falls in the Type VI region, and the curve which has the same first four moments as the above, is

$$y = y_0(x - 164{\cdot}844{,}491)^{4\cdot 222{,}747}\, x^{-138\cdot 135{,}056},$$

where $\log y_0 = 301{\cdot}446{,}5204$.

For the curve: mode $= 170{\cdot}121{,}456$,

mean $= 171{\cdot}471{,}584$.

Hence, equation of curve referred to mean is

$$y = y_0\,(x + 6{\cdot}627{,}093)^{4{\cdot}288{,}747}\,(x + 171{\cdot}471{,}584)^{-138{\cdot}186{,}064} \quad \text{.........(i)}.$$

Terms of series	Midordinates of (i)	Areas under (i)
1·000,000	·864,975	·988,594
2·970,298	2·894,090	2·897,940
4·965,056	5·023,947	4·985,139
6·170,167	6·276,503	6·229,540
6·362,985	6·437,399	6·403,479
5·769,107	5·792,867	5·776,574
4·762,233	4·751,426	4·748,863
3·662,279	3·636,780	3·644,017
2·666,172	2·643,873	2·652,174
1·858,982	1·843,221	1·851,678
1·252,279	1·243,150	1·250,392
·820,494	·816,128	·821,744
·525,629	·524,004	·528,083
·330,620	·330,251	·331,081
·204,877	·204,941	·206,833
·125,420	·125,512	·126,744
·076,022	·076,011	·076,828

II. $\alpha = \beta = 30$,

$e = 50, \quad c = 1$.

Uncorrected moments and betas are

$$\log N = 4{\cdot}974{,}5371,$$
$$\nu_2 = 47{\cdot}020{,}4081,$$
$$\beta_1' = {\cdot}540{,}5824,$$
$$\beta_2' = 3{\cdot}944{,}4733.$$

Corrected moments and betas are

$$\mu_2 = 46{\cdot}937{,}0748,$$
$$\beta_1 = {\cdot}543{,}46676,$$
$$\beta_2 = 3{\cdot}947{,}83375,$$

and the Pearson curve with the same first four moments is

$$y = y_0\,(x - 86{\cdot}467{,}3846)^{13{\cdot}408{,}4663}\,x^{-69{\cdot}792{,}1369},$$
$$\log y_0 = 127{\cdot}806{,}2761.$$

Mode of series measured from start of histogram $= 16{\cdot}173{,}077$.

Mean of series measured from start of histogram $= 18{\cdot}5$.

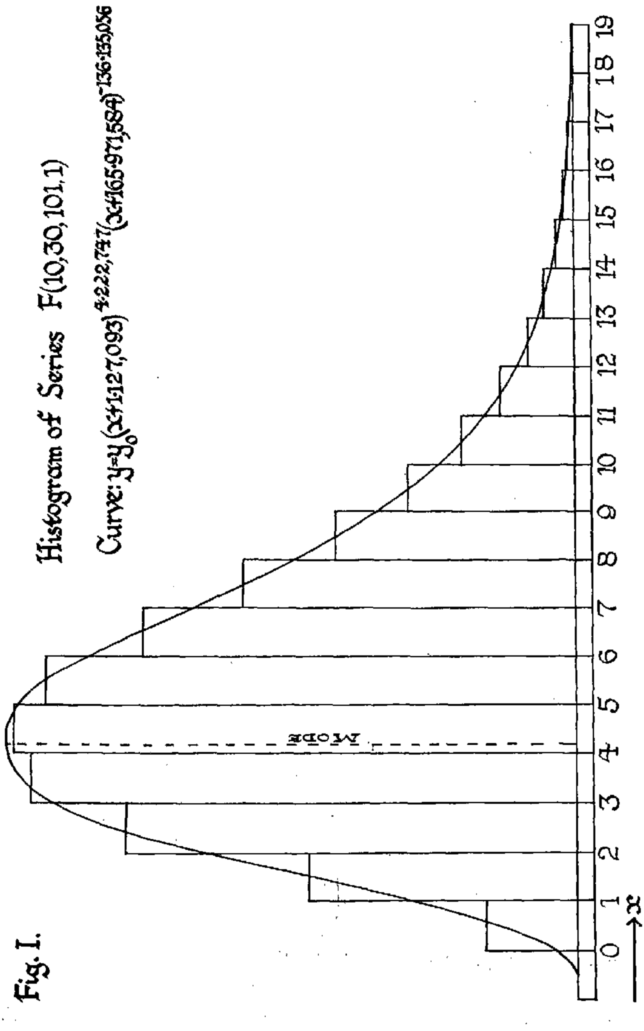

Histogram of Series F(10,30,101,1)
Curve: y=y0(x+1.127,093)^4.222,747 (x+165.971,584)^-136.135,056
MODE
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
x

Fig. I.

For the curve: mode = 107·002,995,

mean = 109·376,152.

Hence, equation of curve referred to mean as origin is

$$y = y_0 (x + 22\cdot 908,7671)^{[illegible]} (x + 109\cdot 376,1517)^{-[illegible]}.$$

Terms of series	Midordinates of (i)	Areas under (i)	Terms of series	Midordinates of (i)	Areas under (i)
1·000	1·661	2·143	2219·294	2210·193	2217·745
8·108	9·338	10·122	1881·510	1878·842	1880·440
34·785	35·842	37·594	1582·046	1579·832	1581·402
105·074	104·401	107·483	1320·269	1318·493	1319·981
250·903	246·924	251·384	1094·251	1092·800	1094·233
504·485	496·498	501·947	901·236	900·190	901·428
887·922	876·844	882·507	738·007	737·243	738·340
1405·064	1393·196	1398·031	601·165	600·631	601·587
2037·641	2027·573	2030·881	487·341	486·964	487·798
2747·294	2741·426	2742·532	393·330	393·072	393·773
3482·197	3481·828	3480·533	316·176	316·000	316·596
4185·962	4191·146	4187·632	253·220	253·098	253·591
4806·421	4816·147	4810·885	202·110	202·003	202·441
5302·393	5314·851	5308·492	160·831	160·788	161·104
5647·539	5660·936	5654·136	127·619	127·564	127·839
5831·270	5843·824	5837·208	101·007	100·957	101·180
5857·304	5867·936	5861·957	79·758	79·708	79·889
5740·646	5748·560	5743·552	62·846	62·796	62·941
5503·945	5508·994	5506·110	49·426	49·375	49·492
5173·842	5176·205	5173·474	38·804	38·751	38·844
4777·844	4777·864	4776·297	30·417	30·365	30·439
4341·916	4340·233	4339·646	23·809	23·758	23·817
3888·884	3886·114	3886·210	18·613	18·564	18·610
3437·577	3434·300	3434·994	14·535	14·488	14·525
3002·535	2999·010	3000·138	......	......	......
2594·189	2590·791	2592·192			

III. $\alpha = \beta = 94, \quad c = 1,$

$$\gamma = 589, \quad e = 400.$$

Raw moments and betas are

$$\log N = 7\cdot 807,6359,$$
$$\nu_2 = 33\cdot 776,6619,$$
$$\beta_1' = \cdot 193,6389,$$
$$\beta_2' = 3\cdot 210,1843.$$

Mode of series = 21·514,925,

mean of series = 22·59,

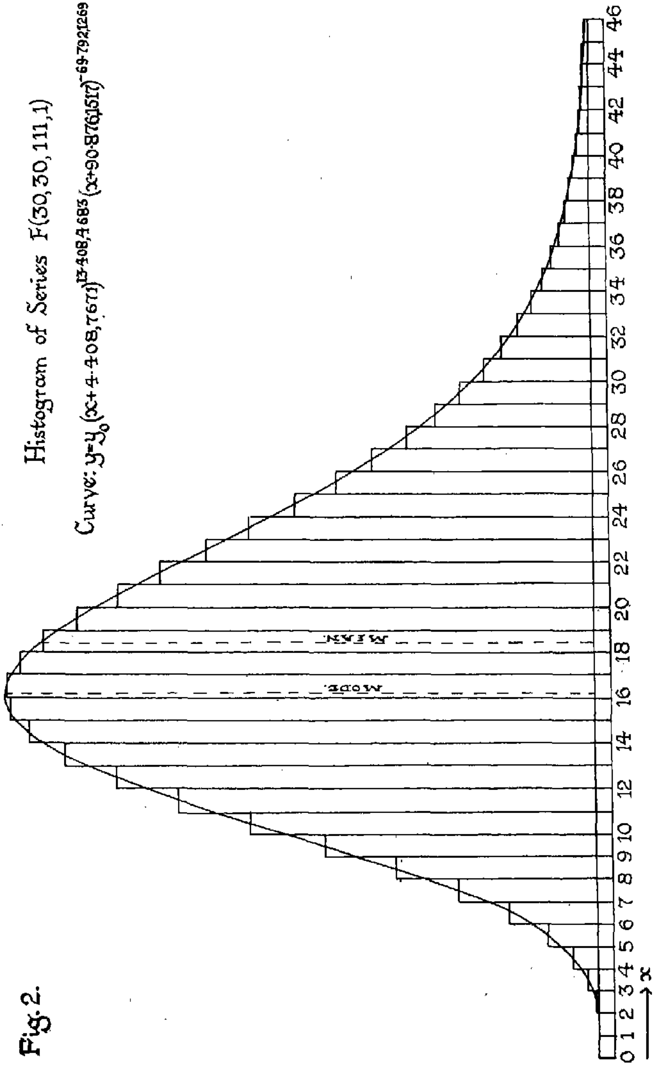
both being measured from the start of the histogram.

The quantity $r = \dfrac{6(\beta_2 - \beta_1 - 1)}{2\beta_2 - 3\beta_1 - 6}$ is very large, of the order 6000. We are justified, therefore, in fitting a Type III curve.

Fig. 2.

Histogram of Series F(30, 30, 111, 1)

Curve: $y = y_0 (x + 4{\cdot}408{,}7671)^{13{\cdot}408{,}4683} (x + 90{\cdot}876{,}1517)^{-69{\cdot}792{,}1269}$



Corrected moments are

$$\mu_2 = 33\cdot 693{,}3286,$$

$$\mu_3 = 73\cdot 354{,}7625,$$

whence $$\beta_1 = \cdot 140{,}67776.$$

The Pearson-type III curve having the same first three moments is

$$y = y_0 e^{-918{,}6405x}\, x^{27\cdot 493{,}8194},$$

$$\log y_0 = 22\cdot 096{,}8526.$$

For the curve: $$\text{mode} = 29\cdot 863{,}491,$$

$$\text{mean} = 30\cdot 952{,}057.$$

Equation of curve referred to mean as origin is

$$y = y_0' e^{-918{,}6405x} (x + 30\cdot 952{,}057)^{27\cdot 493{,}8194}.$$

Significant terms of series $\times 10^{-1}$	Corresponding midordinates of curve $\times 10^{-1}$	Areas under curve $\times 10^{-1}$	Significant terms of series $\times 10^{-1}$	Corresponding midordinates of curve $\times 10^{-1}$	Areas under curve $\times 10^{-1}$
2	3	3	126,924	126,743	126,931
11	15	17	99,959	99,806	99,993
60	68	73	77,439	77,316	77,488
237	250	264	59,060	58,962	59,116
708	778	811	44,377	42,298	44,430
2,111	2,009	2,165	32,874	32,810	32,920
5,068	5,000	5,116	24,025	23,972	24,062
10,843	10,676	10,857	17,332	17,288	17,359
20,996	20,694	20,948	12,350	12,313	12,368
37,249	36,805	37,125	8,696	8,665	8,707
61,144	60,694	60,955	6,055	6,028	6,060
93,627	93,049	93,409	4,171	4,148	4,171
134,647	134,150	134,456	2,843	2,824	2,841
182,911	182,604	182,803	1,919	1,903	1,915
235,873	235,839	235,888	1,283	1,270	1,278
289,984	290,254	290,130	850	839	845
341,158	341,704	341,411	558	550	554
385,351	386,095	385,657	363	357	360
419,131	419,967	419,430	235	230	232
440,122	440,939	440,357	151	147	149
447,246	447,954	447,384	96	93	94
440,748	441,282	440,772	61	58	59
422,024	422,361	421,946	38	36	37
393,320	393,471	393,170	24	23	24
357,363	357,354	357,172	15	14	15
317,002	316,879	316,800	9	9	9
274,907	274,714	274,738	6	5	6
233,355	233,131	233,229	3	3	3
194,112	193,888	194,036	...	...	...
158,399	158,190	158,367			

With the exception of the first few terms we see that the areas under the curve accord very closely with the terms of the series when the standard deviation of the latter is not small, or what is equivalent, when the number of significant terms is not small. (See also pp. 302—304.)

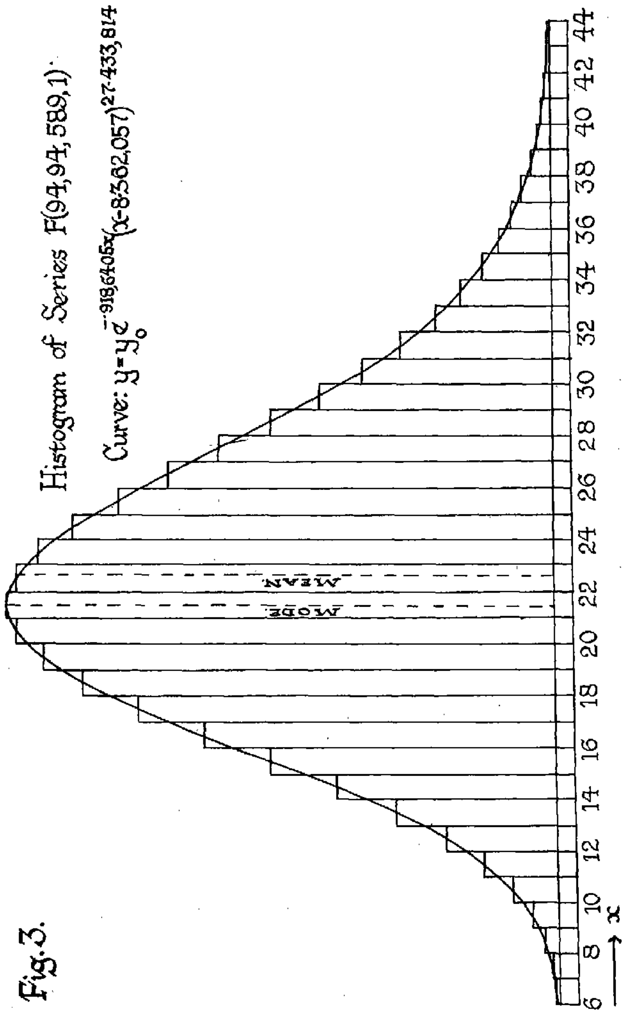

Histogram of Series F(94,94,589,1)
Curve: $y = y_0 e^{-918,6405x}(x-8362,057)^{27\cdot433,814}$
MEAN
MODE
6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44
$\longrightarrow x$

Fig. 3.

I *b*. *Infinite Hypergeometric Series in which two of α, β, γ are Negative Non-integral Numbers.*

(i) α and β negative, γ positive.

Let a be the largest integer in $|\alpha|$ and b the largest integer in $|\beta|$, then

$$\alpha = -a - \delta_1,$$

$$\beta = -b - \delta_2 \qquad 0 < \delta_1,\ \delta_2 < 1.$$

Assume that $b \geqslant a$, i.e. $\beta \leqslant \alpha$. The $(a+1)$st term of the series is

$$\frac{\alpha(\alpha+1)(\alpha+2)\dots(-\delta_1)\beta(\beta+1)\dots(\beta+a)}{(a+1)!\,\gamma(\gamma+1)\dots(\gamma+a)}.$$

This term is positive. The $(a+2)$nd term is

$$\frac{\alpha(\alpha+1)\dots(-\delta_1)(1-\delta_1)\beta(\beta+1)\dots(\beta+a)(\beta+a+1)}{(a+2)!\,\gamma(\gamma+1)\dots(\gamma+a+1)}.$$

In order that this and all subsequent terms be positive, the following condition must be satisfied:

$$\beta + a + 1 > 0.$$

Now $$\alpha = -a - \delta_1 \qquad 0 < \delta_1 < 1,$$

$$\beta = -a - \delta_2' \qquad \delta_2' \geqslant \delta_1.$$

Therefore $$\beta + a + 1 = (1 - \delta_2') > 0,$$

i.e. $$1 > \delta_2' > 0.$$

A necessary and sufficient condition for all terms to be positive is, therefore,

$$a = b.$$

Since the series is infinite, the condition $\epsilon > 3$ must be satisfied if we intend fitting curves by the first four moments.

When $\epsilon \geqslant 4$, the first two betas lie above the Type III line. When also

$$-(\epsilon - 1) \geqslant \frac{\alpha}{\beta} \geqslant -1,$$

the first two betas lie above the Poisson line. In either case the corresponding Pearson curve is of Type I (or the types associated with Type I). (For proof of this statement, see corresponding section under finite series, pp. 298—9.)

(ii) α and γ negative, β positive.

As in the previous case, in order that the terms of the series be all positive, the condition $|\gamma - \alpha| < 1$ must be satisfied.

Now $$\epsilon = \gamma - \alpha - \beta - 1 \quad \begin{array}{l} > 3, \\ < -\beta. \end{array}$$

Hence, for all expressions to be finite, $(-\beta)$ must be at least greater than 3. This implies $\beta < -3$, contrary to the hypothesis that β is positive. A positive ϵ would imply a negative β, hence we cannot have an infinite convergent hypergeometric series with all terms positive and γ negative.

Example.

$$\left.\begin{array}{l} \alpha = \beta = -25{\cdot}6 \\ \gamma = 30, \quad c = 1 \end{array}\right\}.$$

Crude moments and betas are

$$\log N = 5{\cdot}369{,}3566,$$

$$\nu_2 = 3{\cdot}835{,}2290,$$

$$\beta_1' = {\cdot}004{,}688549,$$

$$\beta_2' = 2{\cdot}949{,}4076.$$

$$\text{Mean of series} = 8{\cdot}671{,}571.$$

$$\textit{Mode of series} = 8{\cdot}607{,}786.$$

Corrected moments and betas are

$$\mu_2 = 3{\cdot}751{,}8956,$$

$$\beta_1 = {\cdot}005{,}007951,$$

$$\beta_2 = 2{\cdot}947{,}7272,$$

and the Pearson curve having the same first four moments is

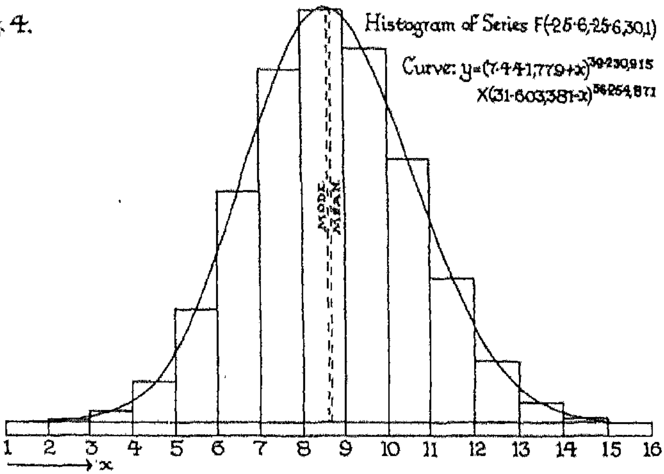
$$y = y_0(16{\cdot}041{,}94 + x)^{39{\cdot}230{,}915}(23{\cdot}00322 - x)^{56{\cdot}254{,}871},$$

$$\log y_0 = \overline{120}{\cdot}789{,}4740.$$

Origin at mode. Mean-mode of curve $= {\cdot}0714{,}082$.

Terms of series	Midordinates of curve	Areas under curve
1·0	1·2	1·6
21·8	19·7	24·5
213·2	188·3	216·6
1,237·1	1,131·3	1,232·5
4,786·7	4,554·5	4,764·9
13,136·9	12,886·0	13,122·7
26,546·5	26,538·9	26,575·1
40,468·6	40,829·6	40,514·2
47,299·1	47,808·2	47,300·7
42,840·2	43,154·0	42,799·3
30,269·4	30,271·0	30,273·0
16,741·7	16,561·8	16,750·5
7,253·4	7,064·9	7,270·7
2,457·1	2,339·8	2,463·8
648·0	596·6	646·6
132·1	115·6	129·8
20·6	16·7	19·6
2·4	1·8	2·2

Fig. 4.



Here again the accordance between the terms of the series and the areas under the curve is very close in the significant parts of the curve. (See also pp. 302—4 and 306.)

II. *Finite Hypergeometric Series.*

The hypergeometric series

$$1+\frac{\alpha\beta}{1!\,\gamma}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}+\dots$$

is finite only when α or β or both are negative integers. In order that the series may represent a frequency distribution, each term must be positive. Hence, either

(*a*) α and β are negative with one at least an integer, or

(*b*) α and γ negative, α a negative integer and $|\gamma|>|\alpha|$.

These two sets of conditions give rise to two distinct types of series which will be considered separately.

(*a*) *α and β negative.*

In order to find the position of the point (β_1, β_2) for such series, relative to the Poisson line, substitute the expressions for the β's in $\beta_2-\beta_1-3\equiv\theta$. We then have

$$\theta=\frac{2}{(\epsilon-2)^2(\epsilon-3)}\,\frac{1}{\alpha(\alpha+\epsilon)}\,\frac{1}{\beta(\beta+\epsilon)}\left[(7\epsilon^2-16\epsilon+12)\,\alpha\beta\,(\alpha+\epsilon)(\beta+\epsilon)\right.$$
$$\left.+\,\epsilon^2(\epsilon-1)\{\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)\}+\epsilon^3(\epsilon-1)^2\right].$$

If $\epsilon > 3$, the coefficient $\frac{2}{(\epsilon-2)^2(\epsilon-3)}\frac{1}{\alpha(\alpha+\epsilon)}\frac{1}{\beta(\beta+\epsilon)}$ is positive. Hence, the sign of $\beta_2-\beta_1-3$ is the same as that of the expression

$$E=\alpha(\alpha+\epsilon)\beta(\beta+\epsilon)\{7\epsilon^2-16\epsilon+12\}+\epsilon^3(\epsilon-1)\{\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)\}+\epsilon^3(\epsilon-1)^2.$$

E is negative when $\alpha=\beta=-1$ and also when $\alpha=\beta=-(\epsilon-1)$ for all permissible values of ϵ. When either α or β is greater than -1, E may be positive, in which case the point (β_1, β_2) will lie below the Poisson line. We will now show that for $\epsilon \geqslant 4$, E is negative when α and β lie in the range $(-1, -\epsilon+1)$. It will be sufficient to prove that under these conditions E is negative at all its points of maxima and minima.

Differentiate E with respect to α and β respectively.

$$\frac{\partial E}{\partial \alpha}=(2\alpha+\epsilon)[\epsilon^3(\epsilon-1)+\beta(\beta+\epsilon)(7\epsilon^2-16\epsilon+12)],$$

$$\frac{\partial E}{\partial \beta}=(2\beta+\epsilon)[\epsilon^3(\epsilon-1)+\alpha(\alpha+\epsilon)(7\epsilon^2-16\epsilon+12)].$$

Clearly, the points of maxima and minima are given by

(i) $\beta(\beta+\epsilon)=\alpha(\alpha+\epsilon)=-\epsilon^3(\epsilon-1)/(7\epsilon^2-16\epsilon+12)$,

(ii) $\alpha=\beta=-\frac{\epsilon}{2}$.

At point (i), E has the value

$$\begin{aligned}\alpha(\alpha+\epsilon)[\epsilon^2(\epsilon+1)+\beta(\beta+\epsilon)(7\epsilon^2-16\epsilon+12)]&\\ +\epsilon^3(\epsilon-1)^2+\beta(\beta+\epsilon)\epsilon^3(\epsilon-1)&\\ =\epsilon^3(\epsilon-1)[(\epsilon-1)+\beta(\beta+\epsilon)]&\\ =-\frac{\epsilon^3(\epsilon-1)^2(\epsilon-2)^2(\epsilon-3)}{(7\epsilon^2-16\epsilon+12)}&.\end{aligned}$$

This is negative for all values of $\epsilon > 3$.

At point (ii), E has the value $-\frac{\epsilon^3}{16}(\epsilon-4)(\epsilon-2)^2$. This is again negative for $\epsilon \geqslant 4$. It follows, therefore, that E is negative when $\epsilon \geqslant 4$ and $-1 \geqslant \begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} \geqslant -(\epsilon-1)$. The corresponding betas then lie above the Poisson line.

When $\epsilon=4$, then $(\alpha+\beta) > -5$, in which case the values of α and β which result in a series with the largest number of terms are, respectively, -2, -3. This is a series with three terms. Hence, for finite series with a minimum of three terms, the first two betas lie above the Poisson line.

As previously (p. 300), if $-\alpha$, $-\beta$ and γ tend to infinity in such a way that $\frac{\alpha\beta}{\gamma}$ remains constant and equal to a finite quantity m, the finite hypergeometric series will, in the limit, become a Poisson series.

When α and β are fractional, the series is infinite (see p. 308). α and β may then be greater than -1 and the corresponding betas may lie below the Poisson line. However, by substituting in $2\beta_2-3\beta_1-6$ it can be readily shown that for $\epsilon \geqslant 4$ and for any negative values of α and β the first two betas lie above the Type III line.

The series (2) (p. 295) is finite. α and β are negative integers. Hence, for $\epsilon = N \geqslant 4$, its first two betas lie above the Poisson line.

(*b*) *Second type of finite series.*

α and γ negative, α an integer; β positive and $|\gamma| > |\alpha|$.

The last condition is introduced to ensure that all terms are positive.

Now
$$\gamma < \alpha, \quad \gamma-\alpha-\beta-1 < -\beta-1,$$
i.e.
$$\epsilon < -(\beta+1).$$

β is positive, hence ϵ is negative.

The position of the first two betas of such series relative to the Poisson line is determined from the sign of the expression

$$\frac{2}{(\epsilon-2)^2}\,\frac{1}{(\epsilon-3)\,\alpha\,(\alpha+\epsilon)}\,\frac{1}{\beta\,(\beta+\epsilon)}\,[(7\epsilon^2-16\epsilon+12)\,\alpha\beta\,(\alpha+\epsilon)\,(\beta+\epsilon)$$
$$+\epsilon^3\,(\epsilon-1)\,\{\alpha\,(\alpha+\epsilon)+\beta\,(\beta+\epsilon)\}+\epsilon^3\,(\epsilon-1)^2].$$

The term outside the square brackets is always positive. It is sufficient, therefore, to consider the sign of the expression inside the brackets. This becomes, after substituting for α, ϵ, $(\epsilon+\beta)$ etc., their absolute values $-\alpha'$, $-\epsilon'$, $-(\epsilon'-\beta)$ etc.,

$$-(7\epsilon'^2+16\epsilon'+12)\,\alpha'\,(\alpha'+\epsilon')\,\beta\,(\epsilon'-\beta)$$
$$+\epsilon'^3\,(\epsilon'+1)\,\{(\epsilon'+\alpha')\,\alpha'-(\epsilon'-\beta)\,\beta\}-\epsilon'^3\,(\epsilon'+1)^2.$$

This expression may be positive, negative or zero. The first two betas may, therefore, lie below, above or on the Poisson line.

Rearrange in the form

$$\alpha'\,(\alpha'+\epsilon')\,[\epsilon'^3\,(\epsilon'+1)-\beta\,(\epsilon'-\beta)\,(7\epsilon'^2+16\epsilon'+12)]-\epsilon'^3\,(\epsilon'+1)\,[(\epsilon'+1)+\beta\,(\epsilon'-\beta)].$$

α' may vary independently of ϵ'. Hence, this expression will be negative for all values of α' only when

$$\epsilon'^3\,(\epsilon'+1)-\beta\,(\epsilon'-\beta)\,(7\epsilon'^2+16\epsilon'+12) \leqslant 0,$$

i.e.
$$\beta^2-\beta\epsilon'+\frac{\epsilon'^3\,(\epsilon'+1)}{7\epsilon'^2+16\epsilon'+12} \leqslant 0,$$

i.e. when β lies between the values

$$\frac{\epsilon'}{2}\left[1 \mp (\epsilon'+2)\sqrt{\frac{3}{7\epsilon'^2+16\epsilon'+12}}\right]$$

When β lies outside these limits, the first two betas of the corresponding series may lie above the Poisson line.

However, by substituting in $2\beta_2-3\beta_1-6$, we can show that for all permissible values of β, the corresponding point (β_1, β_2) lies above the Type III line. For the position of this point relative to the Type III line depends on the sign of the expression

$$[\alpha(\alpha+\epsilon)+\beta(\beta+\epsilon)](\epsilon-1)+\alpha\beta(\alpha+\epsilon)(\beta+\epsilon)-\frac{\epsilon(\epsilon-1)}{12}(\epsilon^2-7\epsilon+4).$$

Putting in for the constants their absolute values, this expression may be rewritten in the form

$$-(\epsilon'+1)\{\alpha'(\alpha'+\epsilon')-\beta(\epsilon'-\beta)\}-\alpha'\beta(\epsilon'-\beta)(\alpha'+\epsilon')-\frac{\epsilon'(\epsilon'+1)(\epsilon'^2+7\epsilon'+4)}{12}.$$

Rearranging the terms,

$$\beta(\epsilon'-\beta)[(\epsilon'+1)-\alpha'(\alpha'+\epsilon')]-\alpha'(\alpha'+\epsilon')(\epsilon'+1)-\frac{\epsilon'(\epsilon'+1)}{12}(\epsilon'^2+7\epsilon'+4) \quad \text{(i)}.$$

This can be positive only when $(\epsilon'+1)>\alpha'(\alpha'+\epsilon')$.

For such values, the maximum is reached when $\beta=\frac{\epsilon'}{2}$. Substitute, therefore, $\beta=\frac{\epsilon'}{2}$ in (i). We have

$$-\alpha'(\alpha'+\epsilon')\left[(\epsilon'+1)+\frac{\epsilon'^2}{4}\right]+\frac{(\epsilon'+1)\epsilon'^2}{4}-\frac{\epsilon'(\epsilon'+1)(\epsilon'^2+7\epsilon'+4)}{12} \quad \text{...(ii)}.$$

This is evidently negative for all permissible values of ϵ'. Hence, the corresponding betas always lie above the Type III line. The series referred to in paragraphs (i) and (ii) (p. 296) and Examples II (p. 315) and IV (p. 318) are illustrations of finite series with negative γ.

When $\beta=1$, the hypergeometric series reduces to

$$1+\frac{\alpha}{\gamma}+\frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}+\dots.$$

This is finite when α is a negative integer. The terms are all positive and finite when γ is also negative and $|\gamma|>|\alpha|$. Substitute for α and γ their absolute values, the series then adopts the form

$$1+\frac{\alpha'}{\gamma'}+\frac{\alpha'(\alpha'-1)}{\gamma'(\gamma'-1)}+\frac{\alpha'(\alpha'-1)(\alpha'-2)}{\gamma'(\gamma'-1)(\gamma'-2)}+\dots\dots\dots\dots \text{(iii)}.$$

When $\gamma'\to\alpha'$, the distribution represented by the terms of the series tends to become rectangular. For all other values the distribution is J-shaped.

In order that a hypergeometric series may represent a U-shaped distribution, the following conditions must be satisfied:

(i) Series finite.

(ii) $\gamma>\alpha\beta$.

(iii) There must exist an antimode, i.e. from some point onwards the terms of the series must be constantly increasing, i.e.

$$\frac{(\alpha-r)(\beta-r)}{(r+1)(\gamma+r)}>1 \quad \text{for} \quad \begin{cases} r>r_0. \\ r_0<|\alpha|. \end{cases}$$

Case (*a*). α and β negative.

$$\frac{(\alpha+r)(\beta+r)}{(r+1)(\gamma+r)}=\frac{(|\alpha|-r)(|\beta|-r)}{(r+1)(\gamma+r)}<\frac{\alpha\beta}{\gamma}.$$

Hence, if initially we have $\gamma>\alpha\beta$, the terms of the series will form a monotone decreasing sequence and in no case can they represent a U-shaped distribution.

Case (*b*). β positive, α and γ negative.

For a U-shaped distribution, we must initially have $\gamma>\alpha\beta$. There must also exist a positive r for which

$$\frac{(\beta+r)(\alpha'-r)}{(r+1)(\gamma'-r)}>1 \quad r<\alpha' \qquad \text{.......................(iv)},$$

where α' and γ' are the absolute values of the constants. (iv) is equivalent to

$$r>\frac{\gamma'-\alpha'\beta}{\alpha'-\beta-\gamma'+1}=r_0<\alpha'.$$

r_0 must be less than α', i.e.

$$\gamma'<\alpha'^2-\alpha'\gamma'+\alpha' \quad\text{or}\quad \gamma'(1+\alpha')<\alpha'(1+\alpha') \quad\text{or}\quad \gamma'<\alpha',$$

contrary to the hypothesis that γ' is greater than α'.

In no case, therefore, can a hypergeometric series with $x=1$ represent a U-shaped distribution. Since the upper branch of the biquadratic on the β_1, β_2 plane forms the lower bound to Pearson U-shaped curves, it also forms the upper limit to the first two betas of hypergeometric series in which the fourth element x is unity.

Examples.

I.

$$\alpha=-\ 30,\quad \beta=-50,$$
$$\gamma=\ \ 100,\quad x=\ \ \ 1.$$

The variance and betas of the series after applying Sheppard's corrections are

$$\mu_2=4{\cdot}971{,}8996,$$
$$\beta_1=\ {\cdot}018{,}306731,$$
$$\beta_2=2{\cdot}956{,}3878.$$

The sum of the series is given by

$$\log N=4{\cdot}740{,}1506.$$

Mean and mode of series, measured from the start of the histogram, are respectively,

$$8{\cdot}879{,}8883;\quad 8{\cdot}734{,}8066.$$

The Pearson-type curve having the same first four moments as the series is

$$y=y_0x^{27{\cdot}737{,}858}(42{\cdot}504{,}879-x)^{53{\cdot}060{,}623} \qquad \text{......................(i)},$$

$$\log y_0=\overline{104}{\cdot}415{,}7762.$$

$$\text{Mode of curve}=14{\cdot}772{,}980.$$

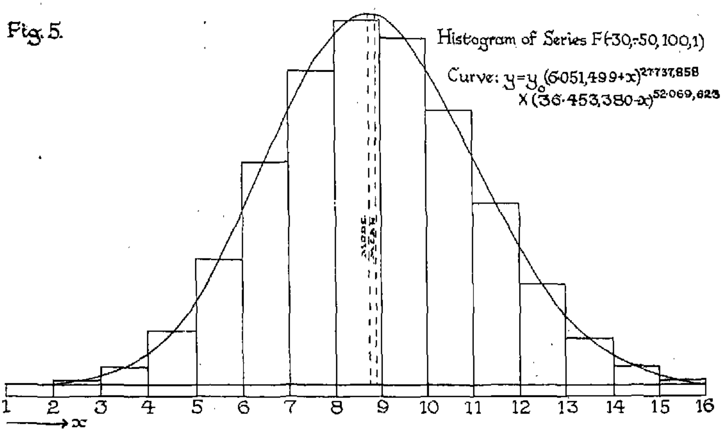
$$\text{Mean of curve}=14{\cdot}931{,}387.$$

Equation of curve referred to the start of the histogram as origin is

$$y = y_0 (6{\cdot}051{,}499 + x)^{27{\cdot}737{,}858} (36{\cdot}453{,}380 - x)^{52{\cdot}069{,}623}.$$

Terms of series	Corresponding midordinates of (i)	Corresponding areas under (i)
1·00	1·17	1·50
15·00	13·81	16·29
105·52	195·89	105·90
463·46	434·52	459·18
1427·50	1362·39	1419·70
3283·25	3253·92	3281·78
5862·95	5887·76	5873·85
8344·04	8424·94	8357·10
9640·48	9736·84	9640·82
9164·40	9231·93	9152·52
7239·04	7262·68	7241·46
4786·14	4778·35	4785·49
2662·56	2641·87	2668·52
1250·82	1230·47	1256·37
497·32	482·74	499·43
167·52	159·16	167·28
47·80	43·90	46·99
11·54	10·06	11·00
2·35	1·90	2·12

Fig. 5.



II. $$\alpha = -30;\ \beta = 60,$$
$$\gamma = -81;\ c = 1.$$

Corrected moments and betas are:

$$\mu_2 = 9{\cdot}293{,}3598,$$
$$\beta_1 = {\cdot}00127{,}2249,$$
$$\beta_2 = 2{\cdot}895{,}9895,$$
$$\log N = 8{\cdot}459{,}5208.$$

Mean and mode of series measured from the start of the histogram are, respectively,

$$16{\cdot}571{,}4286;\ 16{\cdot}627{,}2727.$$

The Pearson-type curve which has the same first four moments as the series is

$$y = y_0 x^{37{\cdot}649{,}947}(45{\cdot}180{,}196 - x)^{24{\cdot}035{,}187} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

$$\log y_0 = \overline{63}{\cdot}565{,}6609.$$

$$\text{Mean of curve} = 24{\cdot}103{,}296.$$

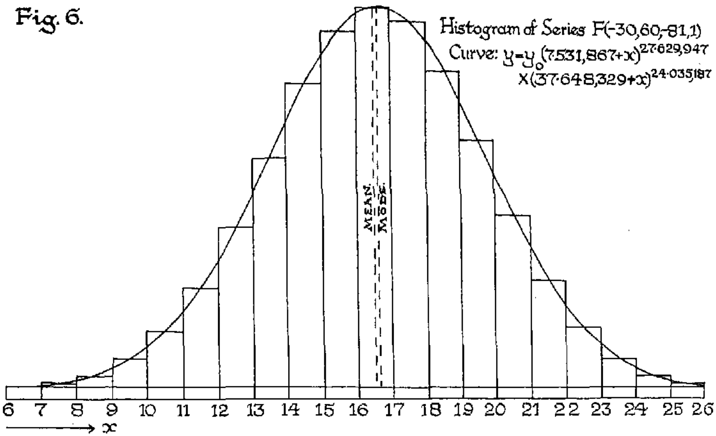
$$\text{Mode of curve} = 24{\cdot}161{,}973.$$

Equation of curve referred to the start of the histogram as origin is

$$y = y_0(7{\cdot}531{,}869 + x)^{37{\cdot}649{,}947}(37{\cdot}648{,}327 - x)^{24{\cdot}035{,}187}.$$

Significant terms of series × 10^{-2}	Corresponding midordinates of (i) × 10^{-2}	Corresponding areas under curve × 10^{-2}
2·5	2·4	2·8
18·0	16·9	19·0
98	91	99
424	390	424
1,511	1,430	1,506
4,560	4,374	4,545
11,869	11,523	11,847
27,025	26,499	27,012
54,388	53,738	54,410
97,495	96,871	97,502
156,573	156,191	156,658
226,219	226,284	226,259
294,892	295,519	294,839
347,416	348,551	347,278
370,116	371,542	369,980
356,383	357,754	356,339
309,670	310,641	309,847
242,148	242,492	242,294
169,699	169,420	169,811
105,979	105,291	106,002
58,829	57,739	58,480
28,294	27,647	28,228
11,810	11,405	11,770
4,177	3,983	4,170
1,219	1,152	1,229
282	268	293
48·7	48·2	55·0
5·6	6·4	7·6

Fig. 6.



III. $\alpha = -100,\ \beta = -100,$
$\epsilon = 200,\ c = 1.$

This series is symmetrical. Its corrected moments and betas are

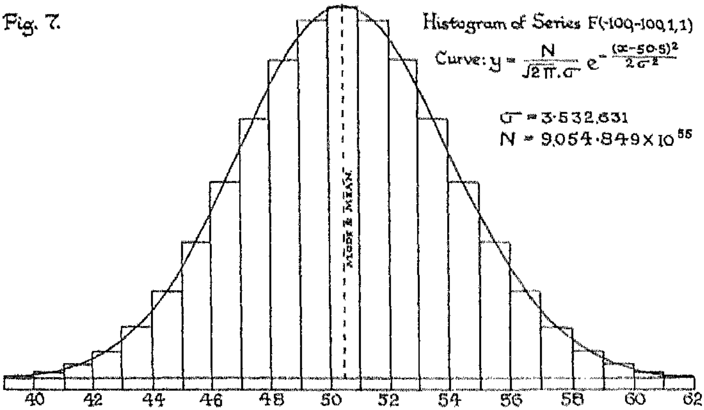
$$\mu_2 = 12{\cdot}479{,}4807,$$
$$\beta_1 = 0,$$
$$\beta_2 = 2{\cdot}989{,}9710,$$
$$N = 9054{\cdot}849 \times 10^{55}.$$

Significant terms of series $\times 10^{-55}$	Corresponding midordinates of normal curve $\times 10^{-55}$	Corresponding areas $\times 10^{-55}$
1017·906	1022·570	1018·286
979·380	982·400	979·825
868·746	871·141	859·173
712·564	713·000	712·344
539·799	538·627	539·081
377·591	375·569	376·852
243·821	241·705	243·237
145·287	143·577	144·966
79·856	78·721	79·806
40·467	39·837	40·570
18·896	18·607	19·042
8·126	8·022	8·256
3·215	3·192	3·305
1·170	1·172	1·256
·391	·397	·418
·120	·124	·130
·034	·036	·039
·009	·009	·010
·002	·002	·003

Mean = mode = 50·5, measured from the start of the histogram. β_2 is sufficiently near 3 to justify our fitting the normal curve $y = \frac{N}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$, $\sigma = 3{\cdot}532{,}631$.

Since the series is symmetrical, the terms after the mode only will be given.

Fig. 7.



IV. $\alpha = \quad 1, \ \beta = -60,$

$\gamma = -65, \ c = \quad 1.$

The hypergeometric series having the above values for its constants is very abrupt, the maximum term being the first. It is necessary, therefore, to apply abruptness corrections to the moments. The corrected moments about the start of the histogram are

$$\mu_1' = \quad 9{\cdot}063{,}5495,$$

$$\mu_2' = \ 143{\cdot}738{,}0331,$$

$$\mu_3' = 3038{\cdot}952{,}2009,$$

and $N = 11.$

The best fit is obtained by fixing the start and fitting a Type I curve by equating the first three moments about the stump. This curve is found to be

$$y = y_0 x^{-{\cdot}000{,}1138} (63{\cdot}349{,}729 - x)^{4{\cdot}966{,}8238} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

This is sufficiently close to the curve

$$y = y_0' \, (63{\cdot}349{,}729 - x)^{4{\cdot}966{,}8238} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$$\log y_0 = \overline{9}{\cdot}028{,}4019$$

for practical purposes.

The area up to a point s is given by the following relation:

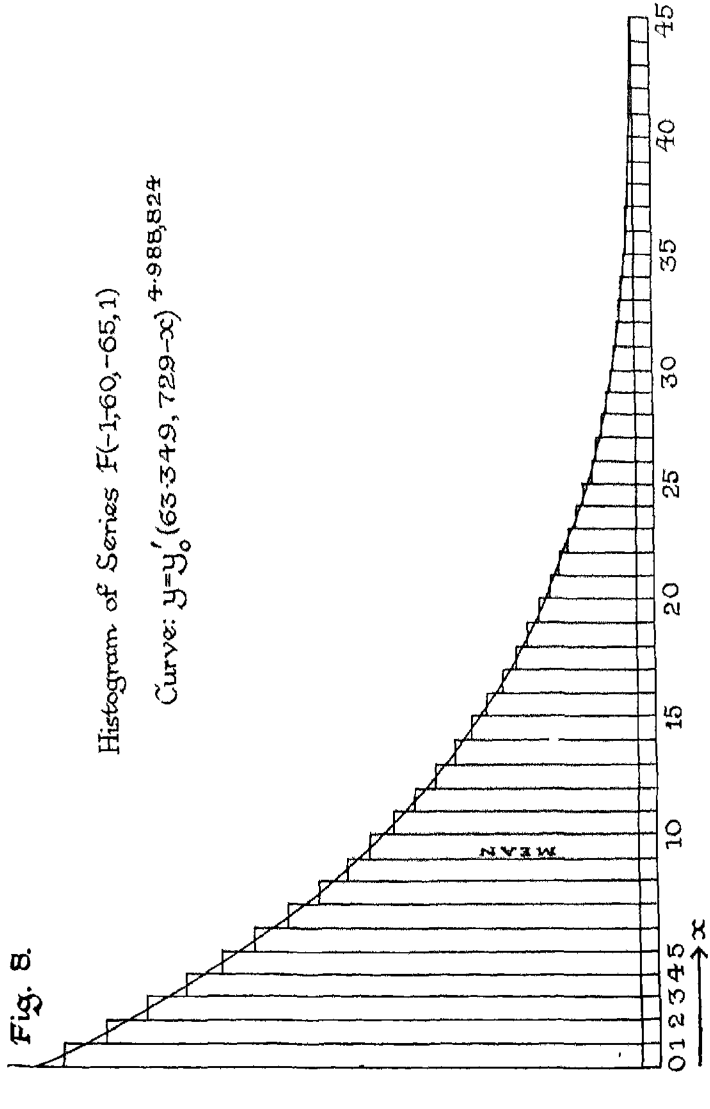
$$N\alpha_s = y_0 \int_0^s (b-x)^p \, dx = \frac{y_0}{p+1} \left[b^{p+1} - (b-s)^{p+1} \right].$$

Terms of series	Midordinates of curve	Areas under curve	Terms of series	Midordinates of curve	Areas under curve
1·000,000	·999,587	·999,980	·033,688	·033,661	·033,690
·923,077	·922,722	·922,922	·028,734	·028,709	·028,733
·850,962	·850,655	·850,855	·024,380	·024,357	·024,381
·783,425	·783,163	·783,344	·020,571	·020,549	·020,569
·720,246	·720,021	·720,194	·017,253	·017,233	·017,252
·661,209	·661,018	·661,183	·014,377	·014,359	·014,376
·606,108	·605,944	·606,097	·011,898	·011,882	·011,897
·554,743	·554,603	·554,748	·009,774	·009,759	·009,773
·506,921	·506,800	·506,939	·007,964	·007,951	·007,962
·462,454	·462,350	·462,479	·006,432	·006,421	·006,432
·421,163	·421,074	·421,200	·005,146	·005,136	·005,145
·383,876	·382,799	·382,914	·004,074	·004,066	·004,074
·347,424	·347,357	·347,472	·003,188	·003,182	·003,188
·314,648	·314,589	·314,690	·002,464	·002,457	·002,464
·284,394	·284,340	·284,442	·001,877	·001,873	·001,877
·256,512	·256,463	·256,557	·001,408	·001,405	·001,409
·230,861	·230,816	·230,900	·001,037	·001,035	·001,038
·207,303	·207,262	·207,346	·000,749	·000,748	·000,751
·185,709	·185,670	·185,745	·000,529	·000,528	·000,531
·165,953	·165,915	·165,987	·000,364	·000,363	·000,365
·147,915	·147,878	·147,944	·000,243	·000,243	·000,244
·131,480	·131,444	·131,507	·000,156	·000,156	·000,157
·116,539	·116,504	·116,562	·000,096	·000,096	·000,097
·102,988	·102,954	·103,007	·000,056	·000,057	·000,057
·090,727	·090,694	·090,745	·000,031	·000,031	·000,031
·079,663	·079,631	·079,675	·000,015	·000,016	·000,016
·069,706	·069,673	·069,716	·000,007	·000,007	·000,007
·060,769	·060,738	·060,778	·000,003	·000,003	·000,003
·052,773	·052,743	·052,778	·000,001	·000,001	·000,001
·045,641	·045,612	·045,646	·000,000	·000,000	·000,000
·039,302	·039,275	·039,304			

With the possible exception of the first example, in which the number of significant terms is small, the goodness of fit of the Pearson curves to the above series is quite close. The fit improves rapidly as the number of significant terms of the series increases, and when this number is fairly large, or, what is equivalent, when the standard deviation of the series is fairly large, the agreement between the terms of the series and the corresponding areas under the Pearson curve fitted to it by moments is sufficiently close to justify our replacing the Probability Integral of the series by that of the Pearson curve.

In the first example (p. 314), where the fit is not very close, the standard deviation is small, being approximately 2. In the following two examples the standard deviations of the series concerned lie between 3 and 3·5 and the corresponding fit has improved appreciably, giving an accuracy of three to four figures in the areas. This in itself is sufficiently close for most statistical purposes. In

Fig. 8.


Histogram of Series F(-1,-60,-65,1)
Curve: y=y₀'(63·349, 729-x) 4·988,824
MEAN
0 1 2 3 4 5
10
15
20
25
30
35
40
45
x

the last example, the standard deviation is larger, being approximately 7·9. The resulting fit, with the exception of the extreme tail, is surprisingly close.

The number of significant terms of the hypergeometric series is approximately equal to six times the standard deviation. For series with less than 24 significant terms—corresponding to S.D.'s less than 4—one would not in actual practice go into the labour of fitting a Pearson curve in order to determine an approximation to the sum of a number of terms because this sum can be obtained accurately and quite readily by calculating the terms directly. It is doubtful, however, whether such a procedure would be adopted if the number of terms is greater than 30. The total sum of the series is, of course, known. Hence, the probable limit where one would calculate the terms directly is a series with 60 significant terms, corresponding to a standard deviation of about 10. For such series, the appropriate Pearson curve should give at least as good a fit as in the previous example (Ex. IV). For instance, the Pearson curve fitted by moments to the series $F(1, -60, -65, 1)$ is (p. 318)

$$y = y_0 x^{-\cdot 0001,138} (63\cdot 349,729 - x)^{4\cdot 988,8238} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{i})\,bis.$$

The ratio of the sum of the remainder of the series after the ninth term, to the total sum of the series, is ·399,3917.

The approximate value of this ratio found from the curve (i) is

$$\int_9^{63\cdot 349,729} y\,dx \Big/ \int_0^{63\cdot 349,729} y\,dx.$$

This is the incomplete beta function

$$I_{\cdot 857,9315}(5\cdot 988,8238;\ \cdot 999,8862).$$

By triple interpolation into the beta function tables, using third differences, this was found to be ·399,3800.

This differs from the true value by less than unity in the fifth place.

In order to appreciate what happens when the number of significant terms is large, the goodness of fit of a Pearson curve to the series $F(10{,}000;\ 10{,}000;\ 1;\ 1)$ is examined within a short range after the mode.

Sum of series $N = \cdot 224,5596 \times 10^{6019}$.

F is symmetrical and β_2 is approximately equal to 3.

$\mu_2 = 1249\cdot 979170$ whence $\sigma = 35\cdot 356,0445$.

In the following table the terms of the series after the mode are compared with the areas under the normal curve

$$y = \frac{N}{\sqrt{2\pi}\,.\,\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

which has the same mean standard deviation and sum as the above series.

The quantities tabulated are equal to 10^{-8017} times their actual values.

Terms of series	Areas under normal curve
·253,393	·253,382
·253,292	·253,281
·252,988	·252,977
·252,488	·252,471
·251,777	·251,765
·250,872	·250,861
·249,771	·249,760
·248,475	·248,465
·246,989	·246,978
·245,316	·245,304
·243,458	·243,446
·241,422	·241,410
·239,211	·239,200
·236,831	·236,820

The true value of β_2 is 2·999906, which is slightly less than 3. This accounts for the constant deficiency of 1 in the fifth place in the areas under the normal curve. A slightly better fit is given by the Type II curve which has the same first four moments as the series. Even so, the normal curve gives sufficiently accurate results.

I am indebted to Professor Pearson for suggesting this problem to me and for his advice and criticism throughout the preparation of this paper.

THE BODY BUILD OF AMERICAN-BORN JAPANESE CHILDREN.

By P. M. SUSKI, M.D.

The better bodily development of American-born Japanese children over the children born in Japan, presumably through environmental influences, is the fact observed by many during the past few years. Ishiwara first called attention, in Japanese medical literature, to the superior height and weight of the second generation Japanese in America. A few years ago, I measured American-born young Japanese ranging in ages from 15 to 25. At the request of Dr Ishiwara, two years ago, I measured a hundred Japanese boys born in America. In the former group the height and weight only were obtained. In the latter, the stature, sitting height, iliac spine height, knee joint height and weight were measured. The ages of these boys were 19 to 22. With the measurements of the latter group, the height and weight of their parents were compared, and it has been found that the children born in America excelled their parents by about 5 % in height and about 6 % in weight. (In several instances, one or both of the parents could not be measured.)

That there is a marked difference in bodily development of the children born and raised under environments foreign to those under which the parents thrived, is found to be true by many authors. American-born descendants of immigrants differ in type from the foreign-born parents, according to Boas(1) who further asserts that, the longer the sojourn of parents in America before the children are born, the more intense the influence is felt. Bakwin and Bakwin(2) demonstrated a striking difference in the body build of infants of two groups from different social environments. Gray and Gower(3) found the height and weight of girls in private schools are better than the average. Studying the bodily growth of Chinese children in Hawaii and those in Chekiang and Kiangsu Provinces in China, Appleton(4) observes a marked influence in the manner of growth and development, such that the growth curves, in general, are more smooth and regular, the period of growth retardation comes in a later period and to a lesser degree in Hawaiian-born Chinese children than those born in China. She thinks that the acceleration of growth in the stature of Chinese boys in Hawaii which is greater than that of boys in East China is mainly due to the more rapid increase in length of lower limbs, in which the height of knee is the more active factor.

If lands foreign to the parents act upon the growth and development of their offspring, we may enquire what may be the influence of America (i.e. Southern California) upon the growth of the children of Japanese parents; this was the

question I had always borne in mind while making measurements of children in Los Angeles, born of Japanese parents.

The material consists of students of private Japanese schools, which are conducted to supplement the public school education of the city of Los Angeles. So, practically, all of the children are public school children. The age range is from 6 to 19, although the youngest and those above 18 years of age are so few in number, that I have omitted them from the tables and charts. The parents of the children are from all walks of life, from unskilled labourers to business men, manufacturers or literary and professional people. With the exception of about one per cent. of the children, who were born in Hawaii, those measured are California born.

These are the first of the series of the intended annual physical measurements of Japanese children born in America. As yet the number is not large enough to show conclusive results, but the chief reasons for the present publication of my findings are to show at least the apparent marked difference in stature, weight, chest circumference, sitting height and leg length which I found, when compared with figures available for the Japanese children in Japan, and also to request the authorities in this line of work to point out any defects in procedure in my work, and to give me suggestions for research work in directions which I may not have taken.

The children will be measured hereafter once a year at the same time of the year as far as possible. Thus far, the measurements were limited to the months of June and July. But the field will be enlarged so that other lots of children will be obtained for other months of the year. That the growth rate is not uniform throughout the year, is pointed out by Orr and Clark (5), Zeiner-Hendriksen (6), Nylin (7) and others. Hejinian and Hatt (8) made monthly measurements of stem-stature index. Sumner and Whitacre (9) followed monthly change in weight of Texas school children. Iowa Child Welfare Research Station (10) reports various measurements made monthly of young children from 3 to 6 years of age. It is usual for the investigators to take measurements weekly or monthly in the case of infants or small children. The figures obtained through these frequent measurements, if computed individually, would certainly reveal the seasonal variation in growth and development of children. But the purpose of my present investigation concerns the annual growth rate. I have planned the examination of all children only once a year, and at the same season of the year for each group of children, so that the seasonal variation in growth would not interfere with results.

The time of measurement was in the afternoon for one school and in the forenoon for others. Martin (11) recommends about 10 A.M. as the ideal time of measurement. Nakadate found the maximum diurnal variation in stature of children of 10 to 15 years of age to be 1·66 cm. for boys, and 1·52 cm. for girls. He finds the maximum in stature to be immediately after rising in the morning, and it drops to the lower level around 11.30 A.M.; from then on till 8 P.M. there is a very small amount of change (12).

The nearest birthday system is employed in classifying the children into different age groups; i.e., the children from 6 years 6 months to 7 years 6 months of age are grouped as 7 years, and so on. When making comparison with figures obtained from Japan, it was necessary to make allowance of a half year, because of the fact that many Japanese investigators classify a child between the 6th and 7th birthday as 7 years old, and so on.

Instruments used are: Martin's anthropometric set, consisting of a measuring rod with arms, callipers, steel tape measure and a slide compass; a box with a square flat top to measure sitting height; a standard spring scale.

I have examined the lists of measurements of Martin (11), Hrdlicka (13), the Geneva Agreement as cited by Hrdlicka (13), Yoshida (12), and Lucas and Pryor (23), and found them to contain 30 to 70 direct measurements of the living body, exclusive of functional and capacity tests. As Hrdlicka points out, the number of practicable measurements on the human body is infinite, and these measurements may be of value if taken by the same method on a sufficiently large number of individuals of various groups. Bearing these things in mind, I made a selection of physical measurements, chiefly through the suggestions of Ishiwara, namely, ten measurements, nine direct and one indirect. They were: 1. Stature, 2. Sitting height, 3. Height of anterior superior spine of ilium, 4. Leg length, 5. Knee joint height, 6. Arm span, 7. Chest circumference, 8. Intercristal width, 9. Acromial width, 10. Body weight.

The stature is measured with a child standing on a flat floor without shoes, and the head erect, keeping the eye-ear line horizontal. The left anterior superior spine of ilium is first determined by palpation, marked with a dermatograph pencil and then measured. Downes (14) reports there are over 50 % of asymmetry in the height of the iliac spine exceeding 0·5 cm. I found only a few cases of asymmetry, and these in a slight degree.

For the knee joint height, a groove between femur and tibia is easily found laterally on the level between the patellar prominence and the tibial tuberosity. By inspection from the left side of the left knee joint, one may easily locate the position of the groove, and verify it by palpation and mark it with a pencil. The distance from the floor is then determined. Martin recommends the internal groove of the knee joint, and it is most subcutaneous there and therefore easiest to measure. But I did not meet any difficulty in locating the groove on the lateral side.

The leg length has been computed from the figures of the anterior superior spine height according to Martin's formula, and is intended to measure the height of the head of femur from the floor, corresponding to Martin's "ganze Beinlänge" (the entire leg length).

The sitting height is obtained by seating a child on a box not too high so that the child can rest the feet on the floor, and measuring the height of the vertex above the seat level. Gray and Root (15) call our attention to the fact that some workers call this and other measurements under various names as stem-length, trunk-length,

rump-length, etc., with analogous terms in other languages, so that it may mean lengths from vertex, shoulder, suprasternal notch, 7th cervical vertebra or acromion down to os pubis, first or last coccyx, perineum, ischial tuberosities or gluteal lines. It is therefore necessary for one to be very careful when referring to any old literature, and it is also very important to make a clear statement as to exactly how the sitting height is obtained. The method I employed is in accord with that of the Geneva Agreement of 1912, used by Hrdlicka, Martin, v. Pirquet, Yoshida, Ishiwara and others. Dreyer(16) urges bending of the knee and pulling up slightly, instead of square sitting with both knee and hip joints at right angles. I tried Dreyer's method in a few of the children after measuring them in the usual way, but I failed to find the 3% difference as pointed out by Dreyer. In the first place, it was difficult to keep the sacrum in contact with the post, if the knees were drawn up as in Dreyer's illustration, besides in nearly all of my cases (which were young children) the ischial tuberosities were quite subcutaneous with no appreciable amount of adipose tissue intervening.

The arm span is measured with a steel tape measure, from the tip of one middle finger to the other, the arms being stretched to both sides of the body horizontally. I found it was the best way to let a child stand against the wall to take this measurement.

The chest circumference is taken at the middle of the normal inspiration and expiration, on the level above nipples in front and below the scapular angle on the back. The bicristal width is obtained with a pair of callipers, the tips of which were firmly pressed on the widest part of the pelvic crest The acromial width is measured in a similar manner, only here as the respiration, shrugging of shoulders or throwing backward or forward of the entire shoulder will give a great change in results. I exercised great care in taking the diameter at the middle of respiration and keeping the shoulders at the proper position.

The body weight is measured nude, allowing only a garment weighing 200 gm., and after emptying the bladder. It was recorded in the nearest ¼ pound, and later converted into kilograms.

The arithmetic mean, for each age sex group, of all measurements is given in the Table I, with the standard deviation and probable error of each, computed according to the formulae: (17)

$$\sigma = \sqrt{\frac{\Sigma(fd^2)}{N} - \left(\frac{\Sigma(fd)}{N}\right)^2},$$

$$\text{P.E.} = \pm\, 0{\cdot}6745\, \frac{\sigma}{\sqrt{N}}.$$

Table II gives the relative measures to the total height or stature, often called the index to stature, or ratio. The relative measures are computed as follows:

$$\text{Index} = \frac{\text{measure other than stature} \times 100}{\text{stature}}.$$

TABLE I.

Measurements of American-born Japanese Children.

Boys (cm.)		Stature (cm.)		Sitting Height (cm.)		Ant. Super. Spine Height (cm.)		Leg Length (cm.)		Knee Joint Height (cm.)		Arm Span (cm.)		Chest Circumference (cm.)		Bicristal Width (cm.)		Acromial Width (cm.)		Weight (kg.)	
n	Age	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
31	7	115·5± ·49	4·1	64·3± ·23	1·8	61·9± ·29	2·4	60·4± ·29	2·4	32·5± ·18	1·5	114·9± ·49	4·1	57·9± ·28	2·3	18·1± ·08	0·6	24·5± ·12	1·0	22·3± ·15	1·2
61	8	121·3± ·46	5·2	66·1± ·22	2·6	66·3± ·36	4·1	64·8± ·20	2·3	30·7± ·20	2·3	120·3± ·50	5·8	55·8± ·27	3·1	18·4± ·00	0·0	25·4± ·11	1·2	23·9± ·21	2·4
76	9	126·2± ·46	5·2	68·2± ·20	2·6	69·3± ·29	3·7	67·7± ·29	3·8	32·3± ·15	2·0	125·5± ·44	5·7	61·0± ·28	3·6	19·2± ·07	0·9	26·2± ·12	1·6	26·5± ·29	3·8
65	10	131·6± ·43	5·4	70·6± ·25	3·0	73·7± ·37	4·4	71·5± ·34	4·1	34·6± ·18	2·1	131·7± ·53	5·3	63·7± ·32	3·8	19·9± ·11	1·3	27·5± ·19	2·3	29·7± ·36	4·3
52	11	135·9± ·65	6·9	72·3± ·34	3·6	76·3± ·43	4·6	74·3± ·42	4·5	35·5± ·25	2·7	136·1± ·78	8·3	66·6± ·51	5·4	20·4± ·15	1·6	27·9± ·18	1·9	32·8± ·45	4·8
51	12	139·7± ·68	7·2	74·2± ·33	3·5	78·7± ·47	5·0	76·7± ·44	4·7	36·7± ·24	2·5	141·1± ·68	7·2	70·5± ·46	4·8	20·9± ·14	1·4	29·2± ·19	2·0	34·8± ·52	5·4
54	13	147·3± ·70	7·7	77·7± ·35	3·9	82·9± ·44	4·8	80·5± ·41	4·5	39·7± ·27	2·9	149·3± ·78	8·5	71·9± ·43	4·7	22·0± ·16	1·8	30·5± ·14	1·5	39·5± ·54	5·8
37	14	155·6± ·79	7·2	81·9± ·46	4·2	87·8± ·46	4·1	85·0± ·45	4·0	41·6± ·28	2·6	158·7± ·99	8·9	77·1± ·56	5·1	23·9± ·23	2·1	32·5± ·26	2·4	46·3± ·78	7·0
32	15	161·3± ·92	7·7	83·9± ·45	3·8	90·4± ·47	3·9	87·2± ·44	3·7	42·8± ·29	2·4	163·6± ·90	7·2	79·7± ·48	4·0	24·3± ·22	1·8	33·3± ·24	2·0	50·0± ·69	5·8
26	16	161·4± ·59	4·5	85·6± ·34	2·5	90·7± ·51	3·9	87·6± ·51	3·8	42·9± ·33	2·5	164·1± ·74	5·6	80·7± ·43	3·3	24·6± ·20	1·5	34·5± ·26	2·0	53·0± ·78	5·9
13	17	161·4±1·23	4·6	86·6± ·48	2·6	89·9± ·72	3·8	86·8± ·73	3·9	42·4± ·39	2·1	165·7±1·25	6·7	85·1±1·28	6·8	24·9± ·30	1·6	34·7± ·29	1·6	56·8±1·46	7·8
Girls																					
56	7	115·6± ·40	4·5	63·9± ·24	2·7	62·4± ·27	3·0	60·9± ·27	3·0	29·5± ·14	1·5	113·0± ·53	5·8	57·2± ·27	3·0	17·9± ·00	0·0	24·8± ·11	1·2	21·1± ·25	2·8
89	8	119·4± ·47	5·3	65·6± ·21	2·9	65·2± ·28	3·9	63·7± ·26	3·6	30·7± ·30	2·1	118·4± ·46	6·5	58·8± ·22	3·1	18·9± ·08	1·2	25·5± ·12	1·6	22·6± ·23	3·3
82	9	124·1± ·40	5·4	67·3± ·24	3·2	68·0± ·31	4·1	66·3± ·30	4·1	31·9± ·15	2·0	122·1± ·45	6·0	60·4± ·22	3·0	19·4± ·08	1·0	26·4± ·13	1·8	24·6± ·25	3·3
74	10	130·4± ·44	3·6	69·8± ·24	3·0	72·9± ·28	3·6	71·1± ·27	3·4	33·9± ·15	1·9	130·9± ·58	7·4	62·5± ·29	3·7	20·1± ·11	1·4	27·4± ·13	1·7	28·2± ·31	4·0
61	11	135·4± ·55	6·3	72·4± ·31	3·6	76·1± ·36	4·2	74·2± ·36	4·1	35·5± ·21	2·4	136·0± ·61	7·1	64·8± ·44	5·0	20·9± ·13	1·5	28·5± ·13	1·5	30·2± ·42	4·9
54	12	143·2± ·54	5·8	76·3± ·37	4·0	80·4± ·30	3·3	78·3± ·29	3·1	37·7± ·19	2·1	144·7± ·61	6·7	70·3± ·59	6·4	22·6± ·14	1·5	30·6± ·17	1·9	38·1± ·82	8·9
58	13	146·0± ·44	4·9	78·3± ·29	3·3	81·3± ·31	3·6	79·2± ·30	3·4	37·9± ·18	2·0	147·5± ·53	6·0	72·4± ·42	4·7	23·3± ·13	1·5	31·3± ·15	1·7	39·8± ·50	5·7
49	14	149·4± ·41	4·2	81·1± ·28	2·9	83·0± ·31	3·2	80·7± ·30	3·1	38·8± ·23	2·4	150·9± ·44	4·6	74·2± ·45	4·7	23·9± ·16	1·6	32·5± ·15	1·5	43·1± ·52	5·4
34	15	150·4± ·55	4·7	81·5± ·41	3·5	83·8± ·37	3·2	81·3± ·35	3·0	38·9± ·22	1·9	153·3± ·60	5·1	74·9± ·57	4·9	24·6± ·17	1·5	32·6± ·13	1·2	43·4± ·84	7·3
16	16	151·8± ·78	4·8	83·3± ·39	2·4	83·8± ·71	4·4	81·3± ·67	4·1	39·0± ·36	2·2	153·9±1·00	6·1	74·8± ·54	3·3	24·8± ·14	0·9	32·5± ·22	1·3	45·7± ·73	4·4

TABLE II.

American-born Japanese Children. Measurements relative to Stature.

Age Boys	Stature	Sitting Height	Ant. Sup. Spine	Leg Length	Knee Joint	Arm Span	Chest Circ.	Crist. Width	Acrom. Width
7	100	55·7	53·6	52·3	25·3	98·7	50·1	15·07	21·21
8	100	54·5	54·7	53·4	25·3	99·2	48·5	15·17	20·94
9	100	54·0	54·9	53·7	25·6	99·4	49·0	15·21	20·76
10	100	53·6	56·0	54·6	26·3	100·0	48·4	15·12	20·90
11	100	53·2	56·1	54·7	26·1	100·1	49·0	15·01	20·53
12	100	53·1	56·3	54·6	26·3	101·0	50·5	14·96	20·90
13	100	52·7	56·3	54·9	26·9	101·6	48·8	14·94	20·71
14	100	52·6	56·4	54·5	26·7	102·0	49·6	15·36	20·89
15	100	52·0	56·0	54·2	26·5	101·4	49·4	15·07	20·65
16	100	53·0	56·2	54·3	26·6	101·6	50·0	15·24	21·38
17	100	53·6	55·6	53·7	26·2	102·5	52·7	15·41	21·47
GIRLS									
7	100	55·3	54·0*	52·7*	25·5*	97·8	49·5	15·48	21·45*
8	100	54·9*	54·6	53·4	25·7*	99·2	49·2*	15·83*	21·36*
9	100	54·2*	54·8	53·6	25·7*	98·4	48·7	15·63*	21·27*
10	100	53·5	55·9	54·8*	26·0	100·4*	47·9	15·41*	21·01*
11	100	53·5*	56·2*	55·1*	26·2*	100·4*	47·9	15·44*	21·05*
12	100	53·3*	56·1	54·7*	26·3	101·0	49·1	15·78*	21·37*
13	100	53·6*	55·7	54·3	26·0	101·0	49·6*	15·96*	21·44*
14	100	54·3*	55·6	54·2	26·0	101·0	49·7*	16·00*	21·75*
15	100	54·2*	55·7	54·4*	25·9	101·9*	49·8*	16·36*	21·08*
16	100	54·9*	55·2	53·2	25·7	101·4	49·3	16·34*	21·41*

* The asterisk shows that in these cases the girls' figures were higher than the boys.

Tables III and IV together with Figures 1 to 10 show the American, German, Japanese, and American-born Japanese figures. The American figures

TABLE III.

Standing Height by different Authorities compared.

BOYS Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.	GIRLS Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.
7	116·7	115·0	115·5	7	116·1	113·0	115·6
8	121·9	120·0	121·3	8	121·4	118·0	119·4
9	126·8	125·0	126·2	9	125·8	123·0	124·1
10	132·0	130·0	131·6	10	130·9	128·0	130·4
11	136·3	135·0	135·9	11	136·1	133·0	135·4
12	141·0	140·0	139·7	12	143·5*	139·0	143·2*
13	146·6	145·0	147·3	13	148·4*	146·0*	146·0
14	153·6	151·0	155·6	14	152·7	153·0*	149·4
15	159·2	157·0	161·3	15	155·6	158·0*	150·4
16	165·9	—	161·4	16	156·5	160·0	151·8

* Those marked with an asterisk are instances in which girls excel boys in height.

are obtained by combining the United States Government Table of Heights and Weights and Holt's Age-Height-Weight Table (18). The German figures are from the

TABLE IV.

Body Weight by different Authors compared.

Boys Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.	Girls Age	Holt's and U.S. Gov. figures combined cm.	Camerer-v. Pirquet cm.	American-born Japanese cm.
7	22·3	23·0	22·3	7	21·7	21·0	21·1
8	24·6	25·0	23·9	8	23·9	23·0	22·6
9	27·1	27·5	26·5	9	26·0	25·0	24·6
10	29·9	30·0	29·7	10	28·7	27·0	28·2
11	32·4	32·5	32·8	11	31·6	29·0	30·2
12	35·4	35·0	34·8	12	36·2*	32·0	38·1*
13	39·3	37·5	39·5	13	40·8*	37·0	39·8*
14	44·0	41·0	46·3	14	45·1*	43·0*	43·1
15	49·4	45·0	49·9	15	48·6	48·0*	43·4
16	55·5	—	53·0	16	51·1	52·0	45·7

* Those marked with an asterisk are instances in which girls excel boys in weight.

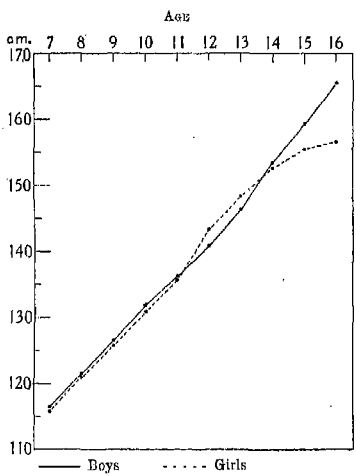


Fig. 1. Standing Height of Americans. (U.S. Government figures and Holt's combined.)

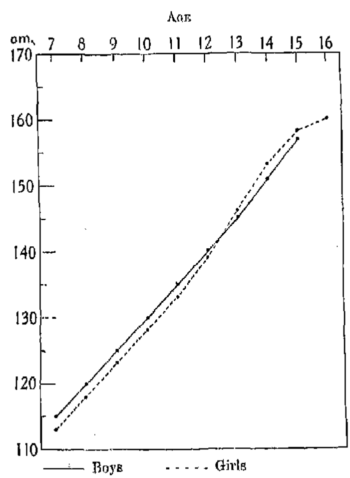


Fig. 2. Standing Height of European Children. (Camerer-v. Pirquet figures.)

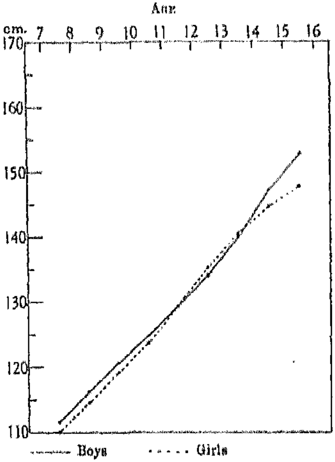


Fig. 3. Standing Height of Japanese.
(Education Ministry figures.)

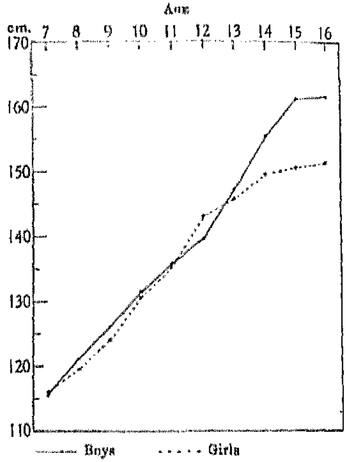


Fig. 4. Standing Height of American-born Japanese

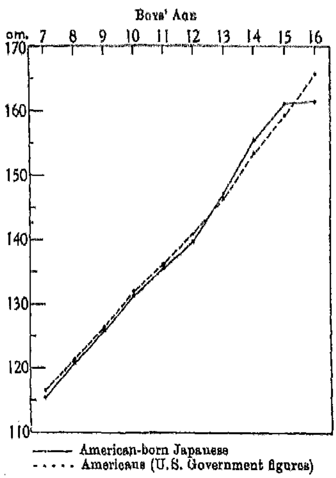


Fig. 5. Standing Height of American-born Japanese and Americans compared.

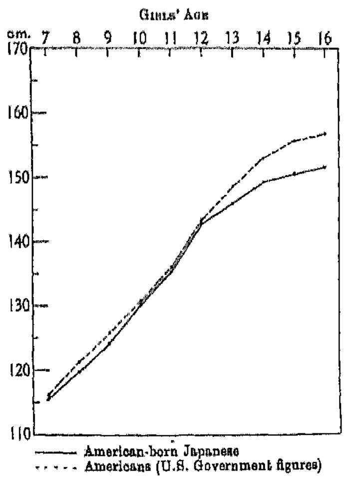


Fig. 6. Standing Height of American-born Japanese and Americans compared.

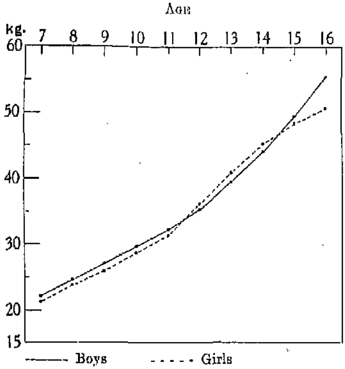


Fig. 7. Body Weight of Americans. (Holt's and U.S. Government figures combined.)

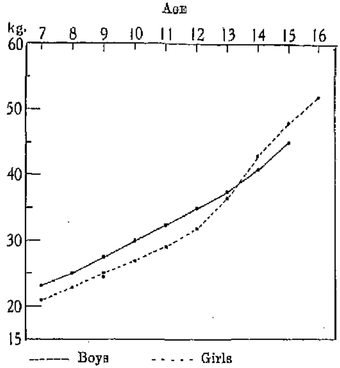


Fig. 8. Body Weight of European Children. (Camerer-v. Pirquet figures.)

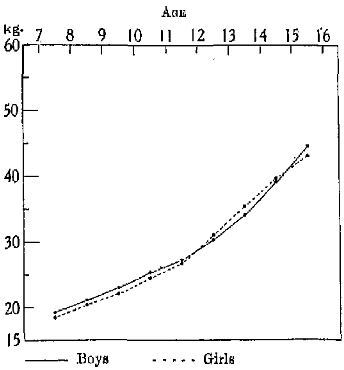


Fig. 9. Body Weight of Japanese Children. (Education Ministry figures.)

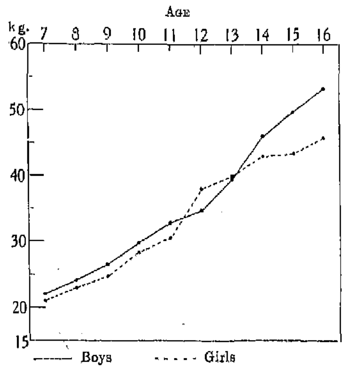


Fig. 10. Body Weight of American-born Japanese.

Age-Length-Weight Table by Camerer and v. Pirquet (19). The Japanese figures are from the table issued by the Japanese Ministry of Education (20). As seen in Figs. 5 and 6, the stature of American-born Japanese boys is very close to that of American boys at least up to 16 years of age, while there exists a wide difference between the former and that of Japan-born Japanese (see Fig. 11). The stature of American-born Japanese girls is nearly the same as that of American girls up to the age of 12, after which it drops off a little, but when compared with the figures of Japan-born Japanese girls, there is quite a space above the latter. The American-born

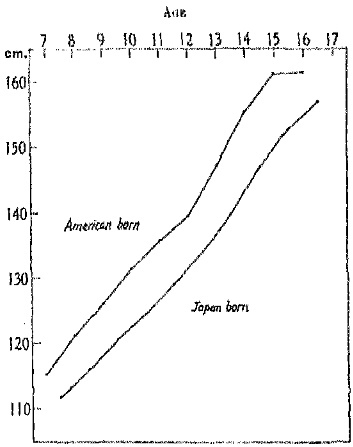


Fig. 11. Standing height of American-born Japanese Boys compared with Japan-born Japanese Boys.

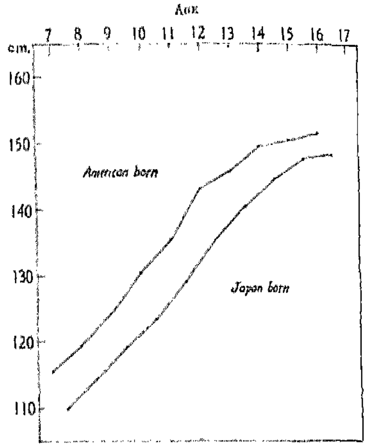


Fig. 12. Standing Height of American-born Japanese Girls compared with Japan-born Japanese Girls.

Japanese also excel their cousins in Japan in body weight and chest circumference (see Table V and Figs. 13—16).

It is a well-known fact that the children from infancy to 11 and 12 years of age, grow up without showing a marked difference with respect to sex. Between 11 and 14, corresponding to the pre-puberty period, girls excel boys in stature and weight, after which boys attain superior weight and height. Our tables and figures show these changes in all groups. But the relationship between girls' curves and boys' curves, which is of slightly different character in different groups, is best studied on the special diagram I have prepared for the purpose. First, I took the boys' height or weight at each age as a standard and figured out a percentage for girls at that age. Thus, for the stature,

$$\frac{\text{Girls' Mean Stature at given age} \times 100}{\text{Boys' Mean Stature at the same age}},$$

and called it "the percentage of girls' stature on that of boys." In the same manner, the body weights of boys and girls are compared (Tables VI and VII, Figs. 17 and 18). The boys' figures are represented by a straight line at the 100 % level, and the girls' curve runs below it while the boys' figures are greater, but when the girls' figures surpass that of boys, the curve runs above the boys' line. When I first drew this type of diagram, I showed it to Drs Lucas and Pryor, who thought it was unique and excellent in indicating the change in relationship of boys' and girls' figures so clearly. Later, however, I found Collins and Clark (21) had already devised

TABLE V.

Comparison between Japan-born and American-born Japanese Children in Stature, Chest Circumference and Body Weight.

Age	Height (cm.)		Chest Circumference (cm.)		Weight (kg.)	
Boys	Japan born	U.S. born	Japan born	U.S. born	Japan born	U.S. born
7	—	115·5	—	57·9	—	22·3
7½	111·7	—	56·1	—	19·3	—
8	—	121·3	—	58·8	—	23·9
8½	116·1	—	57·9	—	21·1	—
9	—	126·2	—	61·9	—	26·5
9½	120·6	—	60·0	—	23·1	—
10	—	131·6	—	63·7	—	29·7
10½	124·8	—	61·8	—	25·1	—
11	—	135·9	—	66·6	—	32·8
11½	129·1	—	63·9	—	27·4	—
12	—	139·7	—	70·5	—	34·8
12½	133·9	—	65·5	—	30·1	—
13	—	147·3	—	71·9	—	39·5
13½	140·0	—	68·4	—	34·0	—
14	—	155·6	—	77·1	—	46·3
14½	147·3	—	71·6	—	39·3	—
15	—	161·3	—	79·7	—	49·9
15½	152·7	—	75·1	—	44·8	—
16	—	161·4	—	80·7	—	53·0
16½	157·3	—	77·9	—	48·6	—
17	—	161·6	—	85·1	—	56·8
17½	159·4	—	79·7	—	51·2	—
Girls						
7	—	115·6	—	57·2	—	21·1
7½	110·0	—	54·2	—	18·5	—
8	—	119·4	—	58·8	—	22·6
8½	114·5	—	55·8	—	20·3	—
9	—	124·1	—	60·4	—	24·6
9½	119·1	—	57·6	—	22·2	—
10	—	130·4	—	62·5	—	28·2
10½	123·6	—	59·4	—	24·4	—
11	—	135·4	—	64·8	—	30·2
11½	129·1	—	61·8	—	27·3	—
12	—	143·2	—	70·8	—	38·1
12½	135·1	—	64·5	—	31·0	—
13	—	146·0	—	72·4	—	39·8
13½	140·4	—	67·6	—	35·2	—
14	—	149·4	—	74·2	—	43·1
14½	144·8	—	71·2	—	39·6	—
15	—	150·4	—	74·9	—	43·4
15½	147·6	—	73·9	—	43·1	—
16		151·8	—	74·8	—	45·7
16½	148·2	—	75·8	—	45·4	—

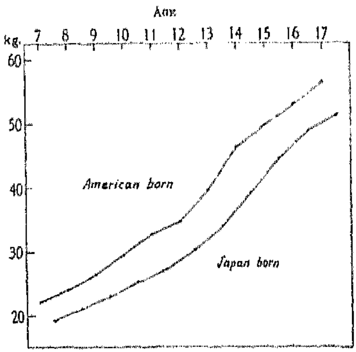


Fig. 13. Body Weight of American-born Japanese Boys compared with Japan-born Japanese Boys.

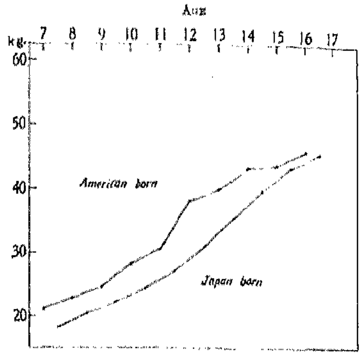


Fig. 14. Body Weight of American-born Japanese Girls compared with Japan-born Japanese Girls.

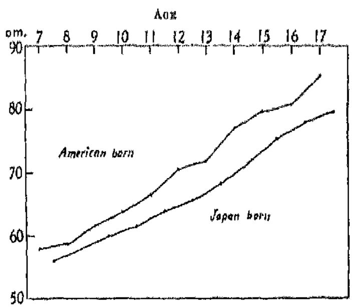


Fig. 15. Chest Circumference of American-born Japanese Boys compared with Japan-born Japanese Boys.

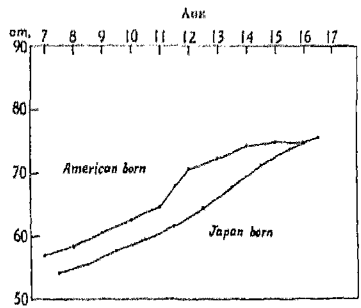


Fig. 16. Chest Circumference of American-born Japanese Girls compared with Japan-born Japanese Girls.

a somewhat similar diagram in 1929*. I have also figured out the percentage of girls' other relative values on those of boys, namely, the relative sitting height, relative iliac spine height, relative leg length, relative knee joint height, relative arm span, relative chest circumference, relative bicristal width, relative acromial width. Among these, I could only make comparison with values of Japanese in relative sitting height, relative leg length, and relative chest circumference, apart from stature and weight (see Tables VI, VII and VIII, also Figs. 17, 18 and Figs. 19 *b*—26 *b*).

* [This seems merely a diagram of the customary sex-ratio multiplied by 100; the sex-ratio is very familiar to anthropometricians and craniometricians. Ed.]

TABLE VI.

The Percentage of Girls' Standing Height to that of Boys.

Age	U.S. Gov. and Holt's Figures combined Americans	Camerer-v. Pirquet Germans	American-born Japanese	Education Ministry Japanese
7	99·5	98·3	100·1	98·5
8	99·6	98·3	98·4	98·6
9	99·2	98·4	98·3	98·8
10	99·2	98·5	99·1	99·0
11	99·9	98·5	99·6	100·0
12	101·9	99·3	102·5	100·9
13	101·4	100·3	99·1	100·2
14	99·4	101·3	96·0	98·3
15	97·7	100·6	93·2	96·7
16	94·4	—	94·0	94·2

TABLE VII.

The Percentage of Girls' Weight to that of Boys.

Age	U.S. Gov. and Holt's Figures combined Americans	Camerer-v. Pirquet Germans	American-born Japanese	Education Ministry Japanese
7	97·1	91·3	94·6	95·9
8	98·3	92·0	94·6	96·2
9	96·2	90·9	92·8	96·1
10	96·0	90·0	95·8	97·2
11	97·4	89·2	92·1	99·6
12	102·3	91·4	109·5	103·0
13	104·0	98·7	100·8	103·5
14	102·4	104·9	93·1	100·8
15	98·4	106·7	87·0	96·2
16	91·9	—	86·0	93·4

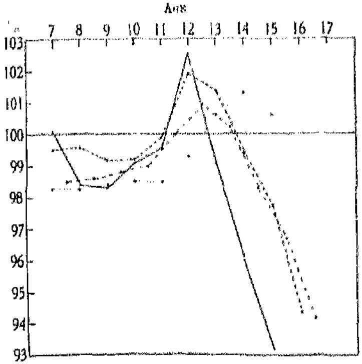


Fig. 17. The Percentage of Girls' Standing Height to that of Boys.

This diagram shows percentage values of girls' standing height, taking the boys' standing height as the standard or 100% for each age. The figures published by the Ministry of Education of Japan are based upon age groups taken from one birthday to the next. For this reason, the points are marked on the diagram half-way between ages.

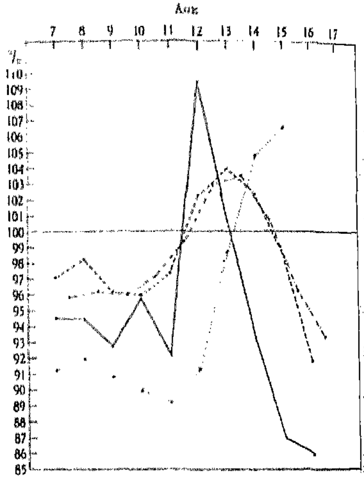


Fig. 18. The Percentage of Girls' Body Weight to that of Boys.

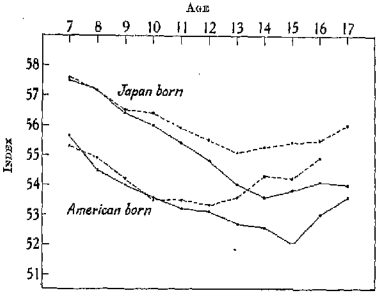
·········· German (Camerer-v. Pirquet)
- - - - - American (U.S. Government-Holt's)
-·-·-·- Japanese (Education Ministry)
——— Japanese (American-born)

TABLE VIII. *Comparison of Relative Sitting Heights of Japan-born and American-born Japanese Children.*

Age	Japanese in Japan (Figures of Drs Tsurumi, Nakadate, Yagi and Toyoda combined)		American-born Japanese	
	Boys	Girls	Boys	Girls
7	57·5	57·6*	55·7	55·3
8	57·2	57·2	54·5	54·9*
9	56·4	56·5*	54·0	54·2*
10	56·0	56·4*	53·6	53·5
11	55·4	55·9*	53·2	53·5*
12	54·8	55·5*	53·1	53·3*
13	54·0	55·1*	52·7	53·6*
14	53·6	55·3*	52·6	54·3*
15	53·8	55·4*	52·0	54·2*
16	54·1	55·5*	53·0	54·9*
17	54·0	56·0*	53·6	—

* Those marked with an asterisk are instances in which girls excel boys in sitting height.

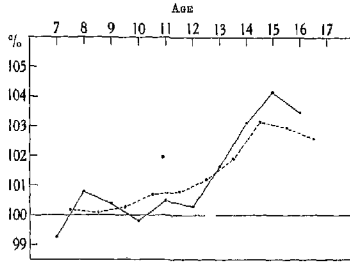
Relative Sitting Height of American-born Japanese and Japan-born Japanese compared



—— Boys ----- Girls

Fig. 19 *a*.

Percentage of Girls' Relative Sitting Height to that of Boys

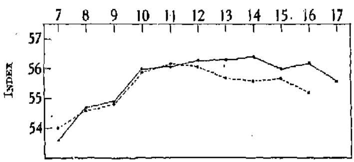


—— American-born ----- Japan-born

Fig. 19 *b*.

Relative Height of Anterior Superior Iliac Spine

$$\frac{\text{Iliac Spine Height}}{\text{Stature}} \times 100$$



—— Boys ----- Girls

Fig. 20 *a*.

Percentage of Girls' Relative Height of Anterior Superior Iliac Spine to that of Boys

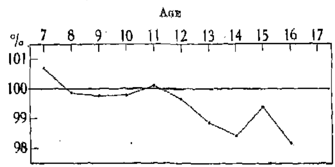
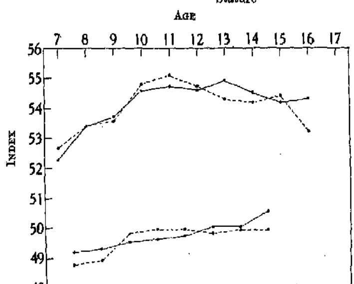


Fig. 20 *b*.

American-born Japanese.

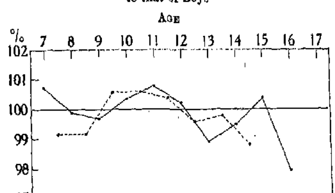
Relative Leg Length, or $\frac{\text{Leg Length}}{\text{Stature}} \times 100$



—— Boys ----- Girls

Fig. 21 *a*.

Percentage of Girls' Relative Leg Length to that of Boys



—— American-born Japanese
----- Japan-born Japanese

(These figures are measurements of Drs Mishima and Minayoshi combined.)

Fig. 21 *b*.

American-born Japanese and Japan-born Japanese.

Relative Knee Joint Height, or $\frac{\text{Knee Joint Height}}{\text{Stature}} \times 100$

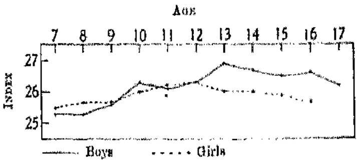


Fig. 22 *a*.

Percentage of Girls' Relative Knee Joint Height to that of Boys

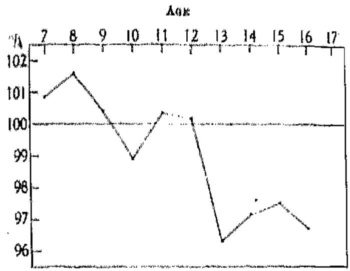


Fig. 22 *b*.

American-born Japanese.

Relative Arm Span, or $\frac{\text{Arm Span}}{\text{Stature}} \times 100$

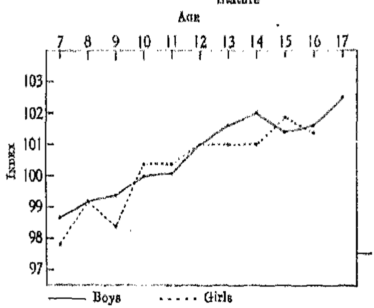


Fig. 23 *a*.

Percentage of Girls' Relative Arm Span to that of Boys.

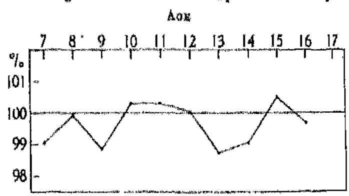
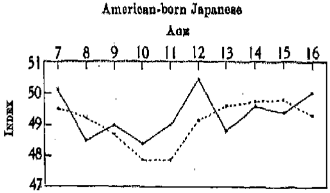


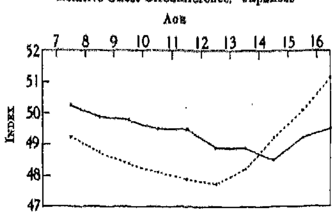
Fig. 23 *b*.

American-born Japanese.

Relative Chest Circumference.
American-born Japanese



Relative Chest Circumference, Japanese



—— Boys

- - - - Girls

Fig. 24 *a*.

Fig. 24 *b*.

Percentage of Girls' Relative Chest Circumference to that of Boys

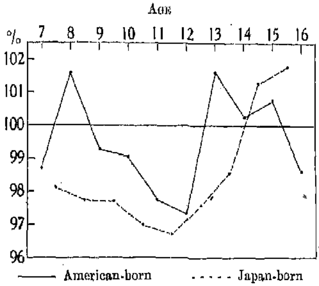


Fig. 24 c.

Relative Bicristal Width, or $\frac{\text{Bicristal Width}}{\text{Stature}} \times 100$

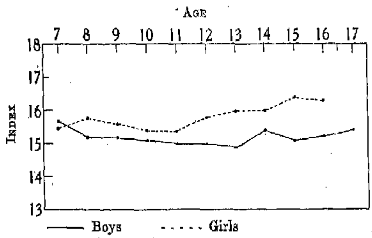


Fig. 25 a.

Percentage of Girls' Relative Bicristal Width to that of Boys

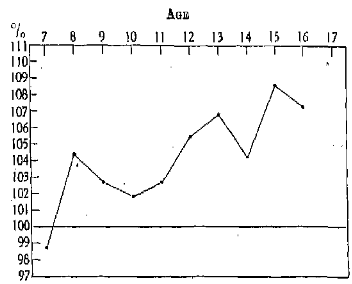


Fig. 25 b.

American-born Japanese.

Relative Acromial Width, or $\frac{\text{Acromial Width}}{\text{Stature}} \times 100$

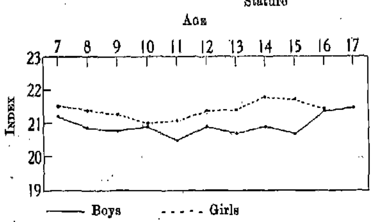


Fig. 26 a.

Percentage of Girls' Relative Acromial Width to that of Boys

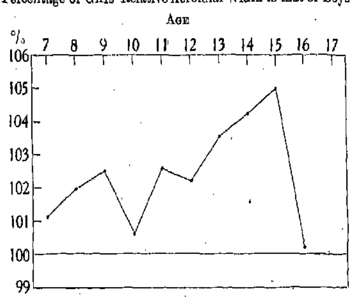


Fig. 26 b.

American-born Japanese.

The curves of the percentages of girls' stature and weight to those of boys (Tables VI, VII, Figs. 17, 18) run the same general course, whether they be for Americans, Japanese, American-born Japanese or Europeans. The curves show the pre-puberty upheaval over the boys' 100% line between the ages of 11 and 14 years in stature, and between 12 and 15 years in weight. It is noticeable that the curve of the American-born Japanese is slightly irregular on account of the smaller number of observations. Figures for Americans as well as for Japanese are based on measurements of millions; those of Germans are thought to be equally large; but those of American-born Japanese are only a little over a thousand. That the curves of the American-born Japanese girls show a sudden and rapid drop after reaching a maximum at 12 years, may be explained as the general poor physical condition of the girls of 13 years or more. After making measurements of the same group of children a number of times in the future, I may be able to offer a better explanation. The curves of the German girls' acceleration tend to be delayed for about 2 years. This is probably caused by the delay in puberty on account of higher latitude and colder climate.

The relative sitting height of American-born Japanese is remarkably low when compared with that of Japan-borns. It is known that this index is highest at birth and goes down with age until it reaches the lowest level at a definite age and then again goes up a little. Bean (22) calls this index a "skeletal index" and finds it inverse in stature (that is smaller in taller people than in shorter). He found Whites and Filipinos to have about the same index and Negroes smaller. The skeletal index, according to Bean, reaches the minimal level between ages 10 to 16 and 9 to 14 in American boys and girls respectively, and between 8 to 10 and 8 to 12 in German-American boys and girls respectively. The minimal level for American-born Japanese boys is reached at 15, and the same for Japan-born Japanese boys at 14, the minimum for girls being reached a year or two earlier in either case. (Table VIII and Fig. 19 *a*.)

The percentage of girls' relative sitting height is nearly at 100% level at 7 years but it rises gradually till about 15 years of age, to the level between 103 and 104 in both the American-born and Japan-born Japanese. (Table IX and Fig. 19 *b*.)

The relative height of anterior superior iliac spine goes up from about 54 at 7 years to around 56 at 10 years of age, from then on there is no marked change, only the girls' figures drop somewhat after 13 years of age.

The percentage of girls' relative height of anterior superior iliac spine to that of boys is nearly at 100% mark up to 11 years, then it drops off gradually. (See Table IX and Fig. 20 *b*.)

The relative leg length of American-born Japanese is, in general, greater than that of Japan-born by 4·0 to 4·5, the general tendency of both being a gradual climb with age up to 10 or 11 years, after which it stays rather stationary. (Table X and Fig. 21 *a*.)

The percentage of girls' relative leg length to that of boys runs just about the

TABLE IX. *The Percentage of Girls' Relative Measures to those of Boys.*

Age	Sitting Height	Iliac Spine Height	Leg Length	Knee Joint Height	Arm Span	Chest Circumf.	Cristal Width	Acromial Width
				American-born Japanese Children				
7	99·29	100·72	100·74	100·94	99·04	98·71	98·81	101·14
8	100·82	99·91	99·87	101·59	99·99	101·59	104·35	101·99
9	100·35	99·78	99·74	100·43	98·94	99·23	102·75	102·47
10	99·78	99·82	100·36	98·88	100·31	99·02	101·93	100·55
11	100·51	100·11	100·77	100·37	100·30	97·66	102·83	102·53
12	100·32	99·66	100·24	100·21	100·04	97·28	105·49	102·23
13	101·67	98·94	98·89	96·32	98·76	101·59	106·85	103·54
14	103·13	98·46	99·48	97·14	99·03	100·23	104·15	104·15
15	104·18	99·42	100·38	97·48	100·50	100·79	108·57	104·99
16	103·47	98·23	97·96	96·66	99·72	98·55	107·19	100·16
				Japan-born Japanese Children				
7½	100·16	—	99·2	—	—	98·1	—	—
8½	100·10	—	99·2	—	—	97·8	—	—
9½	100·30	—	100·6	—	—	97·7	—	—
10½	100·70	—	100·6	—	—	97·0	—	—
11½	100·79	—	100·4	—	—	96·8	—	—
12½	101·21	—	99·6	—	—	97·6	—	—
13½	101·92	—	99·8	—	—	98·6	—	—
14½	103·15	—	99·8	—	—	101·3	—	—
15½	102·97	—	—	—	—	101·8	—	—
16½	102·62	—	—	—	—	—	—	—
	These are from the figures of Drs Tsurumi, Nakadate, Yagi and Toyoda combined		These are from the figures of Drs Mishima and Minayoshi combined			These are from the figures published by the Ministry of Education, Japan		

TABLE X. *Relative Leg Length of American-born Japanese Children compared with that of Japanese Children in Japan.*

Age	American-born Japanese		Japanese in Japan. Measurements of Drs Mishima and Minayoshi combined	
	Boys	Girls	Boys	Girls
7	52·3	52·7	49·2	48·8
8	53·4	53·4	49·3	48·9
9	53·7	53·6	49·5	49·8
10	54·6	54·8	49·6	49·9
11	54·7	55·1	49·7	49·9
12	54·6	54·7	50·0	49·8
13	54·9	54·3	50·0	49·9
14	54·5	54·2	50·5	49·9
15	54·2	54·4	—	—
16	54·3	53·2	—	—

same course in American-born as in Japan-born Japanese. The girls excel boys between 9½ and 12 years of age. (See Table IX and Fig. 21 *b*.)

The relative knee joint height of American-born Japanese shows a slight increase with age. The percentage of girls' relative knee joint height to that of boys has a downward tendency, being above the boys' level at 7 years and dropping down to about 96·5 % at 16 years of age. (See Table IX and Fig. 22 *b*.)

The relative arm span of American-born Japanese climbs up from 98 to 102 in the course of years 7 to 17. The values for boys and girls are nearly the same. (See Table IX and Figs. 23 *a*—*b*.)

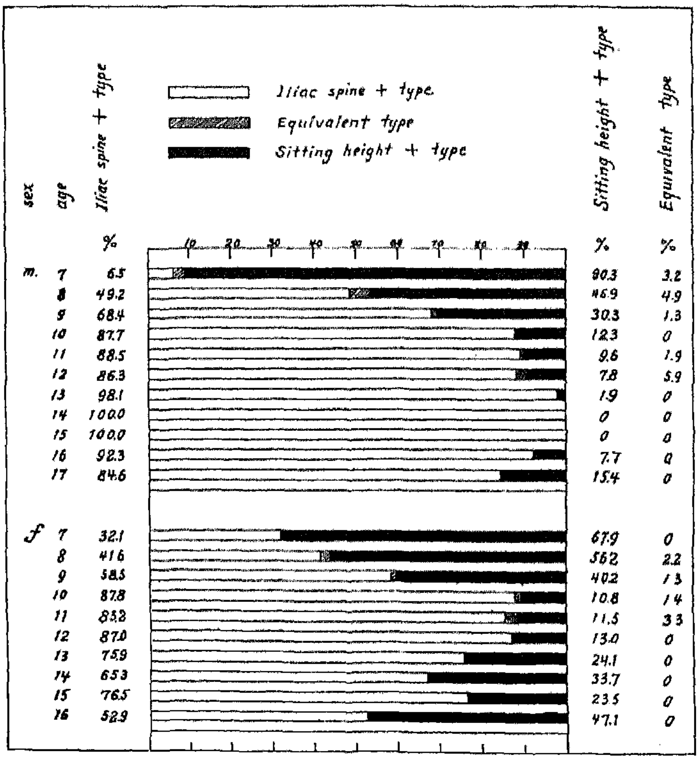


Fig. 27.

The relative chest circumference takes a course first down and then up, the lowest level being at 10 to 11 years in the case of American-born and at about the 13th year in case of Japan-born Japanese. There seems to be a greater difference between boys' and girls' figures in Japan-born Japanese than those of American-born Japanese. (Table IX and Fig. 24 *c*.)

The relative bicristal width and the relative acromial width of American-born Japanese are kept almost at the same level all the way through the ages 7 to 17. The percentage of girls' relative values to those of boys is always above the boys' line, and the general tendency is a steady climb. (Table IX and Figs. 25 *b*, 26 *b*.)

TABLE XI.

American Native White Stock (Third Generation Native-born).

Collins and Clark, *U.S. Public Health Reports* 44; 18, 1059, 1929.

Age	Standing Height (cm.)		Sitting Height (cm.)		Chest Circumf. (cm.)		Weight (kg.)	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
7	119·5	118·5	63·8	63·3	59·3	58·1	22·6	22·1
8	125·1	124·3	66·0	65·5	60·9	59·1	25·1	24·3
9	130·2	129·4	68·0	67·5	62·5	61·3	27·6	27·1
10	135·1	134·5	69·8	69·5	64·3	63·2	30·4	29·8
11	139·9	140·1*	71·5	71·9*	66·2	65·8	33·3	33·7*
12	144·4	146·0*	73·3	74·4*	68·1	68·2*	36·4	37·8*
13	150·0	152·0	75·5	77·6*	70·7	71·1*	40·5	43·0*
14	155·5	155·5	78·2	79·6*	73·6	73·3	45·1	46·8*
15	158·9	157·0	79·8	80·8*	74·4	73·7	47·3	49·5*
Actual Mean Annual Increment								
7— 8	5·6	5·8*	2·2	2·2	1·6	1·3	2·5	2·2
8— 9	5·1	5·1	2·0	2·0	1·6	1·9*	2·5	2·8*
9—10	4·9	5·1*	1·8	2·0*	1·8	1·9*	2·8	2·7
10—11	4·8	5·6*	1·7	2·4*	1·9	2·6*	2·9	3·9*
11—12	4·5	5·9*	1·8	2·5*	1·9	2·4*	3·1	4·1*
12—13	5·6	6·0*	2·2	3·2*	2·6	2·9*	4·1	5·2*
13—14	5·6	3·5	2·7	2·0	2·9	2·2	4·6	3·8
14—15	3·4	1·5	1·6	1·2	0·8	0·4	2·2	2·7*
Percentage Increase per year								
7— 8	4·72	4·87*	3·47	3·55*	2·84	2·25	11·03	9·98
8— 9	4·08	4·11*	3·05	2·97	2·53	3·28*	9·96	11·57*
9—10	3·76	3·94*	2·65	2·97*	2·85	3·05*	10·05	10·27*
10—11	3·54	4·20*	2·45	3·53*	2·98	4·17*	9·77	13·02*
11—12	3·19	4·19*	2·48	3·51*	2·98	3·67*	9·09	12·17*
12—13	3·92	4·13*	3·11	4·23*	3·65	4·25*	11·51	13·60*
13—14	3·67	2·28	3·55	2·58	4·18	2·98	11·37	8·83

* Those marked with an asterisk are instances in which girls excel boys.

The sitting height and the anterior superior spine height are nearly the same in some children, but not in all. A child may be either one of these, i.e., these measurements are nearly equal, or the iliac spine height is greater, or the sitting height is greater. Whether these types, if they may be so called, have something to do with body types of children, is yet to be determined. For the present, I will just divide the boys and girls of the American-born Japanese into these three types, viz.:

1. Iliac spine plus type. (Where the iliac spine height is greater than the sitting height.)

2. Equivalent type. (Where they are equal within 0·5 cm.)

3. Sitting height plus type. (Where the sitting height is greater than the iliac spine height.)

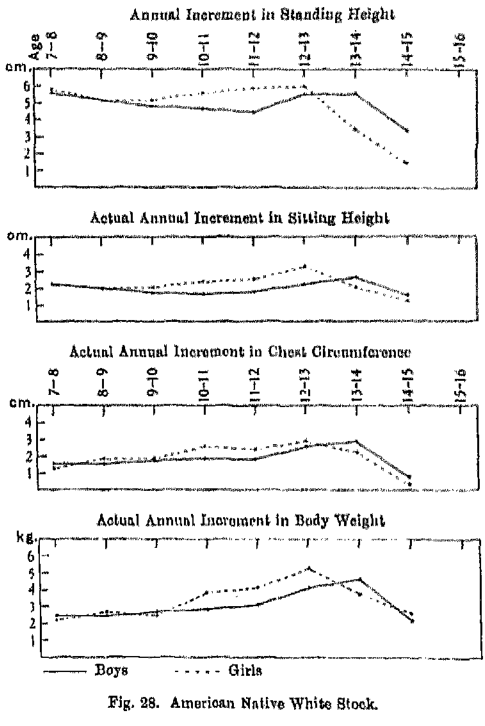


Fig. 28. American Native White Stock.

TABLE XII. *American-born Japanese (Second Generation).*

Age	Standing Height cms.		Sitting Height cms.		Chest Circumf. cms.		Weight kg.	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
7	115·5	115·6*	64·3	63·9	57·9	57·2	22·3	21·1
8	121·3	119·4	66·1	65·6	55·8	58·8*	23·9	22·6
9	126·2	124·1	68·2	67·3	61·0	60·4	26·5	24·6
10	131·6	130·4	70·6	69·8	63·7	62·5	29·7	26·2
11	135·9	135·4	72·3	72·4*	66·6	64·8	32·8	30·2
12	139·7	143·2*	74·2	76·3*	70·5	70·3	34·8	38·1*
13	147·3	146·0	77·7	78·3*	71·9	72·4*	39·5	39·8*
14	155·6	149·4	81·9	81·1	77·1	74·2	46·3	43·1
15	161·3	150·4	83·9	81·5	79·7	74·9	50·0	43·4
				Actual Mean Annual Increment				
7—8	5·8	3·8	1·8	1·7	−2·1	1·6	1·6	1·5
8—9	4·9	4·7	2·1	1·7	5·2	1·6	2·6	2·0
9—10	5·4	6·3*	2·4	2·5*	2·7	2·1	3·2	3·6*
10—11	4·3	5·0*	1·7	2·6*	2·9	2·3	3·1	2·0
11—12	3·8	7·8*	1·9	3·9*	3·9	5·5*	2·0	7·9*
12—13	7·6	2·8	3·5	2·0	1·4	2·1*	4·7	1·7
13—14	8·3	3·4	4·2	2·8	5·2	1·8	6·8	3·3
14—15	5·7	1·0	2·0	0·4	2·6	0·7	3·7	0·3
				Percentage Increase per year				
7—8	5·03	3·23	2·80	2·66	—	2·80	7·18	7·11
8—9	4·04	3·94	3·18	2·59	9·32	2·72	10·88	8·85
9—10	4·28	5·08*	3·52	3·72*	4·43	3·48	12·07	14·63*
10—11	3·27	3·83*	2·41	3·73*	4·55	3·68	10·54	7·09
11—12	2·79	5·76*	2·63	5·39*	5·86	8·59*	6·09	26·15*
12—13	5·44	1·96	4·72	2·62	1·99	2·99*	13·51	4·46
13—14	5·64	2·33	5·41	3·58	7·23	2·49	17·22	8·29
14—15	3·66	0·67	2·44	0·49	3·37	0·94	7·99	0·70

TABLE XIII. *Actual Mean Annual Increment of other Measures than those listed in the last Table.*

Age	Anterior Iliac Sp. Height (cm.)		Leg Length (cm.)		(cm.) Knee Joint Height (cm.)		Arm Span (cm.)		Bicristal Width (cm.)		Acromial Width (kg.)	
	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls	Boys	Girls
7—8	4·4	2·8	4·4	2·8	−1·8	1·2	6·3	5·4	0·3	1·0*	0·9	0·7
8—9	3·0	2·8	2·9	2·6	1·6	1·2	5·2	3·7	0·8	0·5	0·8	0·9*
9—10	4·4	4·9*	3·8	4·8*	2·3	2·0	6·2	8·8*	0·7	0·7	1·3	1·0
10—11	2·6	3·2*	2·8	3·1*	0·9	1·6*	4·4	5·1*	0·5	0·8*	0·4	1·1*
11—12	2·4	4·3*	2·4	4·1*	1·2	2·2*	5·0	8·7*	0·5	1·7*	1·3	2·1*
12—13	4·2	0·9	3·8	0·9	3·0	0·2	8·2	2·8	1·1	0·7	1·3	0·7
13—14	4·9	1·7	4·5	1·5	1·9	0·9	9·4	3·4	1·9	0·6	2·0	1·2
14—15	2·6	0·8	2·2	0·6	1·2	0·1	4·9	2·4	0·4	0·7*	0·8	0·1
15—16	0·3	0·0	0·4	0·0	0·1	0·1	0·5	0·6*	0·3	0·2	1·2	−0·1

* Those marked with an asterisk are instances in which girls excel boys.

The percentage of the number of each type at each age and sex is given in Fig. 27. So far I have not seen reports of this nature by other investigators. I cannot therefore make any comparison or draw any conclusion. One may see,

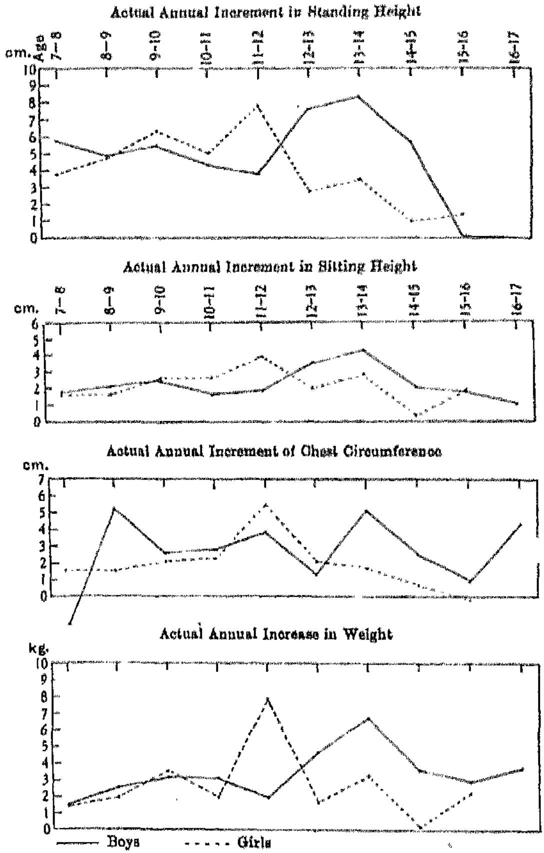


Fig. 29. American-born Japanese Boys and Girls.

however, that the maximum of iliac spine plus type in the boys is reached at ages 13 to 15, while the maximum of the same type for the girls is reached at ages 10 to 12, though less pronounced in degree. The equivalent type seems to have no significance.

Collins and Clark (21) published valuable data on American native white stock (third generation native born children), for which they calculated actual mean annual increment and annual percentage of increase, in stature, sitting height, chest circumference and body weight. (See Table XI and Fig. 28.) Girls seem to have greater actual annual increment than boys during the ages between 10 and 13. *These figures are based on measurements of 28,674 children. My figures on the*

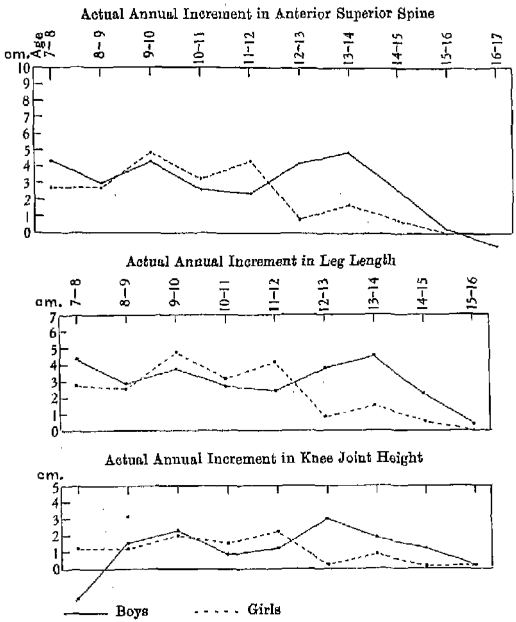


Fig. 30. American-born Japanese.

measurement of a little over 1,000 children may not be worth much, but I publish them here (Tables XII and XIII, Figs. 29, 30 and 31), simply to show that at least the girls' annual increment is decidedly greater than that of boys in all measurements, at the age of 12.

In conclusion, I have computed the index of measurements of the American-born Japanese, taking the figures of the Japan-born Japanese as a standard, thus:

$$\frac{\text{Measurement of American-born} \times 100}{\text{Measurement of Japan-born}}$$

and found that, in stature, the American-born are higher by 6 to 7 %, although girls drop to 3·5 % at the age of 14½. The relative sitting height of American-born is below that of Japan-born by 4 % up to 11½ years, after which it climbs up to about −2 %. The relative leg length of American-born is more than 6 % greater at 7 years than that of Japan-born, and rises to 10 % at 11½ years.

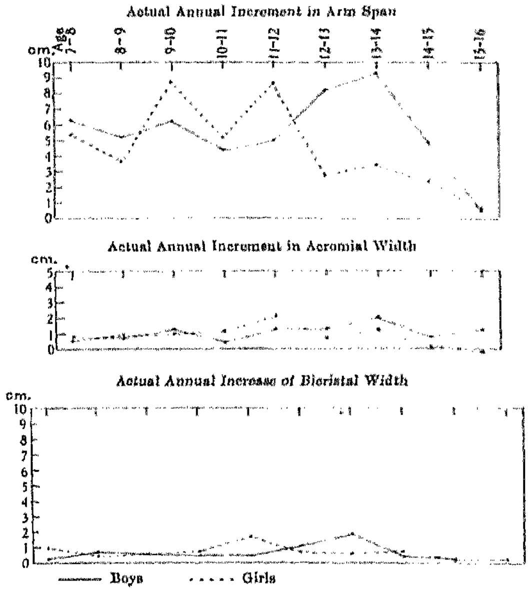


Fig. 31. American-born Japanese

The chest circumference of American-born climbs from 1 % plus to 9 % above that of Japan-born at the age of 14½ in case of boys, while the girls' figures are generally higher, i.e., rise from 7 % at 7 years to over 10 % at 12½ years, then drop to 5 % at 14½ years.

The body weight of American-born ranges from 19 to 26 % above that of Japan-born in the case of boys. The girls' figures are lower than this by 2 to 3 %, but the lowest level at 14½ years is still over 9 % above that of Japan-born. (Table XIV and Figs. 32, 33.)

Taken as a whole, the difference in body build of the American-born Japanese

TABLE XIV.

Index, Values of American-born Japanese, with Values of Japan-born Japanese Children taken as a Standard.

The figures of the American-born Japanese are increased by ½ annual increment in order to raise to the age of the Japanese which is year + ½.

Age	Stature	Rel. Sit.	Rel. Leg	Chest Circ.	Weight
BOYS					
7½	106·00	96·87	106·30	101·25	119·69
8½	106·63	95·28	108·32	100·86	119·43
9½	106·88	95·75	108·48	104·00	121·64
10½	107·13	95·71	110·08	105·50	124·70
11½	106·74	96·03	110·06	106·42	123·36
12½	107·17	96·90	109·45	108·70	123·59
13½	108·22	97·59	112·35	108·92	126·18
14½	107·54	98·14	109·00	109·50	122·65
Average	7·04	− 3·47	9·26	5·64	22·66
GIRLS					
7½	106·82	96·01	108·00	107·01	118·38
8½	106·38	95·98	109·20	106·81	116·26
9½	106·89	95·93	107·63	106·77	118·92
10½	107·20	94·86	109·82	107·24	119·67
11½	107·90	95·71	110·42	109·38	125·28
12½	107·03	96·04	109·84	110·70	125·48
13½	105·20	97·28	108·82	108·44	117·90
14½	103·52	98·19	108·62	104·77	109·34
Average	6·37	− 3·75	9·04	7·64	18·90
Average for both sexes	6·7	− 3·6	9·15	6·64	20·8

(the so-called second generation Japanese, as their parents are of Japan birth) from that of Japan-born children is, in round numbers, 7 % higher in stature, 4 % shorter in relative sitting height, 9 % greater in relative leg length, 7 % greater in chest circumference, and 20 % heavier in weight.

Both parents of these children were full-blooded Japanese born in Japan in all instances. There were only a few pupils of mixed Japanese and Caucasian parentage in the two schools where measurements were made, and they were not included in the present investigation. Even here in America, the intermarriage of Japanese with other races is very rare, as the Japanese community is fairly large and Eurasian children are unusual.

The children in this investigation are certainly growing faster, that is taller and heavier, age for age, in comparison with children in Japan. It seems very likely that they will grow up to be taller and heavier adults than native born Japanese, inferring from the height and weight attained by these children at 16 or 17 years of age, which were already greater than those of adult Japanese.

A few years ago, I measured a hundred young Japanese of American birth whose ages were 19 years and upwards, and found them to surpass their fathers in height by an average of more than 3 inches. When I have measured the children of the present study annually for a few years, I expect to be able to report more definitely what their ultimate stature and weight will be.

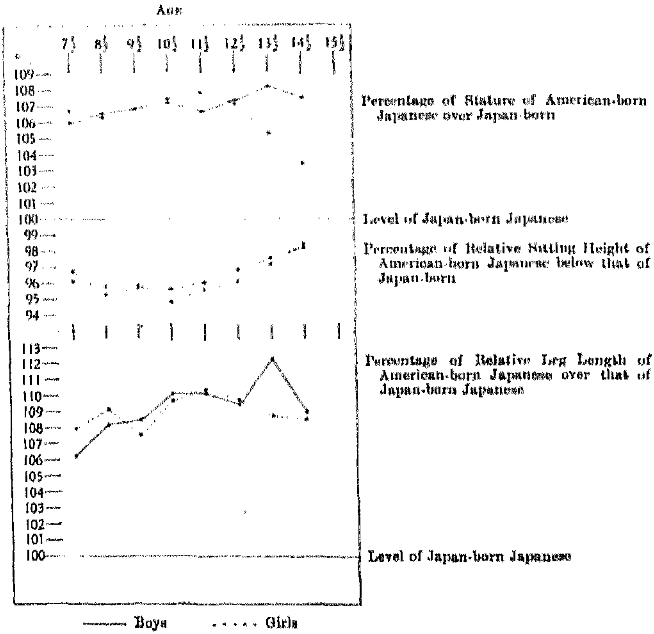


Fig. 32.

Prof. K. Pearson tells me in a personal communication of his experience with two tall Japanese, who said their stature was quite usual in the case of the Japanese island from which they came. I am not aware of any island in Japan which is noted for a higher stature of its inhabitants. As far as I know, the people of Japan cannot be divided into racial groups, with the exception of the Ainus in the extreme north. The latter are not included in the present study. Parents of the children in this work come from all parts of Japan, and they may be regarded as representing the pure Japanese race. The Japanese Government measurements were performed under the supervision of Dr Yoshida, who follows Martin's method as I do. The

Government measured the school children from all over Japan, that is Japan proper, where no different races figured, annually for a number of years. Therefore I think I am justified in making a comparison between my measurements and those of the Japanese Government.

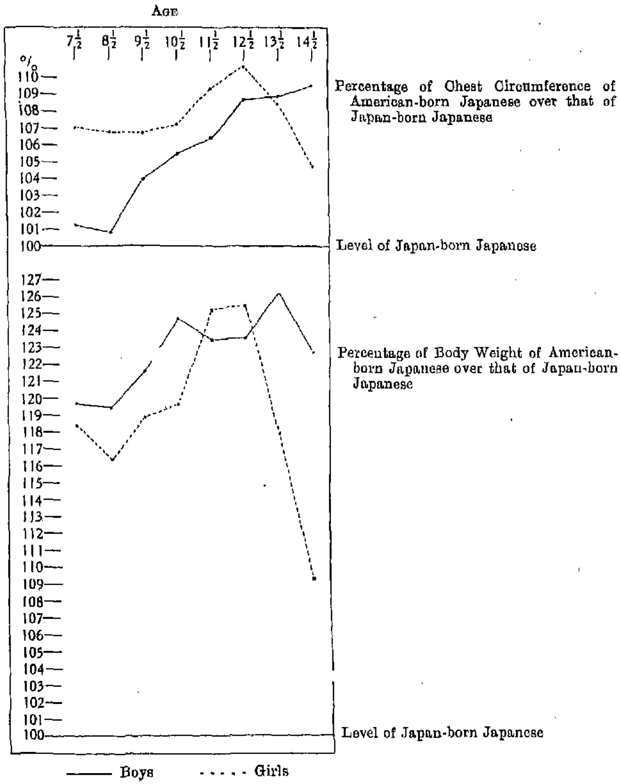


Fig. 33.

No appreciable difference is shown in stature and weight between social and labouring classes in Japan. Of course, the weight and stature must be greater in better nourished children of well-to-do families when compared with those of paupers. In any case, my figures as well as those of the Japanese Government measurements cover all classes.

LITERATURE CITED.

(1) BOAS, FRANZ. "Changes in the Bodily Form of Descendants of Immigrants." *American Anthropologist*, Vol. XIV, pp. 42--53, 1927.

(2) BAKWIN, HARRY and BAKWIN, R. M. "Body build in Infants." *Journal of Clinical Investigations*, Vol. X, pp. 369—402, June, 1931.

(3) GRAY, H. and GOWER, C. "Growth Standards of Height and Weight for Girls in Private Schools." *American Journal for Diseases of Children*, Vol. XXXV, pp. 411—413, March, 1928.

(4) APPLETON, VIVIA B. "Growth of Chinese Children in Hawaii and in China." *American Journal of Physical Anthropology*, Vol. X, p. 237, 1927.

(5) ORR, J. B. and CLARK, M. L. "Seasonal Variation in the Growth of School Children." *Lancet*, Aug. 16, 1930.

(6) ZEINER-HENDRIKSEN, K. "Growth of Children in Summer." *Norsk Magazin f. Laegevidenskab*, Vol. LXXXI, p. 262, March, 1920.

(7) NYLIN, G. "Periodic Variation in Growth." *Acta medica Scandinavica*, Supp. 31, pp. 1—207, 1929.

(8) HEJINIAN, L. and HATT, E. "The Stem-length; Recumbent Length Ratio as an Index of Body Type in Young Children." *American Journal of Physical Anthropology*, Vol. XIII, pp. 287—307, July, Sept. 1929.

(9) SUMNER, E. E. and WHITACRE, J. "Some Factors affecting accuracy in the collection of data on the Growth in Weight of School Children." *Journal of Nutrition*, Vol. IV, p. 15, May, 1931.

(10) IOWA CHILD WELFARE RESEARCH STATION. "Physical Traits of Young Children." *American Journal for Diseases of Children*, Vol. XXXVIII, p. 541, Sept. 1929.

(11) MARTIN, R. *Anthropometrie.*

(12) YOSHIDA. *Physical Measurements.*

(13) HRDLICKA, A. *Anthropometry.*

(14) DOWNES, R. M. "The Interrelationship of some Trunk Measurements and their relation to Stature." *Journal of Anatomy and Physiology*, Vol. XLVIII, p. 299, 1913—1914.

(15) GRAY, H. and ROOT, H. F. "Stem-length and Trunk-length." *Boston Medical and Surgical Journal*, Vol. CLXXXIV, p. 439, 1921.

(16) DREYER, G. *Assessment of Physical Fitness.* 1921.

(17) BOVARD, J. F. and COZENS, F. W. *Tests and Measurements in Physical Education.* 1931.

(18) HOLT, L. E. *The Diseases of Infancy and Childhood.*

(19) LUST, F. *Diagnostik u. Therapie d. Kinderkrankheiten.*

(20) YOSHIDA. "Standard weight of Japanese School Children." *Japan School Hygiene*, Vol. XIX, p. 12, Dec. 1931.

(21) COLLINS, S. D. and CLARK, T. "Physical Measurements of Boys and Girls of native White race Stock (third generation native born) in the United States." *Public Health Report*, Vol. XLIV, p. 1059, May, 1929.

(22) BEAN, R. BENNET. "The adult Sitting Height. The Sitting Height in Children." *Anatomical Record*, Vol. XVIII, p. 222, 1920—1921.

(23) LUCAS, M. P. and PRYOR, H. B. "Physical Measurements and Physiologic Processes in Young Children." *Journal of American Medical Association*, Vol. XCVII, p. 1127, 1931.

METHODS OF STATISTICAL ANALYSIS APPROPRIATE FOR k SAMPLES OF TWO VARIABLES.

By E. S. PEARSON, D.Sc. and S. S. WILKS*, Ph.D.

CONTENTS.

I. Introduction.

(1) *The Testing of Statistical Hypotheses.* Statistical theory which is not purely descriptive is largely concerned with the development of tools which will assist in the determination from observed events of the probable nature of the underlying cause system that controls them. The measured characteristics of quality vary from unit to unit, and the statistical technique is required to analyse this variation and covariation to break it into parts with which may be associated assignable causes, to test and compare alternative hypotheses, and to express the resulting conclusions in terms of measures of probability. It will be found that some of the most recent generalisation of theory has resulted from an attempt to provide critical tests of increasingly complex hypotheses. We may trace the development through a chain of questionings: Is it likely, (a) that this sample has been drawn from a specified population, P; (b) that these two samples have come from a common but unspecified population; (c) that these k samples have come from a common but unspecified population? Again the population P may be (d) completely specified or, (e) only partly specified, e.g. its mean is given but not its standard deviation; or when there are a number of samples we may allow the means in the sampled population to be different and question whether the standard deviations are the same. Another line of advance is from (f) problems dealing only with a single variable, to (g) those in which there may be a number of correlated variables.

Now we may frankly admit that in so far as the technique is to be used in handling data broken up into small groups, the recent theoretical developments assume normal variation. But to place the procedure for testing statistical hypotheses on a firm logical basis under one set of simplified conditions, is in itself an

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achievement of some value, and perhaps the most practical line of advance is the following

(*a*) To establish what we may term "normal theory."

(*b*) To study in a more systematic way than has been attempted the extent of departure from normality met with in different fields of application.

(*c*) To examine how rapidly normal theory tests become inefficient as the form of variation and covariation departs from the normal, and to determine the nature of the errors in judgment that will arise if these tests are still used.

(2) *The Analysis of Variance.* R. A. Fisher's methods of Analysis of Variance may be regarded from the following viewpoint:

(*a*) In any given problem it will generally be possible to specify certain factors which may be the cause of part of the variation, while there will be a residual part which, in the state of knowledge at the moment, must be regarded as due to unidentifiable or chance causes.

(*b*) An experiment may be designed to test whether a certain factor is operative or not; for example:

(i) Do differences in manurial treatment affect the yield of some variety of cereal?

(ii) Does modification in the production process alter the quality of output of some manufactured product?

(*c*) At the same time, in addition to these factors whose influence is under investigation, there may be other assignable causes of variation inevitably present, the effect of which would be obscuring were it not eliminated. Thus, for example, in the illustrations given above, there might be

(i) Variation due to changing soil fertility.

(ii) Variation due to differences in the skill of operatives or in the state of wear of machines.

(*d*) It follows that it is often possible to regard the variation in a character x as made up of parts due to different assignable causes A, B, C, ..., and of a residual part which, for the time being, we must attribute to chance causes. This may be expressed as follows:

$$x_{t.u,v,w,\ldots} = a_u + b_v + c_w + \ldots + X_{t.u,v,w,\ldots} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),^*$$

where $x_{t.u,v,w,\ldots}$ is the character of the t-th individual of a group of $n_{u,v,w,\ldots}$, all of which receive the same contribution a_u from the A factor, the same contribution b_v from the B factor and so on. $X_{t.u,v,w,\ldots}$ represents the residual term. In so far as the causes of variation are assignable, this grouping is possible.

* [The reader must bear in mind that for (1) to be true the effect of the causes A, B, C, ... on the character x must be *additive*. For example, the real effect of A and B might lead to a ratio a_u/b_v in the expression for $x_{t.u,v,w,\ldots}$, in which case the assumption of an additive relation would involve the influence of A, B, C, ... appearing in $X_{t.u,v,w,\ldots}$. Ed.]

(*e*) The technique of analysis consists in arranging the data so as to test separately for the presence of an A factor, or a B factor, etc. as desired. This is effected by obtaining in each case two estimates of the unknown variance, σ^2, of the residuals X, which would differ only through chance fluctuations if changes in the particular factor had no influence on variation within the limits covered by the experiment.

This method of analysis is based upon the assumptions that the residuals X, (*a*) are normally distributed, and (*b*) have the same standard deviation, σ, whatever be the values of the terms a, b, c, etc., that is to say, for all combinations of the assignable causes. We may be justified in accepting this to be the true position in very many practical cases, but it should be recognised that the method outlined above does not put these assumptions to the test. There are indeed a number of problems in which (*b*) is not true and where the discovery of significant differences, from group to group, in the variation among the individuals, X, may lead to the identification of further assignable causes of variation. Such has been found to be the case, for example, in the analysis of variation in quality of articles in mass-production industry. Further, we may be concerned not only with a single variable x, but with a number of correlated variables $x, y, z, \ldots$, and we may then need to examine the stability from group to group of the covariation as well as the variation among residuals $X, Y, Z, \ldots$.

(3) *An Illustration of the Problem.* The purpose of this paper is to develop certain methods recently suggested for dealing with these problems*. We shall treat here only the case of two correlated variables x and y, and shall suppose that the observations have been divided into k samples or groups. The problem will be to test whether these groups can be differentiated owing to significant differences either in the average values of x and y, or in the variation and covariation of the residuals within the groups. The choice of suitable criteria, i.e. of the tests to be applied, has been based upon the use of the principle of likelihood as suggested by J. Neyman and E. S. Pearson. More recently† these writers have suggested a more fundamental method of determining the most efficient test of a statistical hypothesis; this method of choice has been shown in many cases to be identical with the method of likelihood, but in the particular problems we are now considering the correspondence has not yet been established.

The following example, which is treated more fully below, will indicate the nature of the problem. In certain cases of manufacture tests of quality are destructive. Such, for example, is a test for breaking strength; it is therefore important to find an alternative correlated measure which may be used in its place in routine testing. In dealing with metal products a measure of hardness, based on a test which is not destructive, is sometimes used as an index of tensile strength. If,

* For the case of a single variable: see J. Neyman and E. S. Pearson: *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1930 and 1931. For the case of many variables, see S. S. Wilks: *Biometrika*, Vol. xxiv (1932), pp. 471—494.

† *Phil. Trans. of the Royal Soc.*, Series A, Vol. 231, pp. 289—337.

however, we are to predict strength from hardness, using the correlation method, it is essential that the degree of relationship between the two qualities should remain stable. It must not change from one plant to another or from one month to the next; in other words a preliminary research investigation should not only be concerned with changes in average strength and hardness which can be attributed to assignable causes, but also with the stability of the covariation among the residuals X and Y. Table I shows a preliminary statistical analysis of 60 pairs of test results made on a certain aluminium die-casting, divided into 5 groups of 12 pairs. Within a group the assignable causes of variation are believed to be constant, but it is necessary to analyze not only the figures in the 2nd and 4th columns, but also those in the 3rd, 5th and 6th.

TABLE I.

Data for Aluminium Die-Castings. (Samples of 12 observations.)*

Sample No.	Tensile Strength (10^3 lb. per sq. in.)		Hardness (Rockwell's E)		Coefficient of Correlation
	Mean	Standard Deviation	Mean	Standard Deviation	
1	33·399	2·565	68·49	10·19	0·683
2	28·216	4·318	68·02	14·49	0·876
3	30·313	2·188	66·57	10·17	0·714
4	33·150	3·954	76·12	11·08	0·715
5	34·269	2·715	69·92	9·88	0·805

II. Derivation of the Criteria.

(4) We shall suppose that each of k samples, $\Sigma_1, \Sigma_2, \ldots \Sigma_k$, of two variables x and y has been drawn from some normal population. Let π_t be the population from which Σ_t has been drawn and let the means of x and y in π_t be a_t and b_t, the standard deviations σ_{xt} and σ_{yt} and the correlation coefficient $\rho_t\,(t=1, 2, \ldots k)$. Thus, the distribution law of π_t will be

$$\frac{1}{2\pi\sigma_{xt}\sigma_{yt}\sqrt{1-\rho_t^2}}\,e^{-\frac{1}{2(1-\rho_t^2)}\left[\frac{(x-a_t)^2}{\sigma_{xt}^2}+\frac{(y-b_t)^2}{\sigma_{yt}^2}-\frac{2\rho_t(x-a_t)(y-b_t)}{\sigma_{xt}\sigma_{yt}}\right]} \quad \ldots\ldots(2).$$

Therefore, the probability of the joint occurrence of the samples Σ_t from their respective populations $\pi_t\,(t=1, 2, \ldots k)$, with values of x and y falling in the intervals $x_{t\alpha} \pm dx_{t\alpha}$, $y_{t\alpha} \pm dy_{t\alpha}$ $(\alpha=1, 2, \ldots n_t,\ t=1, 2, \ldots k)$ will be given, except for infinitesimals of higher order than $dx_{t\alpha}$ and $dy_{t\alpha}$, by

$$C=\prod_{t=1}^{k}\left(\frac{1}{2\pi\sigma_{xt}\sigma_{yt}\sqrt{1-\rho_t^2}}\right)^{n_t} e^{-\theta}\,dX\,dY \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

* The data are taken from W. A. Shewhart's *Economic Control of Manufactured Product*, Macmillan 1931. Although they would hardly be adequate for a research investigation in practice, they are suggestive and provide a good illustration of method.

in which

$$\theta = \sum_{t=1}^{k} n_t \left[\frac{s_{xt}^2 + (\bar{x}_t - a_t)^2}{2\sigma_{xt}^2 (1 - \rho_t^2)} + \frac{s_{yt}^2 + (\bar{y}_t - b_t)^2}{2\sigma_{yt}^2 (1 - \rho_t^2)} - \frac{2\rho_t [s_{xt} s_{yt} r_t + (\bar{x}_t - a_t)(\bar{y}_t - b_t)]}{2\sigma_{xt}\sigma_{yt}(1 - \rho_t^2)} \right] \quad (4),$$

where $\bar{x}_t$ and $\bar{y}_t$ are the means, s_{xt} and s_{yt} the standard deviations*, r_t the correlation coefficient of x and y, and n_t the number of individuals in the sample Σ_t, and

$$dX\,dY = \prod_{t=1}^{k} \prod_{\alpha=1}^{n_t} dx_{t\alpha}\,dy_{t\alpha} \quad \ldots\ldots\ldots\ldots (5).$$

We shall now consider the derivation of criteria for testing the following three hypotheses concerning the populations π_t:

(i) The hypothesis H that the populations π_t are identical, that is, that

$$\sigma_{xt} = \sigma_x, \quad \sigma_{yt} = \sigma_y, \quad \rho_t = \rho \quad \ldots\ldots\ldots\ldots (6),$$

$$a_t = a, \quad b_t = b \qquad (t = 1, 2, \ldots k) \quad \ldots\ldots\ldots\ldots (7).$$

(ii) The hypothesis H_1 that the samples have come from populations with the same set of variances and correlations but having means with any differing values whatever, that is, that (6) is true whatever may be the values of the means a_t and b_t.

(iii) The hypothesis H_2 that the samples are from populations in which (7) is true, when it is assumed that (6) is true.

These are generalisations to two variables of the three hypotheses considered by Neyman and Pearson† for the case of k samples of a single variable; or they may be regarded as special cases of the more general problem whose solution has been considered more recently by Wilks‡. Thus, it will only be necessary to indicate briefly the steps involved in applying the method of likelihood to determine criteria appropriate in testing H, H_1 and H_2. In each case we must fix:

(*a*) The class Ω of admissible sets of populations π_t $(t = 1, 2, \ldots k)$ from one set of which the set of samples Σ_t is assumed to have been drawn.

(*b*) The subclass ω of Ω to which the set π_t must belong if the hypothesis tested be true.

Then we must find the maximum of C in (3) for variations of the population parameters under the assumption that the set π_t is (i) a member of Ω; call this $C(\Omega \max)$; and (ii) a member of ω; this we call $C(\omega \max)$. Then the expression for the likelihood of the composite hypothesis H has been defined to be

$$\lambda_H = \frac{C(\omega \max)}{C(\Omega \max)} \quad \ldots\ldots\ldots\ldots (8).$$

Let us consider this λ-criterion for each of the hypotheses H, H_1 and H_2.

* Here and throughout the paper the standard deviation in a sample of n will be defined by the relation $ns^2 = \Sigma (x - \bar{x})^2$.

† *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1931.

‡ *Loc. cit.*

(i) Criterion for H. We find that $C(\Omega \max)$ occurs when

$$a_t = \bar{x}_t, \quad b_t = \bar{y}_t \quad \text{.................................(9)},$$

$$\sigma_{xt} = s_{xt}, \quad \sigma_{yt} = s_{yt}, \quad \rho_t = r_t \qquad (t = 1, 2, \ldots k) \quad (10),$$

and $C(\omega \max)$ occurs when

$$a = \bar{x}_0, \quad b = \bar{y}_0 \quad \text{.................................(11)},$$

$$\sigma_x^2 = v_{110} = v_{11a} + v_{11m}, \quad \sigma_y^2 = v_{220} + v_{22m}, \quad \sigma_x \sigma_y \rho = v_{120} = v_{12a} + v_{12m} \quad \ldots(12),$$

where

$$\bar{x}_0 = \frac{1}{N} \sum_{t=1}^{k} n_t \bar{x}_t, \quad \bar{y} = \frac{1}{N} \sum_{t=1}^{k} n_t \bar{y}_t \quad \text{........................(13)},$$

$$\left.\begin{aligned} N v_{110} &= \sum_{t=1}^{k} \sum_{\alpha=1}^{n_t} (x_{t\alpha} - \bar{x}_0)^2 = N s_{x0}^2 \\ N v_{220} &= \sum_{t=1}^{k} \sum_{\alpha=1}^{n_t} (y_{t\alpha} - \bar{y}_0)^2 = N s_{y0}^2 \\ N v_{120} &= \sum_{t=1}^{k} \sum_{\alpha=1}^{n_t} (x_{t\alpha} - \bar{x}_0)(y_{t\alpha} - \bar{y}_0) = N s_{x0} s_{y0} r_0 \end{aligned}\right\} \quad \text{...............(14)},$$

that is to say $\bar{x}_0, \bar{y}_0, s_{x0}, s_{y0}$ and r_0 are the means, standard deviations and correlation coefficient obtained on combining the N pairs of observations from the k samples. Further

$$\left.\begin{aligned} N v_{11a} &= \sum_{t=1}^{k} \sum_{\alpha=1}^{n_t} (x_{t\alpha} - \bar{x}_t)^2 = \sum_{t=1}^{k} n_t s_{xt}^2 \\ N v_{22a} &= \sum_{t=1}^{k} \sum_{\alpha=1}^{n_t} (y_{t\alpha} - \bar{y}_t)^2 = \sum_{t=1}^{k} n_t s_{yt}^2 \\ N v_{12a} &= \sum_{t=1}^{k} \sum_{\alpha=1}^{n_t} (x_{t\alpha} - \bar{x}_t)(y_{t\alpha} - \bar{y}_t) = \sum_{t=1}^{k} n_t s_{xt} s_{yt} r_t \end{aligned}\right\} \quad \text{............(15)},$$

$$N v_{11m} = \sum_{t=1}^{k} n_t (\bar{x}_t - \bar{x}_0)^2, \quad N v_{22m} = \sum_{t=1}^{k} n_t (\bar{y}_t - \bar{y}_0)^2, \quad N v_{12m} = \sum_{t=1}^{k} n_t (\bar{x}_t - \bar{x}_0)(\bar{y}_t - \bar{y}_0) \quad \text{......(16)}.$$

We shall write for each sample ($t = 1, 2, \ldots k$)

$$\left.\begin{aligned} n_t v_{11t} &= \sum_{\alpha=1}^{n_t} (x_{t\alpha} - \bar{x}_t)^2 = n_t s_{xt}^2, \quad n_t v_{22t} = \sum_{\alpha=1}^{n_t} (y_{t\alpha} - \bar{y}_t)^2 = n_t s_{yt}^2 \\ n_t v_{12t} &= \sum_{\alpha=1}^{n_t} (x_{t\alpha} - \bar{x}_t)(y_{t\alpha} - \bar{y}_t) = n_t s_{xt} s_{yt} r_t \end{aligned}\right\} \quad \text{......(17)}.$$

Placing these values in (3) and taking the ratio as defined by (8) we find,

$$\lambda_H = \prod_{t=1}^{k} \left[\frac{|v_{ijt}|}{|v_{ij0}|} \right]^{\frac{n_t}{2}} \quad \text{..............................(18)},$$

where*

$$|v_{ijt}| = \begin{vmatrix} v_{11t} & v_{12t} \\ v_{12t} & v_{22t} \end{vmatrix} = s_{xt}^2 s_{yt}^2 (1 - r_t^2) \quad \text{.....................(19)},$$

$$|v_{ij0}| = \begin{vmatrix} v_{110} & v_{120} \\ v_{120} & v_{220} \end{vmatrix} = s_{x0}^2 s_{y0}^2 (1 - r_0^2) \quad \text{.....................(20)},$$

* For convenience we shall call $|v_{ijt}|$ the generalised variance of the tth sample with elements having $n_t - 1$ degrees of freedom. Similarly, $|v_{ij0}|$, $|v_{ijm}|$ and $|v_{ija}|$ will be generalised variances derived from the combined samples, with elements having $N-1$, $k-1$ and $N-k$ degrees of freedom respectively.

(ii) Criterion for H_1. We find that $C(\Omega \max)$ occurs for the same values of the parameters as in the case of H, namely, those given by (9) and (10). $C(\omega \max)$ occurs when (9) is true and when

$$\sigma_x^2 = v_{11a}, \quad \sigma_y^2 = v_{22a}, \quad \sigma_x \sigma_y \rho = v_{12a} \quad \text{.....................(21).}$$

Thus it follows that

$$\lambda_{H_1} = \prod_{t=1}^{k} \left[\frac{|v_{ijt}|}{|v_{ija}|} \right]^{\frac{n_t}{2}} \quad \text{..............................(22),}$$

where $|v_{ija}|$ is a determinant analogous to those of (19) and (20).

(iii) Criterion for H_2. $C(\Omega \max)$ occurs when (9) and (21) are satisfied and $C(\omega \max)$ when the parameters have the values (11) and (12). Consequently

$$\lambda_{H_2} = \left[\frac{|v_{ija}|}{|v_{ija} + v_{ijm}|} \right]^{\frac{N}{2}} = \left[\frac{|v_{ija}|}{|v_{ij0}|} \right]^{\frac{N}{2}} \quad \text{.....................(23)}$$

As in the case of the single-variable problem we observe that

$$\lambda_H = \lambda_{H_1} \times \lambda_{H_2} \quad \text{.................................(24).}$$

III. INTERPRETATION OF THE CRITERIA.

(5) *The H_2 test.* We note that the structure of each of the λ's given by (18), (22) and (23) differs from that of the corresponding λ of the single-variable problem only in that determinants of the second order matrices of variances and covariances appear in place of the corresponding sums of squares in the single-variable case. In other words, determinants of the second order now take the place of determinants of the first order.

We shall first examine λ_{H_2}; in testing H_2 we have assumed that (6) is true and logically we should first consider the grounds for this assumption, if necessary by testing H_1. The test of H_2 is, however, related to R. A. Fisher's tests in the analysis of variance, and it will be clearer to consider this first.

A λ-criterion must lie between 0 and 1, and if the principle underlying the selection is valid, as it decreases from unity towards zero we should be more and more inclined to reject the hypothesis tested in favour of some one of the admissible alternative hypotheses. How far do our intuitional requirements appear satisfied by λ_{H_2}? The ratio of determinants

$$\lambda_{H_2}^{2/N} = \frac{|v_{ija}|}{|v_{ija} + v_{ijm}|} \quad \text{..............................(25)}$$

is of the form of the ratio ψ of Theorem I of the Appendix, if we set $v_{ija} = A_{ij}$, and

$$\sqrt{\frac{n_t}{N}} (\bar{x}_t - \bar{x}_0) = \eta_{1t}, \qquad \sqrt{\frac{n_t}{N}} (\bar{y}_t - \bar{y}_0) = \eta_{2t} \quad \text{...............(26).}$$

It follows that

(*a*) $0 \leqslant \lambda_{H_2} \leqslant 1$; and when $|v_{ija}| > 0$;

(*b*) a necessary and sufficient condition for $\lambda_{H_2} = 1$ is that $\bar{x}_t = \bar{x}_0$, $\bar{y}_t = \bar{y}_0$ $(t = 1, 2, \ldots k)$, that is, that all the sample means be the same;

(c) a necessary and sufficient condition that $\lambda_{H_2} = 0$ is that at least one of the differences $\bar{x}_t - \bar{x}_0$, $\bar{y}_t - \bar{y}_0$ $(t = 1, 2, \ldots k)$ be infinite.

It can also be shown by the ordinary methods of differentiation that λ_{H_2} cannot have any other maximum but that of unity (occuring when $\bar{x}_t = \bar{x}_0$ and $\bar{y}_t = \bar{y}_0$) for any given values of the v_{ija}. In fact the maximum is the only stationary point. Therefore, if we keep the intra-sample variation the same, and allow the system to vary from one in which all of the sample means are equal to the other extreme in which at least one sample mean differs very greatly from the mean of the whole, λ_{H_2} will at the same time decrease from 1 to 0.

The case where $$|v_{ija}| = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$
needs special consideration. The determinant is essentially of the form

$$\begin{vmatrix} \Sigma\, \xi_{ta}^2 & \Sigma\, \xi_{ta}\,\eta_{ta} \\ \Sigma\, \xi_{ta}\,\eta_{ta} & \Sigma\, \eta_{ta}^2 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(28),$$

where $$\Sigma = \sum_{t=1}^{k} \sum_{a=1}^{m}, \quad \xi_{ta} = \frac{(x_{ta} - \bar{x}_t)}{\sqrt{N}} \text{ and } \eta_{ta} = \frac{(y_{ta} - \bar{y}_t)}{\sqrt{N}};$$

which can vanish only when $\xi_{ta} = c\eta_{ta}$, i.e. when $x_{ta} - \bar{x}_t = c\,(y_{ta} - \bar{y}_t)$ where c is a constant for all t and a, which may be finite or infinite. In this case the observation points (x_{ta}, y_{ta}) for each sample lie on a straight line, and the lines are all parallel. If $c = 0$ the lines will be horizontal as in Fig. 1 (*a*), if $c = \pm\infty$ they will be vertical as in (*b*), otherwise they will be sloping as in (*c*)*. Hypothesis H_2 is exceedingly improbable (and $\lambda_{H_2} = 0$) unless these lines coincide, which will occur only when

$$x_{ta} - \bar{x}_0 = c\,(y_{ta} - \bar{y}_0),$$

that is when $\bar{x}_t - \bar{x}_0 = c\,(\bar{y}_t - \bar{y}_0)$. In this case $\lambda_{H_2} = 0/0$, and we are really reduced to a single-variable problem, and could apply the appropriate H_2 test for that case. It appears therefore that the criterion λ_{H_2} does satisfy our intuitional requirements, at any rate as far as the limiting values 1 and 0 are concerned.

(6) *Alternatives to* λ_{H_2}. In the one-variable problem λ_{H_2} is expressible in terms of η^2, the squared correlation ratio, and hence also in terms of $1 - \eta^2$†. In fact, using the present notation, and considering the x variable only,

$$\frac{v_{11m}}{v_{11a} + v_{11m}} = \eta^2, \qquad \frac{v_{11a}}{v_{11a} + v_{11m}} = 1 - \eta^2 \quad \ldots\ldots\ldots\ldots(29),$$

and consequently it is immaterial which of the two ratios is regarded as the criterion. In the case of two variables the corresponding ratios will be

$$\frac{|v_{ijm}|}{|v_{ija} + v_{ijm}|} = U_2^2, \qquad \frac{|v_{ija}|}{|v_{ija} + v_{ijm}|} = \lambda_{H_2}^{2/N} = L_2^2 \quad \ldots\ldots\ldots\ldots(30),$$

but here U_2 cannot be expressed as a single valued function of L_2. As will be shown below, the sampling distributions of both L_2 and U_2 are very simple functions, and it is natural to ask whether U_2 might be used as an alternative criterion

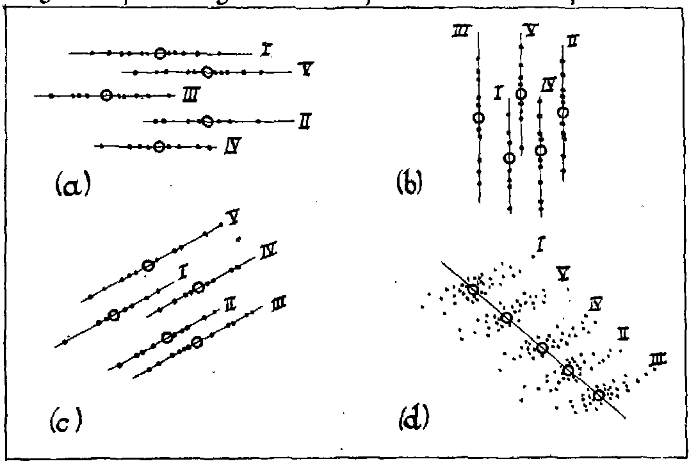
* In the diagram the spots represent observation points (x, y) and the circles represent means of samples.

† Actually $\lambda_{H_2}^{2/N} = 1 - \eta^2$.

for testing hypothesis H_2, decreasing from 1 to 0 as the hypothesis becomes more and more likely. It can be readily seen however that U_2 is not a suitable criterion. Suppose that the $v_{ij\alpha}$ are finite and not zero, so that there is variation within the samples; then $U_2 \to 0$ when $|v_{ijm}| \to 0$. This may occur,

(a) When $\bar{x}_t \to \bar{x}_0$ and $\bar{y}_t \to \bar{y}_0$ for $t = 1, 2, \ldots k$, i.e. when the means of all samples tend to coincide, and hypothesis H_2 is probable.

Diagram representing certain exceptional cases of sample variation



N.B. The numbers are inserted to indicate lines of points corresponding to different samples.

Fig. 1.

(b) But since $|v_{ijm}|$ is of the form (28) it follows that it will also tend to zero when $\bar{x}_t - \bar{x}_0 \to c(\bar{y}_t - \bar{y}_0)$, c being the same constant for all t. This would happen when the sample means tend to lie on a straight line, and when, as suggested in Fig. 1 (d), hypothesis H_2 may be quite untenable. Clearly therefore U_2 is not an acceptable criterion.

There are two other forms of alternative criteria which it is of interest to refer to here. On the assumption that the samples have been drawn from identical normal populations, it is possible to obtain from the data two independent estimates of both,

$$(a) \quad \sigma_x \sigma_y \sqrt{1-\rho^2},$$

$$(b) \quad \sigma_x \sigma_y \rho.$$

Case (*a*). If we write $\theta = |v_{ija}|$, $\phi = |v_{ijm}|$, then it is known* that

$$df(\theta) = \frac{2^{N-k-2}\,\Delta^{\frac{1}{2}(N-k-1)}}{\Gamma(N-k-1)}\,\theta^{\frac{1}{2}(N-k-3)}\,e^{-2\sqrt{\Delta\theta}}\,d\theta \quad \text{.............(31)},$$

$$df(\phi) = \frac{2^{k-3}\,\Delta^{\frac{1}{2}(k-2)}}{\Gamma(k-2)}\,\phi^{\frac{1}{2}(k-4)}\,e^{-2\sqrt{\Delta\phi}}\,d\phi \quad \text{.................(32)},$$

where $\Delta = N^2/[4\sigma_x^2\sigma_y^2(1-\rho^2)]$. Furthermore θ and ϕ are independently distributed, and it may be readily shown from (31) and (32), using the symbol E for "expected" values, that

$$E(\sqrt{\theta}) = \frac{N-k-1}{N}\,\sigma_x\sigma_y\sqrt{1-\rho^2}, \qquad E(\sqrt{\phi}) = \frac{k-2}{N}\,\sigma_x\sigma_y\sqrt{1-\rho^2} \;\text{...(33)}.$$

Hence $\dfrac{N}{N-k-1}\sqrt{|v_{ija}|}$ and $\dfrac{N}{k-2}\sqrt{|v_{ijm}|}$ may be taken as independent estimates of $\sigma_x\sigma_y\sqrt{1-\rho^2}$ with elements having $N-k$ and $k-1$ degrees of freedom respectively. If we now take the ratio of the two estimates, or

$$\psi = \frac{N-k-1}{k-2}\sqrt{\frac{|v_{ijm}|}{|v_{ija}|}} \quad \text{.........................(34)},$$

it is found from (31) and (32) that

$$df(\psi) = \frac{\Gamma(N-3)}{\Gamma(N-k-1)\,\Gamma(k-2)}(k-2)^{k-2}(N-k-1)^{N-k-1}$$
$$\times\,\psi^{k-3}[(k-2)\,\psi + (N-k-1)]^{-(N-3)}\,d\psi \;\text{...(35)}.$$

This is a Pearson Type VI curve; the 5% and 1% sampling limits for ψ could be obtained by taking

$$z = \tfrac{1}{2}\log_e\psi \;\text{....................................(36)},$$

and entering R. A. Fisher's† tables of z with $2(k-2)$ and $2(N-k-1)$ degrees of freedom.

The criterion ψ, will not, however, be suitable for testing the hypothesis H_2. We may write

$$|v_{ijm}| = s_{xm}^2\,s_{ym}^2\,(1-r_m^2) \;\text{..........................(37)},$$

where s_{xm}, s_{ym} and r_m are the standard deviations and coefficient of correlation of the weighted sample means. Then clearly, it would be possible for ψ to be unity and $|v_{ija}|$ fixed and finite, while $r_m \to 1$ and either s_{xm} or $s_{ym} \to \infty$. In such a situation H_2 would be untenable and yet the two independent estimates of $\sigma_x\sigma_y\sqrt{1-\rho^2}$ would be equal.

Case (*b*). It can be easily shown that $\dfrac{Nv_{12a}}{N-k}$ and $\dfrac{Nv_{12m}}{k-1}$ are two independent estimates $\sigma_x\sigma_y\rho$, but their ratio, say ψ', would again be unsuitable for testing H_2 for reasons similar to those holding in the case of ψ. It should also be pointed out that the sampling distribution of ψ' is extremely complicated.

* S. S. Wilks: *loc. cit.* p. 477.

† R. A. Fisher: *Statistical Methods for Research Workers*, 4th edition 1932. Edinburgh: Oliver and Boyd.

These illustrations bring out forcibly an important but often neglected consideration. A critical examination of the efficiency of any statistical criterion is necessary before it is applied to testing a hypothesis. The fact that its sampling distribution is known if the hypothesis be true, does not by itself justify its use. In the present case in using U_2, ψ or ψ' we should be in danger of accepting the hypothesis H_2 in certain cases when it is evidently not true. It is only the likelihood criterion, λ_{H_2}, which appears suitable for our purpose.

(7) *The H_1 test.* λ_{H_1} has been defined by (22); clearly $\lambda_{H_1}^{2/N}$ is of the form of the ratio θ discussed in Theorem II of the Appendix, and it satisfies all of the conditions of that theorem. It follows that $\lambda_{H_1}^{2/N}$ and consequently λ_{H_1}

(*a*) must lie between 0 and 1;

(*b*) will be unity when and only when $v_{ijt} = v_{iju}$, $(i, j = 1, 2)$ for all values of t and u, that is, when the variances and covariances of x and y are respectively equal in all the k samples;

(*c*) will be zero when, (i) $x_{t\alpha} - \bar{x}_t = c_t(y_{t\alpha} - \bar{y}_t)$ for at least one value of t, where c_t is a constant for all values of α; this means that the sample points in at least one sample will lie on a straight line, and there is perfect correlation in some but not all samples. However, if there be a c such that $x_{t\alpha} - \bar{x}_t = c(y_{t\alpha} - \bar{y}_t)$ for all values of α and t, λ_{H_1} has the indeterminant form 0/0, and the points of each of the samples will lie on a straight line and the lines will all be parallel. In this case the problem is reduced to that of a single variable, and the appropriate H_1 test could be applied; (ii) one of the deviations $x_{t\alpha} - \bar{x}_t$ or $y_{t\alpha} - \bar{y}_t$ or both are infinite for at least one value of α and t (but not all values of t, assuming $|v_{ijt}| > 0$ for all t), subject to the condition that the limiting values of the generalised variances remain finite and not zero as these deviations become infinite. Under these conditions it follows by the argument of the proof of (*c*) in Theorem I of the Appendix that $|v_{ij0}|$ becomes infinite while the $|v_{ijt}|$ remain finite and different from zero. The situation is a limiting form of that in which the variation is very much greater in some samples than in others.

(8) *The H test.* Since λ_H is the product of λ_{H_1} and λ_{H_2} it must lie between 0 and 1. It can be unity only when both λ_{H_1} and λ_{H_2} are unity, that is to say, when the means, variances and covariances of x and y in all k samples are respectively identical. It will approach zero when λ_{H_1} or λ_{H_2} or both approach zero.

As in the single-variable problem, the three λ ratios appear therefore to satisfy our intuitive requirements as criteria for testing H, H_1 and H_2, for they are quantities which tend to unity as the corresponding hypothesis becomes intuitively more probable (as far as the information contained in the sample is concerned) and tend to zero as it becomes more likely that the hypothesis is false. Whether the tests based on these criteria satisfy the more fundamental conditions laid down by Neyman and Pearson *, we do not yet know. The problem of testing these hypotheses

* *Phil. Trans. Roy. Soc.*, Ser. A, Vol. 231 (1933).

will be completed by determining the sampling distributions of the λ's on the assumption that the corresponding hypotheses are true, for without these we have no means of testing the significance of an observed value of λ. In the following section we shall first give expressions for the moment coefficients, and then by inverting the moment equations show how the frequency distributions may be obtained. The result is simple only in the case of λ_{H_2}; for λ_H and λ_{H_1} numerical values for the probability integrals can be obtained only by some method of approximation

IV. The Moment Coefficients and Distributions of the Criteria.

(9) *The Moment Coefficients.* In the single-variable problem it was found to be convenient to study the sampling distributions of some fractional power of the λ's, rather than that of the λ's themselves, owing to the extreme skewness of the latter distributions*. The use of $\lambda^{2/N}$ was suggested largely because in this case

$$\lambda_{H_2}^{2/N} = 1 - \eta^2,$$

(where η^2 is the squared correlation ratio) and had a sampling distribution of Type I form. In the present bivariate case we shall find for similar reasons, which will become apparent as we proceed, that some advantage will be gained by using the $\frac{1}{N}$-th power of the λ's. The moments of the λ's have been given by one of the writers† in a recent paper for the case of an n-variate normal system. If we denote by M_{0h}, M_{1h} and M_{2h} the h-th moment *coefficients about zero of* $\lambda_H^{1/N}$, $\lambda_{H_1}^{1/N}$, and $\lambda_{H_2}^{1/N}$ respectively, when the corresponding hypotheses H, H_1 and H_2 are true, then we have at once from the paper just cited, for the case of two variables (i.e. $n = 2$),

$$M_{2h} = \frac{\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-k}{2}\right)\Gamma\left(\frac{N-k-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{N-k+h}{2}\right)\Gamma\left(\frac{N-k-1+h}{2}\right)}{\Gamma\left(\frac{N-1+h}{2}\right)\Gamma\left(\frac{N-2+h}{2}\right)} \quad \text{............(38)},$$

$$M_{1h} = \frac{\Gamma\left(\frac{N-k}{2}\right)\Gamma\left(\frac{N-k-1}{2}\right)}{\Gamma\left(\frac{N-k+h}{2}\right)\Gamma\left(\frac{N-k-1+h}{2}\right)}$$

$$\times \prod_{t=1}^{k} \left\{ \left(\frac{N}{n_t}\right)^{\frac{hn_t}{N}} \frac{\Gamma\left(\frac{n_t-1}{2}+\frac{hn_t}{2N}\right)\Gamma\left(\frac{n_t-2}{2}+\frac{hn_t}{2N}\right)}{\Gamma\left(\frac{n_t-1}{2}\right)\Gamma\left(\frac{n_t-2}{2}\right)} \right\} \quad \text{......(39)},$$

$$M_{0h} = M_{1h} \times M_{2h} \quad \text{..................................(40)}.$$

* Cf. J. Neyman and E. S. Pearson: *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A (1931), pp. 475—476.

† S. S. Wilks: *Biometrika*, Vol. xxiv (1932), pp. 471—494. In this paper the generalisations of λ_H, λ_{H_1} and λ_{H_2} were denoted by $\lambda_{H(n)}$, $\lambda_{H'(n)}$ and λ_H respectively.

These expressions may be considerably simplified by making use of the duplication formula of the Gamma function which can be written

$$\Gamma(a+\tfrac{1}{2})\,\Gamma(a+1)=\frac{\Gamma(\tfrac{1}{2})\,\Gamma(2a+1)}{2^{2a}} \quad\text{.....................(41).}$$

Applying this to (38) and (39) we get

$$M_{2h}=\frac{\Gamma(N-2)\,\Gamma(N-k-1+h)}{\Gamma(N-k-1)\,\Gamma(N-2+h)} \quad\text{.....................(42)}$$

and

$$M_{1h}=\frac{\Gamma(N-k-1)}{\Gamma(N-k-1+h)}\prod_{t=1}^{k}\left[\left(\frac{N}{n_t}\right)^{\frac{nh_t}{N}}\frac{\Gamma\left(n_t-2+\frac{hn_t}{N}\right)}{\Gamma(n_t-2)}\right] \quad\text{.........(43).}$$

(10) *The Distributions of L_2 and U_2.* To find the distribution of $\lambda_{H_2}^{1/N}=L_2$, we use the relation

$$\frac{\Gamma(N-k-1+h)}{\Gamma(N-2+h)}=\frac{1}{\Gamma(k-1)}\int_0^1 u^{N-k-2+h}(1-u)^{k-2}\,du \quad\text{.........(44)}$$

in (42). Accordingly, we find that the h-th moment of L_2 is identical with that of u, where the distribution of u is

$$\frac{\Gamma(N-2)}{\Gamma(N-k-1)\,\Gamma(k-1)}\,u^{N-k-2}(1-u)^{k-2} \quad\text{..................(45).}$$

Therefore it follows from the uniqueness of the solution of the moment problem for a finite interval* that the distribution of L_2 must be identical with that of u, and is given by

$$df(L_2)=\frac{\Gamma(N-2)}{\Gamma(N-k-1)\,\Gamma(k-1)}\,L_2^{N-k-2}(1-L_2)^{k-2}\,dL_2 \quad\text{.........(46).}$$

In a similar manner, it follows that the distribution of U_2 is given by

$$df(U_2)=\frac{\Gamma(N-2)}{\Gamma(N-k)\,\Gamma(k-2)}\,U_2^{k-3}(1-U_2)^{N-k-1}\,dU_2 \quad\text{.........(47).}$$

In both these cases the probability integral is an incomplete B-Function.

(11) *The Distribution of L_1.* Let us consider the sampling distribution of

$$\lambda_{H_1}^{1/N}=L_1,$$

say. The h-th moment of L_1 about zero is given by (43). If we multiply and divide (43) by $\Gamma(N-2k+h)$ and then use the following relations,

$$\frac{\Gamma(N-2k+h)}{\Gamma(N-k-1+h)}=\frac{1}{\Gamma(k-1)}\int_0^1 u^{N-2k-1+h}(1-u)^{k-2}\,du \quad\text{.........(48)}$$

and

$$\frac{\prod_{t=1}^{k}\Gamma(n_t-2+hp_t)}{\Gamma(N-2k+h)}$$
$$=\int_0^1\int_0^{l_1}\dots\int_0^{l_{k-2}}(1-v_1-v_2-\dots-v_{k-1})^{n_k-3+hp_k}\prod_{t=1}^{k-1}(v_t^{n_t-3+hp_t}\,dv_t) \quad\text{...(49),}$$

* See W. Stekloff: *Mémoires de l'Académie Impériale des Sciences de St Pétersbourg*, Vol. XXXIII, No. 9 (1915).

where $l_i = 1 - v_1 - v_2 - \dots - v_i$ $(i = 1, 2, \dots k-2)$ and $p_t = \frac{n_t}{N}$ $(t = 1, 2, \dots k)$, we find that the h-th moment $(h = 0, 1, 2, \dots)$ of L_1 is identical with that of

$$\phi = \frac{1}{p_1^{p_1} p_2^{p_2} \dots p_k^{p_k}} u v_1^{p_1} v_2^{p_2} \dots v_{k-1}^{p_{k-1}} (1 - v_1 - v_2 - \dots - v_{k-1})^{p_k} \quad \dots\dots(50),$$

where u and the v's are distributed according to the function

$$C u^{N-2k-1} (1-u)^{k-2} (1 - v_1 - v_2 - \dots - v_{k-1})^{n_k-2} \prod_{i=1}^{k-1} v_i^{n_i-2} \quad \dots\dots\dots(51),$$

where $0 \leqslant u \leqslant 1$, $v_i \geqslant 0$, and $v_1 + v_2 + \dots + v \leqslant 1$ and C is a constant depending only on k and the n's. Therefore, it follows from the argument used in establishing the uniqueness of (46) that the distribution of L_1 is identical with that of ϕ*. The problem of finding the distribution of ϕ is equivalent to that of solving (50) for the u or one of the v's and substituting in (51) and integrating with respect to all variables except ϕ. This process is extremely complicated, even when the p's are all equal, that is, when $n_1 = n_2 = \dots = n_k = n$; in this case we can find an expression for the distribution of the L_1 by considering a transformation of M_{1h}. The new form of M_{1h} is found by applying the transformation†

$$\Gamma(mz) = \frac{m^{mz-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(m-1)}} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \dots \Gamma\left(z + \frac{m-1}{m}\right) \quad \dots\dots\dots(52)$$

to $\Gamma(N - k - 1)$ and $\Gamma(N - k - 1 + h)$ in (43), by writing $m = k$ and $z = n - 1 - 1/k$ in the first and $m = k$ and $z = n - 1 - 1/k + h/k$ in the second. Accordingly, we get

$$M_{1h} = C \frac{\Gamma^k\left(n - 2 + \frac{h}{k}\right)}{\Gamma\left(n - \frac{k+1}{k} + \frac{h}{k}\right) \Gamma\left(n - \frac{k}{k} + \frac{h}{k}\right) \dots \Gamma\left(n - \frac{2}{k} + \frac{h}{k}\right)} \quad \dots\dots(53),$$

where
$$C = \frac{\Gamma\left(n - \frac{k+1}{k}\right) \Gamma\left(n - \frac{k}{k}\right) \dots \Gamma\left(n - \frac{2}{k}\right)}{\Gamma^k(n-2)}.$$

Distribution functions with moments of this type have been considered by one of the authors‡, from which we can write at once as the distribution of L_1

$$df(L_1) = C' L_1^{N-2k} (1 - L_1^k)^{2k - \frac{(k+1)(k+2)-2}{2k} - 1} d(L_1^k)$$
$$\times \int_0^1 \int_0^1 \dots \int_0^1 \theta_1^{\frac{k-1}{k} - 1} \theta_2^{\frac{k}{k} - 1} \dots \theta_{k-1}^{\frac{2k-3}{k} - 1}$$

* In this connection we note that a simple alternative proof of (*a*) in Theorem II of the Appendix can be constructed at once for the case where the p's are rational numbers and the a_{ijt} are product moments, by showing that the maximum of ϕ is unity for variations of u and the v's in the region over which (51) is defined. Indeed, for a given value of u we find that the only stationary point with respect to the v's is the true maximum which occurs where $v_t = p_t$ $(t = 1, 2, \dots k-1)$. Therefore, ϕ, which is necessarily positive, has a maximum of unity, and since the range of ϕ and L_1 must be the same we have $0 \leqslant L_1^2 \leqslant 1$, that is, $0 \leqslant \theta \leqslant 1$.

† See Whittaker and Watson: *Modern Analysis* (4th edition), p. 240.

‡ Wilks, *loc. cit.* pp. 474—475.

$$\times (1-\theta_1)^{2(k-1)-\frac{k(k+1)-2}{2k}-1}(1-\theta_2)^{2(k-2)-\frac{(k-1)k-2}{2k}-1}\dots$$
$$\dots(1-\theta_{k-1})^{2-\frac{2\cdot 3-2}{2k}-1}$$
$$\times [1-\theta_1(1-L_1^k)]^{-\frac{k}{k}}[1-\{\theta_1+\theta_2(1-\theta_1)\}(1-L_1^k)]^{-\frac{k+1}{k}}\dots$$
$$\times [1-\{\theta_1+\theta_2(1-\theta_1)+\dots+\theta_{k-1}(1-\theta_1)\dots$$
$$\dots(1-\theta_{k-2})\}(1-L_1^k)]^{-\frac{2k-2}{k}}\,d\theta_1\dots d\theta_{k-1}\quad\dots\dots(54),$$

where (using formula (52))

$$C'=\frac{C}{\Gamma\left(\frac{k-1}{k}\right)\Gamma\left(\frac{k}{k}\right)\dots\Gamma\left(\frac{2k-2}{k}\right)}=\frac{\Gamma(N-k-1)}{\Gamma^k(n-2)\,\Gamma(k-1)\,k^{N-2k}},$$

a slightly more condensed form of (54) can be obtained by setting

$$\theta_t=1-\phi_t\quad(t=1,2,\dots k-1).$$

Thus
$$df(L_1)=C'L_1^{nk-3k}(1-L_1^k)^{2k-\frac{(k+1)(k+2)-2}{2k}-1}\,d(L_1^k)$$
$$\times\int_0^1\int_0^1\dots\int_0^1\phi_1^{2(k-1)-\frac{k(k+1)-2}{2k}-1}\phi_2^{2(k-2)-\frac{(k-1)k-2}{2k}-1}\dots\phi_{k-1}^{2-\frac{2}{k}-1}$$
$$\times[(1-\phi_1)^{\frac{k-1}{k}-1}(1-\phi_2)^{\frac{k}{k}-1}(1-\phi_3)^{\frac{k+1}{k}-1}\dots(1-\phi_{k-1})^{\frac{2k-3}{k}-1}]$$
$$\times[1-(1-\phi_1)(1-L_1^k)]^{-1}[1-(1-\phi_1\phi_2)(1-L_1^k)]^{-1-\frac{1}{k}}\dots$$
$$\dots[1-(1-\phi_1\phi_2\dots\phi_{k-1})(1-L_1^k)]^{-2+\frac{2}{k}}\,d\phi_1d\phi_2\dots d\phi_{k-1}\quad\dots\dots(55).$$

The distribution of L_1 for two samples ($k=2$) turns out to be

$$df(L_1)=\frac{\Gamma(2n-3)}{\Gamma^2(n-2)\,2^{2n-6}}L_1^{2n-5}\log\left(\frac{1+\sqrt{1-L_1^2}}{L_1}\right)dL_1\quad\dots\dots\dots(56),$$

and the significance of an observed value L_1 can be obtained from the probability integral (57), which results from integrating (56) by parts.

$$P(L_1<l_1)=\int_0^{l_1}df(L_1)$$
$$=\frac{\Gamma(2n-3)}{\Gamma(n-1)\,\Gamma(n-2)\,2^{2n-5}}\left\{(l_1^2)^{n-2}\log\left(\frac{1+\sqrt{1-l_1^2}}{l_1}\right)+\tfrac{1}{2}\int_0^{l_1^2}y^{n-3}(1-y)^{-\frac{1}{2}}dy\right\}\quad\dots\dots(57).$$

This expression depends on the Incomplete Beta Function. When $k>2$ we have so far been unable to find any simple expression for $df(L_1)$, and some method of approximation seems necessary. Approximate methods are discussed below.

(12) *The Distribution of L.* In a similar manner, if we let $\lambda_H^{1/N}=L$ then the h-th moment of L when $n_1=n_2=\dots=n_k=n$ is

$$M_{0h}=\frac{\Gamma(nk-2)}{\Gamma(nk-2+h)}k^h\frac{\Gamma^k\left(n-2+\frac{h}{k}\right)}{\Gamma^k(n-2)}\quad\dots\dots\dots\dots\dots\dots(58),$$

and the distribution of L can be expressed as

$$df(L) = \frac{\Gamma(nk-2)}{\Gamma^k(n-2)\,\Gamma(2k-2)\,k^{nk-2k}} L^{n-3k} (1-L^k)^{2k+\frac{(k-3)(k-2)-6}{2k}-1} d(L^k)$$

$$\times \int_0^1 \int_0^1 \dots \int_0^1 \phi_1^{2(k-1)+\frac{(k-3)(k-2)-2}{2k}-1} \phi_2^{3(k-2)+\frac{(k-3)(k-2)}{2k}-1} \dots \phi_{k-1}^{2+\frac{k-3}{k}-1}$$

$$\times (1-\phi_1)^{2-\frac{2}{k}-1} (1-\phi_2)^{2-\frac{1}{k}-1} \dots (1-\phi_{k-1})^{2+\frac{k-4}{k}-1} [1-(1-\phi_1)(1-L^k)]^{-2+\frac{1}{k}}$$

$$\times [1-(1-\phi_1\phi_2)(1-L^k)]^{-2} \dots [1-(1-\phi_1\phi_2\dots\phi_{k-1})(1-L^k)]^{-2-\frac{k-3}{k}}$$

$$d\phi_1 \dots d\phi_{k-1} \quad \dots\dots(59).$$

For the case $k=2$, (59) becomes

$$df(L) = \frac{\Gamma(2n-2)}{\Gamma^2(n-2)\,2^{2n-4}} L^{2n-5} \left\{ \log\left(\frac{1+\sqrt{1-L^2}}{L}\right) - \sqrt{1-L^2} \right\} dL \quad \dots(60).$$

The probability integral of (60) assumes the form

$$(L < l) = \int_0^l df(L) = \frac{\Gamma(2n-2)}{\Gamma(n-1)\,\Gamma(n-2)\,2^{2n-5}} \left\{ (l^2)^{n-2} \left[\log\left(\frac{1+\sqrt{1-l^2}}{l}\right) \right.\right.$$

$$\left.\left. - (1-l^2)^{\frac{1}{2}} \right] + \tfrac{1}{2}\int_0^{l^2} y^{n-3}(1-y)^{\frac{1}{2}}\, dy \right\} \quad \dots(61).$$

Again, if $k > 2$ some approximate solution appears necessary.

We note from the distributions of L_1 and L that (55) and (59) are actually the distributions of the n-th roots of λ_{H_1} and λ_H respectively.

(13) *Approximate Solution for testing H_1.* When $k > 2$ it appears necessary to employ some approximate method to calculate the probability integral of the sampling distribution of λ_{H_1} or of $L_1 = \lambda_{H_1}^{1/N}$. To establish the best and simplest method of procedure fuller investigation is required, but we believe that the relatively simple form of approximation which has been used in the single variable case is also suitable here. This involves the assumption that the sampling distribution of L_1 may be represented by the law

$$f(L_1) = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\,\Gamma(m_2)} L_1^{m_1-1} (1-L_1)^{m_2-1} \quad \dots\dots\dots\dots\dots\dots(62),$$

where m_1 and m_2 are determined so that the first and second moment coefficients of $f(L_1)$ about $L_1 = 0$ have the values given by (43) for $h = 1$ and 2 respectively. In other words we represent the distribution of L_1 by a Type I curve having the correct terminals and first two moment coefficients. In many practical applications it is possible to plan for the number of individuals in each sample to be the same, i.e. for $n_t = n\,(t = 1, 2, \dots k)$. This is the situation considered in the illustrations which follow. The equations for determining m_1 and m_2 then become

$$m_1 = \frac{M_{11}(M_{11}-M_{12})}{M_{12}-M_{11}^2}, \qquad m_2 = \frac{(1-M_{11})(M_{11}-M_{12})}{M_{12}-M_{11}^2} \quad \dots\dots\dots(63),$$

from which, since $N = nk$, we obtain from (43)

$$M_{11} = \frac{k}{N-k-1}\left\{\frac{\Gamma\left(n-2+\frac{1}{k}\right)}{\Gamma(n-2)}\right\}^k \quad \text{.....................(64)},$$

$$M_{12} = \frac{k^2}{(N-k-1)(N-k)}\left\{\frac{\Gamma\left(n-2+\frac{2}{k}\right)}{\Gamma(n-2)}\right\}^k \quad \text{..............(65)}.$$

For the probability integral of (62), we have

$$P(L_1 \leqslant l_1) = I_{l_1}(m_1, m_2) \quad \text{..........................(66)},$$

which is the Incomplete Beta Function. This may be obtained from the *Tables of the Incomplete Beta Function** if m_1 and m_2 are $\leqslant 50$ or by means of R. A. Fisher's z-transformation as has been suggested elsewhere†. If m_1 and m_2 are both large or nearly equal, (62) will approach the normal form and the ratio

$$\frac{L_1 - \text{Mean } L_1}{\sigma_{L_1}} = \frac{L_1 - M_{11}}{\sqrt{M_{12} - M_{11}^2}} \quad \text{.......................(67)}$$

can be used as an index of significance to be interpreted on the normal probability scale.

The probability integral of $L = \lambda_H^{1/N}$ could be obtained by a similar approximation, but in general it is likely that the hypotheses H_1 and H_2 will be tested separately.

V. PRACTICAL ILLUSTRATIONS.

(14) *Example* 1. *Relation between Tensile Strength and Hardness in Aluminium Die-Castings.*

This example has been referred to above in Section (3). We shall proceed first to test H_1, that is, the hypothesis that there is no significant difference between the samples as regards variation and covariation in strength and hardness. A summary of the necessary calculations is shown in Table II; we have $N = 60$, $k = 5$, $n = 12$. The unit for x (strength) is 1000 lb. per square inch, and for y it is Rockwell's E.

From (22) we have

$$L_1 = \lambda_{H_1}^{1/N} = \sqrt{\frac{\prod_{t=1}^{k}\left(|v_{ijt}|\right)^{\frac{1}{k}}}{|v_{ija}|}} \quad \text{....................... (68)},$$

and as indicated in the table it is found that $L_1 = \cdot 9065$. From (64) and (65) it is found that if H_1 were true‡, then Mean $L_1 = M_{11} = \cdot 889274$, $M_{12} = \cdot 792592$,

* To be issued shortly as a *Biometrika* publication.

† *Biometrika*, Vol. xxiv. p. 415.

‡ Brownlee's Seven-Figure Tables of the Logarithm of the Gamma Function were used, *Tracts for Computers*, No. IX.

TABLE II.

Strength (x) and Hardness (y) in Aluminium Die-Castings. Test of H_1 (bivariate).

t (Sample No.)	Sums of Squares $nv_{11t} = \sum_{a=1}^{n}(x_{ta} - \bar{x}_t)^2$	$nv_{22t} = \sum_{a=1}^{n}(y_{ta} - \bar{y}_t)^2$	Sums of Products $nv_{12t} = \sum_{a=1}^{n}(x_{ta} - \bar{x}_t)(y_{ta} - \bar{y}_t)$	Generalized Variances $\lvert v_{tw}\rvert = v_{11t}v_{22t} - v_{12t}^2$	$\log \lvert v_{tw}\rvert$
1	78·948	1247·18	214·18	365·204	2·56254
2	223·695	2519·31	657·82	910·401	2·95923
3	57·448	1241·78	190·83	243·029	2·38566
4	187·618	1473·44	375·91	938·451	2·97241
5	88·456	1171·73	259·18	253·281	2·40360
Totals	636·165 $= Nv_{11a}$	7653·44 $= Nv_{22a}$	1697·52 $= Nv_{12a}$		13·28344 $= \sum_{t=1}^{k} \log(\lvert v_{tw}\rvert)$

$$\lvert v_{tw}\rvert = v_{11a}v_{22a} - v_{12a}^2 = 552{\cdot}018,$$

$$\log L_1 = \tfrac{1}{2}\left\{\frac{1}{k}\log\prod_{t=1}^{k}(\lvert v_{tw}\rvert) - \log(\lvert v_{aw}\rvert)\right\} \qquad \text{(from definition (64))}$$

$$= \bar{1}{\cdot}957367,$$

$$L_1 = {\cdot}9065.$$

$$\left.\begin{array}{c}\text{Estimate of correlation}\\ \text{within samples}\end{array}\right\} r_a = \frac{Nv_{12a}}{\sqrt{Nv_{11a} \times Nv_{22a}}} = +{\cdot}7693.$$

$\sigma_{L_1} = {\cdot}04223$. The observed value of L_1 is therefore nearer to unity than the mean value expected in repeated samples, and the ratio (67) is only $+0{\cdot}41$. Therefore there is clearly no reason for rejecting H_1. If, however, we were to proceed in more detail we should find from (63) that $m_1 = 48{\cdot}210$, and $m_2 = 6{\cdot}003$, and by interpolating in the Tables of the Incomplete Beta Function that $P(L_1 < {\cdot}9065) = {\cdot}621$.

We may now proceed to test H_2, the hypothesis that neither mean strength nor mean hardness differs significantly from sample to sample. Table III contains a summary of the calculations in the form of an analysis of variance table. It is seen that $L_2 = {\cdot}6896$, while if H_2 were true, the probability law for L_2 is obtained from (46) as

$$P(L_2) = \frac{\Gamma(58)}{\Gamma(54)\,\Gamma(4)} L_2^{53}(1 - L_2)^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(69).$$

The mean of (69) is ·9310 and the standard error* is ·0330, so that the observed value differs from the mean by more than seven times the standard error. By actually integrating (69) it is found that $P(L_2 < {\cdot}6896) = {\cdot}0000019$. H_2 must clearly be rejected. To discover whether this is due to significant differences in mean strength or in mean hardness or in both, we must consider the two single-variable problems

* For (46): Mean $L_2 = \dfrac{N-k-1}{N-2}$, $\sigma_{L_2} = \dfrac{1}{N-2}\sqrt{\dfrac{(N-k-1)(k-1)}{N-1}}$

TABLE III.

Strength (x) and Hardness (y) in Aluminium Die-Castings. Tests of H_2.

	Degrees of Freedom	Sums of Squares (x)	Sums of Squares (y)	Sums of Products (x, y)	Generalised Variances
Between Samples	$k-1=4$	$Nv_{11m}=306{\cdot}089$	$Nv_{22m}=\ 662{\cdot}77$	$Nv_{12m}=\ 214{\cdot}86$	$\lvert v_{ijm}\rvert=\ \ 43{\cdot}528$
Within Samples	$N-k=55$	$Nv_{11a}=636{\cdot}165$	$Nv_{22a}=7653{\cdot}42$	$Nv_{12a}=1697{\cdot}52$	$\lvert v_{ija}\rvert=\ 552{\cdot}018$
Totals	$N-1=59$	$Nv_{11o}=942{\cdot}254$	$Nv_{22o}=8316{\cdot}19$	$Nv_{12o}=1912{\cdot}38$	$\lvert v_{ijo}\rvert=1160{\cdot}77$

$$L_2=\lambda_2^{1/N}=\sqrt{|v_{ija}|/|v_{ijo}|}={\cdot}6896,$$

$$\eta_{xt}^2=v_{11m}/v_{11o}={\cdot}3248, \qquad \eta_{yt}^2=v_{22m}/v_{22o}={\cdot}0797.$$

	Estimates of Variance	$\log_{10}$ (est.)	Estimates of Variance	$\log_{10}$ (est.)
Between Samples	$\dfrac{Nv_{11m}}{k-1}=76{\cdot}522$	$1{\cdot}883786$	$\dfrac{Nv_{22m}}{k-1}=165{\cdot}69$	$2{\cdot}219296$
Within Samples	$\dfrac{Nv_{11a}}{N-k}=11{\cdot}566$	$1{\cdot}063183$	$\dfrac{Nv_{22a}}{N-k}=139{\cdot}15$	$2{\cdot}143483$
Difference		${\cdot}820603$	Difference	${\cdot}075813$
$z=1{\cdot}15129\times$ Difference		${\cdot}9448$	z	${\cdot}0873$

separately. Tests may be applied to the squared correlation ratios η_{xt}^2 and η_{yt}^2, or R. A. Fisher's z-transformation can be used. The necessary calculations are shown in Table III.

Using Woo's tables*, it is found that η_{xt}^2 is clearly significant while η_{yt}^2 is not. Alternatively, referring to Fisher's z-tables† with $n_1=k-1=4$, $n_2=N-k=55$, it is seen that the 5 % point lies at about ·47 and the 1 % at about ·65, showing, as before, that mean strength differs significantly from sample to sample but not mean hardness.

The limited amount of test records available would therefore suggest the following tentative conclusions:

(a) Within the samples the relationship between the two qualities is stable, and represented by $\sigma_x=3{\cdot}401\times10^3$ lb. per sq. in., $\sigma_y=11{\cdot}80$ in Rockwell's E,

* *Tables for Statisticians and Biometricians*, Part II, Table IV.

† *Statistical Methods for Research Workers*, Table VI.

$r_{xy} = +\cdot 769$. (The first two values are the square roots of $Nv_{11a}/(N-k)$ and $Nv_{22a}/(N-k)$ respectively, and the last is the value of r_a given in Table II.)

(*b*) While the variation in mean strength from sample to sample is imperfectly controlled, the variation in hardness appears no more than might be expected through chance.

From the practical point of view this is not an altogether satisfactory result and further investigation into the anomaly (*b*) would be necessary before hardness could be used with confidence as an index of strength.

(15) *Example* 2. *Relation between Length and Breadth of Human Skulls.*

The data consist of standard measurements of length and breadth of skull in millimetres obtained for 20 adult males from each of 30 different races or groups*, i.e. $N = 600$, $n = 20$, and $k = 30$. That there would be considerable inter-racial variation for mean length and breadth was obvious, but it seemed to be of interest to examine the hypothesis H_1, that is to say, to test the extent of inter-racial uniformity in the relationship of length to breadth. These characters appear sufficiently nearly normally distributed within a race for the normal-theory tests to be applicable. Length will be denoted by x and breadth by y; a summary of the calculations is shown in Table IV. We find from these that

$$|v_{ija}| = 656\cdot 369 \qquad \frac{1}{k}\sum_{t=1}^{k} \log\{|v_{ijt}|\} = 2\cdot 644429$$

$$\log|v_{ija}| = 2\cdot 817148$$

$$\text{Difference} \quad \bar{1}\cdot 827281$$

$$\log L_1 = \tfrac{1}{2} \times \text{difference} = \bar{1}\cdot 913640, \; L_1 = \cdot 8197.$$

From (64) and (65) we obtain Mean $L_1 = M_{11} = \cdot 923678$, $M_{12} = \cdot 853317$, $\sigma_{L_1} = \cdot 0117$. The observed L_1 is below the expected mean value, and the ratio (67) is $-8\cdot 9$. This is so clearly significant that, without further refinement in calculation, we can say that H_1 is untenable. We must now examine whether this lack of uniformity is present both in the group standard deviations s_{xt} and s_{yt}, and in the correlations r_{xyt}.

For the first problem Neyman and Pearson's single variate test for H_1 may be applied†. This involves the calculation of the sums of the logarithms of the quantities nv_{11t} and of v_{22t} given in Table IV, since in this case

$$L_1 = \lambda_{H_1}^{2/N} = \prod_{t=1}^{k} (v_{iit})^{1/k}/(v_{iia}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(70),$$

where $i = 1$ for length and $i = 2$ for breadth. The calculations are shown in Table V, v_{11a} and v_{22a} being obtained from Table IV. It is found that were H_1 true, then:

$$\text{Mean } L_1 = \cdot 9496, \quad \sigma_{L_1} = \cdot 0129.$$

* We are indebted to Dr G. M. Morant for providing us with the necessary sources of information.

† For an illustration of the use of this test, see *Biometrika*, Vol. xxiv, p. 415.

TABLE IV.

Length (x) and Breadth (y) of Skulls. Tests of H_1; data for separate samples.

t	Race	nv_{11t}	nv_{22t}	nv_{12t}	$\lvert v_{ijt}\rvert$	$\bar{x}_t$	$\bar{y}_t$	s_{xt}	s_{yt}	r_t
1	Aëtas	612·00	300·14	41·50	454·91	172·00	143·92	5·532	3·874	+·097
2	Aleuts	218·20	391·20	57·80	205·05	183·30	149·20	3·303	4·423	+·198
3	Andamanese	216·45	311·25	149·50	112·55	167·95	136·25	3·290	3·945	+·576
4	Anglo-Saxons	975·44	777·45	−12·88	1895·47	191·62	142·45	6·984	6·235	−·015
5	Armenians	551·20	320·20	72·80	427·99	176·80	145·70	5·250	4·001	+·173
6	Australians. (South Australia)	1114·20	280·95	427·30	326·12	195·70	127·05	7·464	3·748	+·764
7	Australians. (Victoria)	1131·20	472·95	−27·20	1335·65	182·80	132·45	7·521	4·863	−·037
8	Copts	722·20	558·55	423·90	559·23	174·30	135·85	6·009	5·285	+·667
9	Dayake. (Borneo)	431·50	881·24	8·50	950·46	174·50	138·97	4·645	6·638	+·014
10	Easter Islanders	834·74	331·64	−59·16	683·33	191·72	133·42	6·460	4·072	−·112
11	Egyptians, 1st Dynasty. (Abydos) ...	518·71	319·07	89·03	393·95	185·62	136·50	5·093	3·994	+·219
12	Egyptians, Modern. (Cairo)	1115·45	382·30	−99·28	1041·45	186·86	133·90	7·468	4·372	−·152
13	Egyptians, Predynastic. (Badari) ...	542·05	356·64	140·23	434·13	184·65	131·42	5·206	4·223	+·319
14	English, Mediaeval. (Hythe)	714·95	748·14	227·08	1208·29	179·45	148·07	5·979	6·116	+·310
15	English, Roman? (Spitalfields)	1021·80	550·75	14·25	1406·38	181·10	143·75	7·148	5·248	+·019
16	English, 17th century. (Farringdon St)	697·80	312·24	207·80	436·75	192·10	145·97	5·907	3·951	+·445
17	Eskimos. (Greenland)	325·75	278·95	123·75	188·88	192·75	133·55	4·036	3·735	+·410
18	Gaunche. (Canary Islands)	327·75	158·55	215·75	13·54	189·25	141·65	4·048	2·816	+·946
19	Hindus. (Bengal)	650·55	700·55	11·95	1139·00	176·85	131·65	5·703	5·918	+·018
20	Indonesians. (Ceram)	887·75	1033·80	153·50	2235·48	177·25	134·90	6·662	7·190	+·160
21	Javanese	1018·14	392·14	112·61	966·43	171·57	141·42	7·135	4·428	+·178
22	Mongols. (Urga)	488·95	212·20	245·70	108·47	189·05	145·30	4·944	3·257	+·763
23	Moriori. (Chatham Islands)	232·45	488·24	33·97	280·84	186·05	142·27	3·409	4·941	+·101
24	Negroes, Teita. (Kenya Colony) ...	562·74	368·50	204·50	413·87	187·72	130·00	5·304	4·292	+·449
25	Papuans. (New Guinea)	404·80	448·95	104·60	426·98	188·40	132·05	4·499	4·738	+·245
26	Tagals. (Philippine Islands)	454·64	303·20	133·70	299·93	178·92	138·80	4·768	3·894	+·360
27	Tasmanians	698·00	237·75	241·00	269·67	190·00	139·25	5·908	3·448	+·592
28	Turks	381·80	393·20	−199·60	275·71	168·10	141·80	4·369	4·434	−·515
29	Swiss. (Münster)	405·75	587·20	11·00	595·34	177·75	151·20	4·504	5·418	+·023
30	Venezuelians	480·00	239·20	86·00	268·55	178·00	140·80	4·899	3·458	+·254
	Totals	18736·96 $=Nv_{11a}$	13137·14 $=Nv_{22a}$	+3139·60 $=Nv_{12a}$						

Consequently we have

	Length (x)	Breadth (y)
Observed L_1	·9000	·9074.
$(L_1 - \text{Mean } L_1)/\sigma_{L_1}$	$-3·84$	$-3·27$

TABLE V.

Length and Breadth of Skulls. Tests of H_1 (Single-variate).

	Length (x) $i=1$	Breadth (y) $i=2$
$\frac{1}{30}\sum_{t=1}^{30} \log(20v_{it})$	2·749820	2·599191
$\log 20$	1·301030	1·301030
$\frac{1}{30}\sum_{t=1}^{30} \log(v_{it})$	1·448790	1·298161
$\log(v_{i\alpha})$	1·494548	1·340349
$\log L_1$	$\bar{1}$·954231	$\bar{1}$·957812
L_1	·9000	·9074

The divergence shown by the ratios is significant, and it does not seem necessary to enter here into the approximate calculation of the probabilities $P(L_1 < ·9000)$ and $P(L_1 < ·9074)$, (which are both under ·01), since examples have been discussed elsewhere and it is hoped to publish shortly convenient tables for use with the test.

We must now examine the variation among the 30 correlation coefficients r_{xyt} $(t = 1, 2, \ldots 30)$; the best method of procedure is probably as follows:

If x and y are normally correlated it is known that in repeated samples of n that

$$z' = \tfrac{1}{2}\{\log_e(1+r) - \log_e(1-r)\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(71)$$

is approximately normally distributed with a standard error of $1/\sqrt{n-3}$*. Consequently, we may test whether k independent values of r differ only through chance fluctuations from some unknown population value of ρ, by calculating

$$\chi^2 = \sum_{t=1}^{k} \{(n_t - 3)(z_t' - \bar{z}')^2\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(72),$$

where $\bar{z}' = \sum_{t=1}^{k} (z_t'/k)$, and entering the (χ^2, P) tables with $k-1$ degrees of freedom (i.e. $n' = k$ in the notation of Elderton's Table). In the present instance it is found that $\chi^2 = 96·01$, while $n' = 30$, which is evidently significant. χ, in fact, deviates from the expected value by about 6·3 times the standard error†.

We have found therefore that the covariation in length and breadth of skull within a race cannot be considered as uniform from race to race; further, that while

* R. A. Fisher: *Metron*, Vol. I. No. IV. p. 13. An illustration of using this test with k values of r has been given by L. H. C. Tippett: *The Methods of Statistics* (1931), p. 143.

† This result is obtained by using the rule that when $f = n' - 1$ is large, $\sqrt{2\chi^2} - \sqrt{2f-1}$ is approximately normally distributed about zero with unit standard error.

the standard deviations certainly differ significantly, the lack of uniformity is due in much greater degree to the instability of the correlation coefficient. Having regard to the great variety in the data, these results were to be expected since a "race" is a loosely defined term, and the coefficient of correlation between two measures of size within a group will depend upon the homogeneity of that group. The more similar the skulls are in shape the higher is r likely to be. In considering the possible value of the criterion L_1 in anthropometric work, it should be remembered that, although the present paper is concerned only with the case of two correlated variables, the general theory developed by one of us* is applicable in the case of any number of variables.

Although the difference in mean length and breadth for the different races is so obvious that a statistical test is hardly required, it may be useful to summarise what would be the formal method of approach:

(1) Considering the two variables together it is found that H_1 (x and y) is quite untenable; therefore we should not proceed to test H_2 (x and y).

(2) $H_1(x)$ and $H_1(y)$ are also improbable, but the differences in the standard deviations s_x and s_y are hardly sufficient to invalidate the tests $H_2(x)$ and $H_2(y)$.

(3) From two tables for analysis of variance similar to those contained in Table III, it is found that $\eta_{xt}^2 = \cdot 6489$ and $\eta_{yt}^2 = \cdot 6294$. If H_2 were true for the case of a single variable, then Mean $\eta^2 = \cdot 0484$, and $\sigma_{\eta^2} = \cdot 0124$. Clearly, therefore, $H_2(x)$ and $H_2(y)$ are untenable, that is, the 30 samples of 20 provide convincing evidence that the racial mean characters differ significantly.

VI. CONCLUSION.

Certain general methods of analysis of multivariate data have been developed by one of us elsewhere. In the present paper the special case of two correlated variables has been taken, in order to illustrate (a) the process of reasoning underlying the methods, (b) the practical application of the resulting tests, (c) their relation to other tests in use. The following points may be emphasised:

(1) It is necessary to recognise that in many problems, hypotheses of the H_1 type need to be tested, as well as those of the H_2 type. The technique of Analysis of Variance does not appear suited to deal with the former when more than two samples are concerned.

(2) In the multivariate problem it would be possible to deal with the variation of each character and the correlation of each pair of characters separately, but the application in the first instance of a single comprehensive test has several advantages. If, for example, on the evidence available H can be accepted, it is unnecessary to proceed to test H_1 and H_2. Similarly, if H_1 (using p variates) can be accepted there should be no need to proceed to the p single-variate H_1 tests and the $\frac{1}{2}p(p-1)$

* Wilks, *loc. cit.*

correlation tests. The same situation arises in dealing with H_2. Even when the comprehensive test is not satisfied, and it is necessary to apply the separate tests in order to locate the source of disturbance, relatively little labour will have been wasted in applying the comprehensive test first. For example, in the case of two variables which has been illustrated, the calculation of $\log|v_{ijt}|$ needed to test H_1 (two variables) involves little extra work when once nv_{11t}, nv_{22t} and nv_{12t} have been computed. But the latter quantities are in any case required if the sample variances and correlations are to be considered separately.

(3) The methods suggested for calculating the significance of a given value of the λ or L criteria are admittedly not in final form. For convenience in practical working, tables to be entered with n and k are needed, which would show certain levels of significance of these criteria. The possibility of forming such tables is under consideration.

(4) It has been assumed throughout that the variables are normally distributed. Some investigations on the stringency of this assumption are in progress.

VII. Appendix.

To assist in the interpretation of λ_H, λ_{H_1} and λ_{H_2} as criteria for testing the corresponding hypotheses H, H_1 and H_2 we shall find it convenient to prove the following theorems.

Theorem I. *Let η_{ik} $(i=1, 2;\ k=1, 2, \ldots m)$ be any set of real numbers, and let the matrix*

$$\begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix}$$

be real and positive definite with $A_{12}=A_{21}$ and

$$\psi=\frac{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11}+\Sigma\eta_{1k}^2 & A_{12}+\Sigma\eta_{1k}\eta_{2k} \\ A_{21}+\Sigma\eta_{1k}\eta_{2k} & A_{22}+\Sigma\eta_{2k}^2 \end{vmatrix}},$$

where the moment products of η's are summed for k from 1 to m; then

(*a*) $0\leqslant\psi\leqslant 1$,

(*b*) *a necessary and sufficient condition that $\psi=1$, when*

$$|A_{ij}|>0, \text{ is that } \eta_{ik}=0 \qquad (i=1, 2;\ k=1, 2, \ldots m),$$

(*c*) *a necessary and sufficient condition that $\psi=0$ when $|A_{ij}|>0$, is that at least one of the η's be infinite.*

Proof: Let the determinants in the numerator and denominator of ψ be called A and B respectively. Then (*a*) can be shown at once by induction, for suppose $B\geqslant A$ for $k=t$, then for $k=t+1$, B can be written as B_t+q_{t+1}, where B_t is the value of B for $k=t$ and q_{t+1} is a positive definite quadratic form in $\eta_{1,t+1}$ and $\eta_{2,t+1}$. Therefore, setting $t=0, 1, 2, \ldots m$, we get

$$B\geqslant\ldots\geqslant B_{t+1}\geqslant B_t\geqslant\ldots\geqslant B_0\equiv A>0,$$

which is equivalent to the proof of (*a*).

The sufficient conditions in (b) and (c) are obvious. To prove the necessary condition of (b) we observe that for $B_{t+1}=B_t$ it is necessary that $\eta_{1,t+1}=\eta_{2,t+1}=0$ since q_{t+1} is positive definite in these two variables. Setting $t=0, 1, 2, \dots m$, we see that a necessary condition for

$$B=\dots=B_{t+1}=B_t=\dots=B_0\equiv A,$$

or $\psi=1$ is that $\eta_{ik}=0$ $(i=1, 2;\ k=1, 2, \dots m)$.

To prove the necessary condition in (c), we note that at least one member of the non-decreasing set $B \dots B_{t+1}, B_t \dots B_1$ must become infinite. Let B_{i+1} be the first B which is infinite, then clearly, $B, \dots B_{i+3}, B_{i+2}$ will also be infinite. But $B_{i+1}=B_i+q_{i+1}$, where B_i is finite. Therefore q_{i+1} must be positively infinite which can occur only when at least one of the numbers $\eta_{1,i+1}$ and $\eta_{2,i+1}$ is numerically infinite.

The proof of this theorem for the case when A and B are determinants of the n-th order can be carried out in essentially the same way as the one just given for two variables.

THEOREM II. *Let a_{ijt} $(i, j=1, 2;\ t=1, 2, \dots k)$ be any set of real numbers in which $a_{ijt}=a_{jit}$ for $i\neq j$, and such that the matrix $||a_{ijt}||$ is positive definite, and let $A=|A_{ij}|$, where*

$$A_{ij}=\sum_{t=1}^{k} p_t a_{ijt} \quad\dots\dots\dots\dots(\text{i}),$$

and the p's are positive such that $\Sigma p_t=1$. Then, if

$$\theta=\frac{\prod_{t=1}^{k}|a_{ijt}|^{p_t}}{A} \quad\dots\dots\dots\dots(\text{ii}),$$

we have

(a) $0\leqslant\theta\leqslant 1$,

(b) *a necessary and sufficient condition that $\theta=1$ when $|a_{ijt}|>0$ $(t=1, 2, \dots k)$ is that $a_{ijt}=a_{ijt'}$ $(i, j=1, 2;\ t, t'=1, 2, \dots k)$; that is, that the matrices $||a_{ijt}||$ be identical.*

PROOF: For the one-variable case, that is, when $i=j=1$, the theorem simply states that the weighted geometric mean of a set of positive numbers cannot exceed the weighted arithmetic mean of the set, and that the means can be equal only when all of the numbers are equal. The proof for this case can be found in a number of advanced algebra text-books and will be assumed.

The sufficient condition in (b) is obvious. Thus, let us consider the necessary condition. For convenience let

$$\phi=A-G \quad\dots\dots\dots\dots(\text{iii}),$$

where

$$G=\prod_{t=1}^{k} d_t^{p_t} \quad\text{and}\quad d_t=|a_{ijt}|.$$

Then the theorem reduces to the problem of showing that $\phi\geqslant 0$, where the equality will hold only when the matrices $||a_{ijt}||$ are identical. Consider the minimum of ϕ for variations of the a's. If it exists, it will be given by the equations $(t=1, 2, \dots k)$,

$$\frac{\partial\phi}{\partial a_{11t}}=p_t A_{22}-p_t\frac{a_{22t}}{d_t}G=0 \quad\dots\dots\dots\dots(\text{iv}),$$

$$\frac{\partial\phi}{\partial a_{22t}}=p_t A_{11}-p_t\frac{a_{11t}}{d_t}G=0 \quad\dots\dots\dots\dots(\text{v}),$$

$$\frac{\partial\phi}{\partial a_{12t}}=-2p_t A_{12}+2p_t\frac{a_{12t}}{d_t}G=0 \quad\dots\dots\dots\dots(\text{vi}).$$

By a straightforward combination of the first equation for all values of t,

$$A_{22}=\prod_{t=1}^{k} a_{22t}^{p_t} \quad\dots\dots\dots\dots(\text{vii}).$$

Similarly, with respect to (v),

$$A_{11}=\prod_{t=1}^{k} a_{11t}^{n_t} \qquad \text{.......................(viii)}.$$

From (vi)

$$a_{12t}d_{t'}=a_{12t'}d_t \qquad (t,\ t'=1,\ 2,\ \ldots\ k) \qquad \text{.......................(ix)}.$$

But from the case of a single variable (vii) and (viii) can hold only when

$$a_{11t}=a_{11t'},\quad a_{22t}=a_{22t'} \qquad (t,\ t'=1,\ 2,\ \ldots\ k) \qquad \text{.......................(x)}.$$

Call their common values a_{11} and a_{22} respectively.

Placing these values in (ix) which must hold for all values of a_{11} and a_{22}, we get at once that

$$a_{12t}=a_{12t'} \qquad (t,\ t'=1,\ 2,\ \ldots\ k) \qquad \text{.......................(xi)}.$$

Let the common value be a_{12}. Therefore (iv), (v) and (vi) are satisfied only when the matrices $||a_{ijt}||$ are identical. The matrix of second order derivatives of ϕ with respect to the a_{ijt} can be shown to be positive definite when (iv), (v) and (vi) are satisfied, provided $||a_{ij}||$ is positive definite. Thus, ϕ has a true minimum, and since the minimum is zero, and G is positive (*a*) follows at once. The generalisation of this theorem to the case where $i, j=1, 2, \ldots n$ is straightforward.

ON A METHOD OF DETERMINING WHETHER A SAMPLE OF SIZE n SUPPOSED TO HAVE BEEN DRAWN FROM A PARENT POPULATION HAVING A KNOWN PROBABILITY INTEGRAL HAS PROBABLY BEEN DRAWN AT RANDOM.

By KARL PEARSON.

PROBABILITY integrals have now been tabled for the following curves, all of which occur frequently in statistics, either as accurate or approximate distributions of statistical quantities:

(i) $y = y_0 e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$, the "normal" curve (probability integral provided by Sheppard's Tables*).

(ii) $y = y_0 x^{p-1}(1-x)^{q-1}$ (Pearson's Type I, *Tables of the Incomplete* B-*Function*†).

(iii) $y = y_0 \left(1 - \frac{x^2}{a^2}\right)^m$ (Pearson's Type II, ditto).

(iv) $y = y_0 \Big/ \left(1 + \frac{x^2}{a^2}\right)^m$ (Pearson's Type VII, "Student's" z-Curve, *Tables of the Incomplete* B-*Function*).

(v) $y = y_0 e^{-\frac{\gamma x}{a}} \left(1 + \frac{x}{a}\right)^p$ (Pearson's Type III, *Tables of the Incomplete* Γ-*Function*).

(vi) $y = y_0 (x-a)^{m_1}/x^{m_2}$ (Pearson's Type VI, by transformation to the Incomplete B-function).

(vii) $y = y_0 e^{-\gamma/x} x^{-p}$ (Pearson's Type V, by transformation to the Incomplete Γ-function).

(viii) $y = y_0 e^{-x/\sigma}$ (Pearson's Type X, Newman and Glaisher's tables‡ of the Exponential Function).

In these and a number of other cases the probability integral can be found or easily computed. The probability integral of the $T_\nu(x)$ Bessel function curve has also been provided by Miss F. N. David§. The sole outstanding case among the Pearson curves is Type IV, $y = y_0 e^{-\nu \tan^{-1}\frac{x}{a}} \Big/ \left(1 + \frac{x^2}{a^2}\right)^m$, where a table of

$$\int_{-\frac{\pi}{2}}^{\theta} e^{-\nu\theta} \cos^n \theta \, d\theta$$

* *Biometrika*, Vol. II. pp. 174—190, or *Tables for Statisticians*, Part I. pp. 1—10.

† Published by *Biometrika*, 1933.

‡ *Camb. Phil. Soc. Trans.* Vol. XIII. Part iii. pp. 145—272.

§ *Biometrika*, Vol. XXIV. pp. 344—346.

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for values of ν and n is required. By aid of these probability integral tables any single value of a variate x_s supposed to belong to one of these curves can have its probability integral p_s determined, which measures the frequency of values arising as great as or greater than x_s. Let us suppose a sample of size n containing the variates $x_1, x_2, x_3, \dots x_s, \dots x_n$ to be drawn from a distribution $y = \phi(x)$ and let $p_1, p_2, p_3, \dots p_s, \dots p_n$ be their respective probability integrals. What will be the distribution of these probability integrals?

Now p has a definite value for each value of x and by definition of the probability integral

$$p = \int_a^x \phi(x)\, dx \text{.......................................(1)},$$

where $x = a$ is the start of $\phi(x)$ and the whole area $\int_a^b \phi(x)\, dx = 1$.

Now the frequency with which p_s occurs may be represented by $F(p)\,dp$, but this is the same as the frequency of $x = \phi(x)\,dx$, or

$$F(p)\,dp = \phi(x)\,dx,$$

but by (i) $\qquad dp = \phi(x)\,dx.$

Accordingly $F(p) = 1$, or *the distribution for the probability integrals of any frequency curve is a rectangle on the base $p = 0$ to $p = 1$**.

We note that this is not a discrete but a perfectly continuous distribution of frequency. Since all the x's are supposed to be obtained by random sampling, all the p's will likewise be a random sampling distribution, and $P_{1\dots n}$ the probability of that random sample occurring or one with the individual variates having a greater probability, will be given by

$$P_{1\dots n} = p_1 p_2 \dots p_s \dots p_n = \prod_1^n (p_s).$$

We now add: Is this an improbable sample? In other words we enquire: What is the probability of a sample with as great as or a greater value than $P_{1\dots n}$ occurring? We need to find out the sum of all the samples with a probability $P = > P_{1\dots n}$†.

* Bayes' hypothesis of the rectangular distribution of probabilities, although it clearly does not apply to all probabilities, certainly does apply to all probability integral values.

† The reader must assure himself at this point that we are adopting methods of treatment with which he is already familiar. That is to say, we are extending the argument he is accustomed to use with regard to the probability integral for a single value x_1; for if p_1 be extremely small, he argues that it is reasonable to suppose that x_1 was not a random sample from the curve he has used to compute its probability integral. Now p_1 is the chance that a certain variate x_1 will not exceed a value ξ_1, p_2 that a second variate x_2 will not exceed a value ξ_2 and so on. Then, since $x_1, x_2, \dots x_n$ are supposed to be independent random samples, it follows that $P_{1\dots n} = p_1 p_2 \dots p_n$ is the chance that the combined system $x_1, x_2, \dots x_n$ will not exceed the system $x_1, x_2, \dots x_n$. We then turn the problem round so to speak and ask what other series of $x_1, x_2, \dots x_n$ will give a probability exceeding $P_{1\dots n}$. That is to say, we divide up the possible field of $p_1, p_2, \dots p_n$'s by a contour surface which separates the sub-field of $p_1, p_2, \dots p_n$'s which gives less combined probabilities from the sub-field which gives greater probabilities than $P_{1\dots n}$. If the sub-field of less probabilities be very small, then we judge that it is unreasonable to suppose the observed $x_1, x_2, \dots x_n$ form a random sample from the system or systems of distribution

To answer this we must first solve the following problem:

We have a system of n rectangular axes in Euclidean space and a point $z_1, z_2, \ldots z_n$. This point is constrained to lie in the "n-cuboid" with all its edges equal to unity. A surface is given by the equation

$$\lambda_n = z_1 z_2 \ldots z_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)\ bis,$$

which may be termed the "n-hyperboloid," we require to find out what volume of the n-cuboid is cut off by the n-hyperboloid.

Let us denote the volume of the n-cuboid *inside* the hyperboloid by V_n. Let us assume that

$$V_n = 1 - \lambda_n \left(1 - \frac{\log_e \lambda_n}{1!} + \frac{(\log_e \lambda_n)^2}{2!} - \frac{(\log_e \lambda_n)^3}{3!} + \ldots + (-1)^{n-1} \frac{(\log_e \lambda_n)^{n-1}}{(n-1)!}\right) \quad \ldots\ldots(2).$$

Now add another variate z_{n+1}, so that

$$z_1 z_2 z_3 \ldots z_n z_{n+1} = \lambda_{n+1}.$$

If z_{n+1} be made constant the section of the $(n+1)$-hyperboloid is an n-hyperboloid and the "volume" of this is given by (2) above. We must integrate these strips of "volume" from $z_{n+1} = \lambda_{n+1}$ to 1 and accordingly

$$\begin{aligned} V_{n+1} &= \int_{\lambda_{n+1}}^1 V_n \, dz_{n+1} \\ &= \int_{\lambda_{n+1}}^1 \left[1 - \frac{\lambda_{n+1}}{z_{n+1}} \left(1 - \frac{\log_e \dfrac{\lambda_{n+1}}{z_{n+1}}}{1!} + \frac{\left(\log_e \dfrac{\lambda_{n+1}}{z_{n+1}}\right)^2}{2!} \right.\right. \\ &\qquad \left.\left. - \ldots + (-1)^{n-1} \frac{\left(\log_e \dfrac{\lambda_{n+1}}{z_{n+1}}\right)^{n-1}}{(n-1)!}\right)\right] dz_{n+1}. \end{aligned}$$

Suppose
$$y = \log_e \frac{\lambda_{n+1}}{z_{n+1}},$$
then

$$\begin{aligned} V_{n+1} &= \Big[z_{n+1}\Big]_{\lambda_{n+1}}^1 + \lambda_{n+1} \int_0^{\log_e \lambda_{n+1}} \left(1 - \frac{y}{1!} + \frac{y^2}{2!} - \ldots + \frac{(-1)^{n-1} y^{n-1}}{(n-1)!}\right) dy \\ &= 1 - \lambda_{n+1} + \lambda_{n+1} \left(\frac{\log_e \lambda_{n+1}}{1!} - \frac{(\log_e \lambda_{n+1})^2}{2!} + \frac{(\log_e \lambda_{n+1})^3}{3!} \right. \\ &\qquad \left. - \ldots + \frac{(-1)^{n-1}}{n!} (\log_e \lambda_{n+1})^n\right) \\ &= 1 - \lambda_{n+1} \left(1 - \frac{\log_e \lambda_{n+1}}{1!} + \frac{(\log_e \lambda_{n+1})^2}{2!} - \ldots + \frac{(-1)^n}{n!} (\log_e \lambda_{n+1})^n\right), \end{aligned}$$

upon which we have based the calculation of their probability integrals. We do not assert that our test is the most stringent test, but that it is a very general and often easily applied systematic test of a given hypothesis. It may, indeed, be doubted whether any test is the most stringent throughout the whole area in which it has become customary to apply it. We may say of most tests, as of the present, that they may disprove any hypothesis, but that if they merely render a reasonable probability for the hypothesis, we cannot be certain that a more stringent test may not exist which would render the hypothesis very unlikely, or that a second hypothesis may not have a still greater probability.

or, if (2) holds for V_n, it will hold for V_{n+1}. But for $n=2$, $z_1 z_2 = \lambda_2$, and clearly

$$V_2 = \int_{\lambda_2}^{1} \left(1 - \frac{\lambda_2}{z_1}\right) dz_1 = 1 - \lambda_2 - \lambda_2 \left[\log_e z_1\right]_{\lambda_2}^{1}$$
$$= 1 - \lambda_2 (1 - \log_e \lambda_2).$$

Hence the formula holds for V_2 and will therefore hold for V_3 and so on by induction, and is thus generally true.

We see accordingly that

$V_n = 1 - \lambda_n \times$ sum of first n terms of the exponential series for $e^{-\log_e \lambda_n}$.

But the first n terms of the exponential series e^x

$$= e^x (1 - I(n-1, x)),$$

where I denotes the incomplete Γ-function ratio. Thus

$$V_n = 1 - \lambda_n e^{-\log_e \lambda_n} (1 - I(n-1, -\log_e \lambda_n))$$
$$= I(n-1, -\log_e \lambda_n) \quad \text{...(3).}$$

Since λ_n must always be less than unity the incomplete Γ-function ratio is real. If we use the *Tables of the Incomplete* Γ*-Function**, we must look out $I(n-1, u)$, where

$$u = -\frac{\log_e \lambda_n}{\sqrt{n}} = -\frac{\log_{10} \lambda_n}{\sqrt{n} \log_{10} e} \quad \text{..........................(4).}$$

Now let us return to our original problem. We have a sample of n variates $x_1, x_2, \ldots x_n$ taken from a population following a given or supposed law of distribution of which we know the probability integral. The values of this for our n sample variates $x_1, x_2, \ldots x_n$ are respectively $p_1, p_2, \ldots p_n$. These probabilities follow a rectangular distribution. If we take $\lambda_n = p_1 p_2 \ldots p_n$—the probability of the occurrence of the particular independent set of probabilities $p_1, p_2, \ldots p_n$—then the probability P_{λ_n} of a combination occurring with a probability value as great as or greater than λ_n is given by

$$P_{\lambda_n} = I\left(n-1, -\frac{\log_{10} \lambda_n}{\sqrt{n} \log_{10} e}\right),$$

or if Q_{λ_n} be the probability of a lower probability occurring,

$$Q_{\lambda_n} = 1 - P_{\lambda_n} = 1 - I\left(n-1, -\frac{\log_{10} \lambda_n}{\sqrt{n} \log_{10} e}\right) \quad \text{..................(5),}$$

where $I(p, u)$ is the function tabled in the *Tables of the Incomplete* Γ*-Function*.

The effectiveness of this method of approaching the problem of small random samples lies in the facts, (i) that grouping the individuals of small samples to obtain a (P, χ^2) test is a somewhat hazardous proceeding when n is small, (ii) we do not

* H.M. Stationery Office, 1922.

make the somewhat rash assumption that binomials $(p+q)^s$ in which p is not nearly equal to q, and s is small, may be replaced by a normal curve.

Given that the probability integral of the supposed (or real) parent population is known, then we find Q_{λ_n} without any approximations or hypotheses.

If Q_{λ_n} be very small, we have obtained an extremely rare sample, and we have then to settle in our minds whether it is more reasonable to suppose that we have drawn a very rare sample at one trial from the supposed parent population, or that our hypothesis as to the character of the parent population is erroneous, i.e. that the sample $x_1, x_2, \dots x_n$ was not drawn from the supposed population.

This "settling of what is reasonable" occurs not only with the (P, χ^2) test*, but with the application of most criteria in statistics. It is not peculiar to the present, or, as I propose to call it, the (P, λ_n) test.

A table of P_{λ_n} for $n=2$ to 30 by units and $-\log_{10}\lambda_n$ from 0 upwards by ·125 or ·250 can be formed from the *Incomplete* Γ-*Function Tables*, and when published will expedite the application of the (P, λ_n) test. Such a table is completed, and will shortly be published.

Another very important attribute of this (P, λ_n) test may now be mentioned. Let us suppose we have any number of parent populations each following its own law of distribution, then if x_s be an individual drawn *at random* from the sth population, its probability integral will be also a random sample from a rectangular distribution. Hence the individuals $x_1, x_2, \dots x_n$ need not be drawn from the same parent population, but may be drawn from any number of populations, one or more from each. Their probability integrals $p_1, p_2, \dots p_n$ will all be random samples from a rectangular distribution and may be combined to serve as a random sample of n from such a distribution. Our (P, λ_n) test is accordingly a test of randomness, and not a test of whether the series of individuals have been drawn from a single particular type of distribution—z-distribution, σ^2 distribution, normal distribution or what not. If the sample of n gives a highly improbable result, then we doubt its randomness. This want of randomness may arise because the selection of them or a certain number of them has not been random, or because their probability integrals or some of them have been calculated on the basis of a hypothesis as to their parent population or parent populations which is in itself incorrect, e.g. a sample of x's might have their probability integrals calculated from a normal curve, whereas in fact they were really a random sample from a Type I curve. The probability integrals would not then appear as a reasonable random sample from a rectangular distribution. When we conclude that our sample of probability integrals is very improbably random, we must turn to other sources to determine whether it is owing to the

* There exists a relation between the (P, χ^2) and (P, λ_n) tests for which, I think, some explanation must be forthcoming. Namely, P_{χ^2} can be found from the I (p, u) Tables by taking

$$1 - P_{\chi^2} = I\left(\tfrac{1}{2}(n'-3),\ \frac{\tfrac{1}{2}\chi^2}{\sqrt{\tfrac{1}{2}(n'-1)}}\right) \quad \dots\dots\dots\dots (6).$$

Hence if we take out the P_{χ^2} corresponding to $n'=2n+1$ and $\chi^2 = -2\log_{10}\lambda_n/\log_{10}e$, then P_{χ^2} will equal Q_{λ_n}; and thus within their range the (P, χ^2) Tables may be used to find Q_{λ_n}.

selection being biassed, or to an erroneous hypothesis as to the parental distributions on the basis of which we have computed our probability integrals*.

As a result of the above we see that this new (P, λ_n) method enables us to form *combined* tests, some of the x's may be means, others standard deviations, others correlation coefficients, etc. etc. All we need do, if these quantities are uncorrelated, is to calculate their probability integrals from the appropriate distributions using when necessary the most probable values, as determined from the samples, of the constants required in the parental distributions. Illustrations of such combined tests are given below.

If we suppose the probability integrals, p_s ($s = 1, 2, \ldots n$), to be uniformly distributed, for example, by dividing the range 0 to 1 into n equal sections, and placing a value of p_s in the centre of each of them, so that p_s takes the value $(2s-1)/2n$, then the corresponding χ^2 would be zero, and $P_{\chi^2} = 1$. There is something difficult about this. The result agrees with the mathematical expectation, "one ball in each compartment," but the basis of the *theory* is really too narrow†. The probability of the result must depend upon *where* the p's fall in each compartment. Let us consider what happens in the corresponding case of P_{λ_n}.

We have

$$e^{-\sqrt{n}\,u} = \lambda_n = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(2n)^n} = \frac{(2n)!}{n!\,2^{2n} n^n}$$

* This dilemma as to randomness occurs as far as I can see in most published tests, e.g. we assume as hypothesis that the two samples were drawn from the same normal population, and find it very improbable, this may really be due to bias in the taking of one or other sample and not to the absence of normality. Only further knowledge or investigation can lead to a discrimination between the two possibilities. For example, we have the means and standard deviations of two samples, and we wish to ascertain whether their parent populations are differentiated. We adopt the z-test. In doing so we make two hypotheses, (i) that of normal distributions, (ii) that both samples are truly random. Hence we have at least three possibilities, (a) that one or both parent populations have not normal distributions, (b) that one or both samples are not random and (c) that the two samples do not come from the *same* normal population. It seems to me that many users of the test, when they get an improbable result assume straight off that (c) must be the origin of it, and do not question the possibility of (a) or (b) being the source of the observed improbability. The P_{λ_n} test is no worse (or no better) than the other tests in this respect, i.e. that we have to consider whether the observed want of randomness in the distribution of the probability integrals is due to the hypothesis as to the nature of the distribution curve or to want of the other randomness in the process of sampling. It is unfortunate that the same word "randomness" has to be used in two places in the same investigation. The want of "randomness" in the distribution of probability integrals may or may not be evidence of want of randomness in the sampling.

† Another point is common to other tests as well as our present test, although at times the point is overlooked, namely a result may—to speak paradoxically—be so highly probable as to be wholly improbable. My memory carries me back many years to a memoir I read in manuscript. It concerned the distribution of a character in over a thousand offspring of certain sets of parents; the character being treated as involving five or six Mendelian factors. The classified offspring distributed themselves absolutely accurately according "to the mathematical expectation"; but the odds of course against such an occurrence were immense, and this was pointed out to the writer. The memoir afterwards appeared in print, the "improbable high probability" having disappeared. This development not only convinced me of the elasticity of Mendelian categories, but led me to realise how great is the risk that a biologist, ignorant of mathematical statistics, may heedlessly run in his enthusiasm for a hypothesis. His results may be "too good to be true."

or $$u = \frac{\sqrt{n}}{\log_{10} e}\left(\frac{1}{n}\log_{10}(n\,!) + \log_{10} n + 2\log_{10} 2 - \frac{1}{n}\log_{10}((2n)\,!)\right) \quad\text{.........(7).}$$

Applying Sterling's Theorem to the value of $e^{-\sqrt{n}u}$, we find for the asymptotic value of u,

$$u_{n\to\infty} = \sqrt{n} - \frac{\log 2}{2\sqrt{n}\times \cdot 434{,}2945} \quad\text{........................(8).}$$

Accordingly we have the following results, the first two being obtained by linear interpolation from the *Tables of the Incomplete* Γ-*Function* and liable to an error of a unit in the last decimal of P_{λ_n}.

n	u	$I(n-1, u)$	P_{λ_n}
5	2·08479	I(4, 2·08479)	·5018
10	3·05400	I(9, 3·05400)	·5008
100	9·96561	I(99, 9·96561)	·5004
500	22·34523*	I(499, 22·34523)	·5002
1000	31·62763†	I(999, 31·62763)	·5001
∞	∞	I(∞, ∞)	·5000

Thus we see that P_{λ_n} rapidly approaches the value ·5; in other words, in an absolutely uniform distribution of the probability integrals p_s there would be as many distributions above as below this value. This is what we might *a priori* expect, but it is suggestive in endeavouring to interpret the relation of P_{λ_n} to P_{χ^2}. We will return to this point shortly.

In order to determine the values of the incomplete Γ-function for 99, 499 and 999, recourse was had to the E method of p. xvi of the *Instructions as to the use of the Tables*‡. This method is peculiarly advantageous in the present case both practically and theoretically.

If $p = n - 1$, and $v = \sqrt{n}u = \sqrt{p+1}\,u = p + d'$,

$$\int_0^v \frac{x^p e^{-x}dx}{\Gamma(p+1)} = 0{\cdot}5 + y_2(d'-d) + \frac{9y_1 - 185y_2 + 180y_3 - 4y_4}{540}(d'^3 - d^3)$$
$$- \frac{6y_1 + 10y_2 - 15y_3 - y_4}{360}(d'^4 - d^4) + \frac{3y_1 + 25y_2 - 30y_3 + 2y_4}{900}(d'^5 - d^5) \quad\text{...(9),}$$

where $$y_1 = \frac{(p-2)^p e^{-(p-2)}}{\Gamma(p+1)}, \quad y_2 = \frac{p^p e^{-p}}{\Gamma(p+1)},$$
$$y_3 = \frac{(p+1)^p e^{-(p+1)}}{\Gamma(p+1)}, \quad y_4 = \frac{(p+3)^p e^{-(p+3)}}{\Gamma(p+1)},$$

and $$d = \cdot 6666{,}6667 + \frac{\cdot 0197{,}5309}{p+1} + \frac{\cdot 0072{,}1144}{(p+1)^2} + \frac{\cdot 0003{,}8554}{(p+1)^3} \quad\text{......(10)}$$

gives the distance from mode to median.

* Differs by five units in the last figure from the asymptotic value.

† Asymptotic value.

‡ *Tables of the Incomplete* Γ-*Function*.

Working to four figures only the first two or three terms in (9) and the first two in (10) suffice when p is of the order 100 and more, and the values of the y's are easily determined. Now $v = p + d'$ and the asymptotic value of

$$v = \sqrt{n}\,u = \sqrt{n}\left(\sqrt{n} - \frac{\log 2}{2\sqrt{n} \times \cdot 434{,}2945}\right)$$

$$= n - 1 + \left(1 - \frac{\log 2}{\cdot 868{,}5890}\right) = p + \cdot 653{,}4264.$$

Thus $d' - d$ = a very small quantity, i.e. approaches ·01324, and since y_2 is of the order ·01 to ·02 even the second term of (9) only influences the fourth decimal place, and we see how the approximation to ·5000 arises.

Now while the value of P_{χ^2} remains the same wherever we place the p_s in the n divisions, the value of P_{λ_n} will vary according to their position in those divisions, and it seems very proper that it should. We will consider three cases of the p's for $n = 5$. Instead of taking the values $p_1 = \cdot 1$, $p_2 = \cdot 3$, $p_3 = \cdot 5$, $p_4 = \cdot 7$ and $p_5 = \cdot 9$ we will:

(A) Give the p's larger values close up to the boundaries of the divisions, namely: $p_1 = \cdot 19$, $p_2 = \cdot 39$, $p_3 = \cdot 59$, $p_4 = \cdot 79$ and $p_5 = \cdot 99$.

(B) Give the p's smaller values close up to the opposite boundaries of the five divisions, namely: $p_1 = \cdot 01$, $p_2 = \cdot 21$, $p_3 = \cdot 41$, $p_4 = \cdot 61$ and $p_5 = \cdot 81$.

(C) Take values drawing up the p's closely to the central value ·50, namely: $p_1 = \cdot 19$, $p_2 = \cdot 39$, $p_3 = \cdot 50$, $p_4 = \cdot 61$, $p_5 = \cdot 81$.

We have $\sqrt{n} \log_{10} e = \cdot 971{,}1120$, and accordingly

$$\text{(A)}\quad \log \lambda_n = -1\cdot 466{,}0675,\quad u = 1\cdot 5096,\quad Q_{\lambda_n} = \cdot 7486,$$

$$\text{(B)}\quad \log \lambda_n = -3\cdot 371{,}1820,\quad u = 3\cdot 47147,\quad Q_{\lambda_n} = \cdot 1142,$$

$$\text{(C)}\quad \log \lambda_n = -1\cdot 737{,}3970,\quad u = 1\cdot 78908,\quad Q_{\lambda_n} = \cdot 6287.$$

Now we notice that while $P_{\chi^2} = 1$ for all these cases*, Q_{λ_n}—while in none of the three cases giving an improbable result—is far from giving identical measures for the three distributions of the p's. When we move the central points towards the greater values of p_s, i.e. in (A), we find that 75 °/。 of cases have a less degree of probability; when we move them towards the lesser degrees of probability, i.e. in (B), we find that only about 11 °/。 have a lesser degree of probability, and finally when we endeavour to concentrate towards the centre of the entire range, i.e. in (C), we find the intermediate value, namely about 63 °/。 of cases have a lesser probability.

This illustration will I think suffice to indicate some advantages of the P_{λ_n} test over the P_{χ^2} test.

Illustration 1. *Use of the Probability Integral from the Table of the Normal Curve.*

The mean length of life of 15 samples of five electric lamps is provided by E. S. Pearson.

* This is quite apart from the questionable replacing of a binomial by a normal curve in such a case as $(\frac{1}{2} + \frac{1}{2})^5$.

TABLE I.

*Length of Life of Lamps in Hours**.

Sample No.	Mean	Standard Deviation	Sample No.	Mean	Standard Deviation
1	1295	440	9	1715	385
2	2005	435	10	1650	460
3	2445	580	11	1935	560
4	1900	345	12	1760	280
5	2570	290	13	2175	465
6	1980	510	14	1570	505
7	1990	445	15	1670	380
8	1990	315	—	—	—

Is it reasonable to suppose that these 15 five lamp means have been drawn from the same normal population?

In order to answer this question we must determine what are the most probable values to assign to the mean M and the standard deviation Σ of this supposed common parent population. Let the samples be v in number, and of different sizes n_t, the mean and standard deviation of the tth sample being m_t and s_t. Then the most probable values of M and Σ^2 will be obtained by making the following expression a maximum:

$$E = \prod_{t=1}^{v} (n_t) \frac{1}{(\sqrt{2\pi})^v \Sigma^v} e^{-\underset{t=1}{\overset{v}{S}} \frac{1}{2} \frac{n_t (m_t - M)^2}{\Sigma^2}} \times \frac{1}{2^{N-2v}} \prod_{t=1}^{t=v} \frac{1}{\Gamma(\frac{1}{2}(n_t - 2))}$$

$$\times \frac{1}{\Sigma/\sqrt{2n_t}} \left(\frac{s_t}{\Sigma/\sqrt{2n_t}}\right)^{n_t - 2} e^{-\frac{1}{2}\left(\frac{s_t}{\Sigma_t/\sqrt{2n_t}}\right)^2} \quad \text{......(11).}$$

Taking the logarithmic differentials with regard to M and Σ we find

$$\underset{t=1}{\overset{t=v}{S}} n_t (m_t - M)/\Sigma^2 = 0,$$

or

$$M = \underset{t=1}{\overset{t=v}{S}} \frac{(n_t m_t)}{N}, \text{ where } N = S(n_t) \quad \text{....................(12),}$$

thus M is the weighted mean of the sample means. Further, after differentiating and collecting terms we find

$$\Sigma^2 = \frac{S n_t (s_t^2 + (m_t - M)^2)}{N} \quad \text{........................(13).}$$

Clearly (12) and (13) simply amount to saying that the best value to give M and Σ is that obtained by pooling all the individual values in the samples and finding the mean and standard deviation of the combination. We may note here that if our

* *Tables of the Incomplete* B-*Function*, Introduction, p. lii.

hypothesis were that our samples were drawn from independent parent populations, but with the same variability, we should have to replace the M in (13) by M_t and differentiate with respect to every M_t. This gives us $M_t = m_t$, and

$$\Sigma^2 = \frac{Sn_t(s_t^2)}{N} \qquad \text{..............................(14)},$$

i.e. the weighted square of the variances. The hypothesis that M_t varies from sample to sample may indicate, however, a secular change in M_t and so involve correlation between successive samples, which has not been allowed for in (11) where the samples are considered independent.

We proceed to find M and Σ for the data of the electric lamps. The needful calculations are provided in Table I. If we have determined M and Σ, then the m's are distributed normally with a mean M and standard deviation $\Sigma/\sqrt{n}$, so that we have only to look up $(M - m_t)/(\Sigma/\sqrt{n})$ in the table of the normal probability integral to obtain the series of probability integrals, p_t, in column (h) of our table. Great care must be taken to regard the sign of $(M - m_t)/(\Sigma/\sqrt{n})$ as the areas must all be measured from *one* end of the normal curve.

TABLE II.

(a) Sample No.	(b) Mean	(c) $m_t - M$	(d) $(m_t - M)^2$	(e) s_t	(f) s_t^2	(g) $(c)/(\Sigma/\sqrt{n})$	(h) p_t	(i) $\log_{10} p_t$
1	1295	−615	378225	440	193600	−2·555	·00052	$\bar{4}$·716,0033
2	2005	+ 95	9025	435	189225	+ ·395	·65536	$\bar{1}$·816,4799
3	2445	+535	286225	580	336400	+2·222	·98686	$\bar{1}$·994,2555
4	1900	− 10	100	345	119025	− ·042	·48352	$\bar{1}$·684,4144
5	2570	+660	435600	290	84100	+2·742	·96946	$\bar{1}$·986,5299
6	1980	+ 70	4900	510	260100	+ ·291	·61447	$\bar{1}$·788,5007
7	1990	+ 80	6400	445	198025	+ ·332	·63005	$\bar{1}$·799,3750
8	1990	+ 80	6400	315	99225	+ ·332	·63005	$\bar{1}$·799,3750
9	1715	−195	38025	385	148225	− ·810	·20897	$\bar{1}$·320,0839
10	1650	−260	67600	460	211600	−1·080	·14007	$\bar{1}$·146,3451
11	1935	+ 25	625	560	313600	+ ·104	·54141	$\bar{1}$·733,5263
12	1760	−150	22500	280	78400	− ·623	·26664	$\bar{1}$·425,9253
13	2175	+265	70225	465	216225	+1·101	·86455	$\bar{1}$·936,7901
14	1570	−340	115600	505	255025	−1·412	·12631	$\bar{1}$·101,4377
15	1670	−240	57600	380	144400	− ·997	·15963	$\bar{1}$·203,1145
$\frac{1}{15}$ sum $= M = 1910$		$\frac{1}{15}$ sum $= 99937$		$\frac{1}{15}$ sum $= 189,812$		—	$\log \lambda_n = -8·547,8434$	

$\Sigma^2 = 99,937 + 189,812 = 289,749$, $\sqrt{n}\log_{10}\sigma = 1·682,0153$, $\Sigma = 538·28$, and $\Sigma/\sqrt{n} = 240·7261$.

Thus $u = 5·081,906$, and we require $I(14,\ 5·081,906)$; we find this from the *Tables of the Incomplete* Γ-*Function* to be ·88209. Hence $Q_n = ·118$: or, the number of more improbable sets of 15 samples is 11·8 %.

We cannot on the result of this test assert that the lamps' lives were certainly not samples from the same population.

Illustration 2. Use of the Incomplete B*-Function Table to determine the Probability Integrals.* We may use the data of Illustration 1 to exemplify this method. "Student" deduced* that if M was the mean of the parent population and m_t the mean, s_t the standard deviation of a sample of size n_t, then the distribution of $z_t = (m_t - M)/s_t$ is given by the curve

$$y = y_0 \frac{1}{(1 + z_t^2)^{\frac{1}{2}n_t}} \quad \text{.................................(15).}$$

We require the probability integral of this curve, and various tables have been computed for it, either for z_t or some modified form of z_t. I personally have found nothing so comprehensive and convenient as the *Incomplete* B*-Function Table.*

The probability integral is

$$p(n, z) = \int_{-\infty}^{z} \frac{dz}{(1 + z^2)^{\frac{1}{2}n}} \Big/ \int_{-\infty}^{+\infty} \frac{dz}{(1 + z^2)^{\frac{1}{2}n}}$$

$$= \cdot 5 + \int_0^z \frac{dz}{(1 + z^2)^{\frac{1}{2}n}} \Big/ \left(2 \int_0^\infty \frac{dz}{(1 + z^2)^{\frac{1}{2}n}}\right).$$

Put $$z^2 = \frac{x}{1 - x} \quad \text{or} \quad x = \frac{z^2}{1 + z^2},$$

then $$p(n, z) = \cdot 5 \left\{1 + \int_0^x x^{-\frac{1}{2}} (1 - x)^{\frac{1}{2}(n-3)}\, dx \Big/ \int_0^1 x^{-\frac{1}{2}} (1 - x)^{\frac{1}{2}(n-3)}\, dx\right\}.$$

But the ratio of the two integrals $= I_x(\frac{1}{2}, \frac{1}{2}(n - 1))$ or the incomplete B-function ratio, the quantity $I_x(p, q)$ tabled in the corresponding book of tables. Thus we may write

$$p(n, z) = \cdot 5 \,\{1 + I_x(\tfrac{1}{2}, \tfrac{1}{2}(n - 1))\}$$
$$= \tfrac{1}{2} \{2 - I_{1-x}(\tfrac{1}{2}(n - 1), \tfrac{1}{2})\}$$
$$= \tfrac{1}{2} \{2 - I_{\frac{1}{1+z^2}}(\tfrac{1}{2}(n - 1), \tfrac{1}{2})\} \quad \text{.....................(16).}$$

The reason for this change of expression is that the *Tables of the Incomplete* Γ*-Function* are provided for $p \geqslant q$. We must remember, however, as especially important that we have got to allow for z being negative. Accordingly we take

$$\left.\begin{aligned} p(n, +z) &= 1 - \tfrac{1}{2} I_{\frac{1}{1+z}}(\tfrac{1}{2}(n - 1), \tfrac{1}{2}) \\ p(n, -z) &= \quad\ \tfrac{1}{2} I_{\frac{1}{1+z^2}}(\tfrac{1}{2}(n - 1), \tfrac{1}{2}) \end{aligned}\right\} \quad \text{..................(17).}$$

For present purposes linear interpolation into the B-function table will suffice. The following table indicates the needful work. M remains $= 1910$.

* *Biometrika*, Vol. VI. p. 8.

TABLE III.

Sample No.	$m_t - M$	σ_t	z_t	$1+z_t^2$	$1-x_t$ $=1/(1+z_t^2)$	p_{x_t}	$\log p_{x_t}$
1	−615	440	−1·39772	2·95362	·33856	·0021	$\bar{3}$·322,2193
2	+ 95	435	+0·21839	1·04769	·95448	·7089	$\bar{1}$·850,5850
3	+535	580	+0·92241	1·85084	·54030	·9776	$\bar{1}$·990,1612
4	− 10	345	−0·02899	1·00084	·99916	·4915	$\bar{1}$·691,5235
5	+660	290	+2·27586	6·17954	·16182	·9997	$\bar{1}$·999,8697
6	+ 70	510	+0·13725	1·01884	·98151	·6355	$\bar{1}$·803,1156
7	+ 80	445	+0·17978	1·03232	·96869	·6753	$\bar{1}$·829,4967
8	+ 80	315	+0·25397	1·06450	·93940	·7384	$\bar{1}$·868,2317
9	−195	385	−0·50649	1·25653	·79584	·1111	$\bar{1}$·045,7141
10	−260	460	−0·56522	1·31947	·75788	·0893	$\bar{2}$·950,8515
11	+ 25	560	+0·04464	1·00199	·99801	·5201	$\bar{1}$·716,0809
12	−150	280	−0·53571	1·28698	·77701	·0997	$\bar{2}$·998,6925
13	+265	465	+0·56989	1·32477	·75485	·9123	$\bar{1}$·960,1377
14	−340	505	−0·67327	1·45329	·68809	·0590	$\bar{2}$·770,8520
15	−240	380	−0·63158	1·39889	·71485	·0693	$\bar{2}$·840,7332

As before $\qquad \sqrt{n}\log_{10}e = 1·682,0153, \quad S \log p_{x_t} = -9·301,0094 = \log \lambda_n.$

Thus $\qquad u = 5·565746$, and $Q_{\lambda_n} = 1 - I\,(14, 5·56575)$,

where the latter function is the Γ-function ratio. Interpolating linearly from the *Incomplete* Γ-*function Table*, we have

$$Q_n = 1 - ·93041 = ·0696.$$

Thus some 7 °/ₒ of series of sample would provide a greater degree of improbability.

Dr Egon S. Pearson has applied to the same data in the Introduction to the *Tables of the Incomplete* B-*Function* a third test, namely a modification of Fisher's test for determining whether the regression of one character on a second is a horizontal straight line. This involves the computing of $\eta^2 = \overset{s}{\underset{t=1}{S}}\, n_t \dfrac{(m_t - M)^2}{\Sigma^2}$, and the determination of $I_{1-\eta^2}\left(\tfrac{1}{2}(N-v), \tfrac{1}{2}(v-1)\right)$ from the *Tables of the Incomplete* B-*Function*. Thus as in our *Illustration* 1 we have to determine the most probable values of M and Σ, but there is only one value to be found from the table, not 15 from that table, and one from the *Incomplete* Γ-*function Table*. In this case $\eta^2 = ·3449$ and we have $Q_{\lambda_n} = I_{·6551}(30, 7) = ·0152$, or there are 1·52 °/ₒ of cases only more improbable. It will be seen that this result is more stringent than either of our previous tests. Can we find a reason for this? Fisher's test is based on a triple hypothesis: (i) linearity of regression, (ii) homoscedasticity of arrays and (iii) their normality. Now the approach to a normal distribution of means is fairly rapid even for distributions not absolutely normal. Further the formulae (12) and (13) for M and Σ^2, while we have deduced them from a normal distribution, are most reasonable formulae to take in non-normal distributions, and lastly no assumption is made as to homoscedasticity of arrays. Accordingly we might

anticipate that our test in *Illustration* 1 would cover a wider range than Fisher's triple hypothesis does, and so it appears from the result.

Again in the z-test of the present *Illustration* we do suppose that each sample is taken from a normal distribution, and that all these normal parent populations have the same mean, but we do not insist on the standard deviations of all these parent populations being the same. The probability integrals of the several z_t can be combined, as they are random selections of p_t between 0 and 1. Thus again the hypotheses involved do not seem so stringent as in Fisher's case.

With regard to the comparison of the tests in *Illustrations* 1 and 2, it is not at once obvious why the selection from v normal populations having the same means but not *necessarily* the same standard deviations should be more stringent than the selection from v populations having the same means and standard deviations but not *necessarily* strictly normal.

Illustration 3. In an experiment* in which a certain number of children were given raw milk for four months and the same number of children of closely the same age, stature and weight were given pasteurised milk for the same time, the following system of mean growth differences, standard deviations of those differences and the ratios (z's) of those differences to their standard deviations were obtained. The numbers are sufficiently large to admit of our computing the probability integral of z from the normal curve table:

TABLE IV.

Boys, Central Age in years	No. of Pairs	Raw Mean—Pasteurised	Standard Deviation of Difference	z_t	Probability Integral of $z_t = p_t$	$\log_{10} p_t$
$6\frac{3}{4}$	73	− ·066	·054	− 1·22	·888,7676	$\bar{1}$·948,7882
$7\frac{3}{4}$	76	+ ·022	·053	+ 0·41	·340,9030	$\bar{1}$·532,6308
$8\frac{3}{4}$	71	− ·003	·052	− 0·06	·523,9222	$\bar{1}$·719,2669
$9\frac{3}{4}$	77	+ ·011	·055	+ 0·20	·420,7403	$\bar{1}$·624,0141
$10\frac{3}{4}$	60	+ ·002	·057	+ 0·04	·484,0466	$\bar{1}$·684,8872
(*The units are inches.*)					Total	$\bar{2}$·509,5872

Thus: $\log_{10} \lambda_5 = 1·490{,}4128$, $\sqrt{n} \log_{10} e = \sqrt{5} \times 434{,}2945 = ·971{,}1120$ and accordingly $u = 1·53475$.

The volume within the 5-hyperboloid $= I(4, 1·53475) = ·262$, or the probability of a set of values of z with a less probability than those observed is ·738. That is to say that if the system of z's were really drawn from normal populations about 74 % of the cases would be less probable. We cannot accordingly assert that there is any difference in growth in these boys according as to whether they took raw or pasteurised milk.

* E. M. Elderton, "The Lanarkshire Milk Experiment," *Annals of Eugenics*, Vol. v, pp. 326—336.

Illustration 4. In the experiment referred to in the previous *Illustration* the following results were obtained in the same manner for the Weight of Girls when Raw Milk was administered and when no milk was given, the pairs being taken of closely the same Age, Stature and Weight.

TABLE V.

Girls. Central Age in years	No. of Pairs	Raw Milk— Control	Standard Deviation of Difference	z_t	Probability Integral of $z_t = p_t$	$\log_{10} p_t$
6½	144	+ 0·13	2·02	·06	·476,0778	$\bar{1}$·677,6799
7½	128	+ 1·12	2·41	·46	·322,7581	$\bar{1}$·508,8771
8½	133	+ 7·98	2·66	3·00	·001,3499	$\bar{3}$·130,3016
9½	133	+ 5·62	2·77	2·03	·021,1783	$\bar{2}$·325,8911
10½	115	+11·66	3·27	3·57	·000,1785	$\bar{4}$·251,6382
(*The units are ounces.*)					Total	$\overline{10}$·894,3870

Thus: $\log_{10}\lambda_5 = +9{\cdot}105{,}6121$, $\sqrt{\pi}\log_{10} e = {\cdot}971{,}1120$ and accordingly $u = 9{\cdot}37648$, and the volume inside the hyperboloid $= I\,(4,\ 9{\cdot}37648) = {\cdot}999{,}9922$, or the probability of a set of z's occurring with as great as, or a greater improbability than this set is only ·000,0078. We should accordingly argue that the raw milk feeders and the controls can only be, with the highest degree of improbability, random samples of the same population, i.e. the raw milk accelerated the growth of the girls (especially the elder girls) in weight.

Illustration 5. *Use of Probability Integral of Correlation Coefficient.* A number of coefficients of correlation, $r_1, r_2, \ldots, r_t, \ldots r_u$, are found from samples of sizes $n_1, n_2, \ldots n_t, \ldots n_u$. The corresponding means and standard deviations for the samples, the variates being x and y, are $\bar{x}, \bar{y}, \sigma_{x_1}, \sigma_{y_1}$; $\bar{x}_2, \bar{y}_2, \sigma_{x_2}, \sigma_{y_2}$; …; $\bar{x}_t, \bar{y}_t, \sigma_{x_t}, \sigma_{y_t}$; …; $\bar{x}_u, \bar{y}_u, \sigma_{\bar{x}_u}, \sigma_{\bar{y}_u}$. Each correlation may come from a parent population means m_{x_t}, m_{y_t}, standard deviations $\Sigma_{x_t}, \Sigma_{y_t}$, but these parent populations are supposed to have the same value of the correlation coefficient ρ. What is the most probable value to give to ρ, and what is the chance that the populations from which the u samples are drawn really have the same correlation?

We suppose the distributions in the case of each of the u parent populations to be normal, and we will take $Sn_t = N$. The distribution of the tth population, if we suppose x_{t_a}, y_{t_a} to be any member of it, will be

$$z = \frac{M}{2\pi\Sigma_{x_t}\Sigma_{y_t}(1-\rho^2)^{\frac{1}{2}}}\, e^{-\frac{1}{2(1-\rho^2)}\left\{\frac{(x_{t_a}-\bar{x}_t)^2}{\Sigma^2_{x_t}} + \frac{(y_{t_a}-\bar{y}_t)^2}{\Sigma^2_{y_t}} - \frac{2\rho\,(x_{t_a}-\bar{x}_t)\,(y_{t_a}-\bar{y}_t)}{\Sigma_{x_t}\Sigma_{y_t}}\right\}} \quad \ldots(18).$$

If we take the product of such expressions for all values of a for the tth sample, we find that the chance of such a sample arising from values of x and y lying

between x_{t_a} and $x_{t_a}+\delta x_{t_a}$, y_{t_a} and $y_{t_a}+\delta y_{t_a}$ and similar values for the n_t other pairs of coordinates may be thrown into the familiar form

$$\left(\frac{1}{2\pi\Sigma_{x_t}\Sigma_{y_t}}\right)^{n_t}\frac{1}{(1-\rho^2)^{\frac{1}{2}n_t}}$$
$$\times e^{-\frac{n_t}{2(1-\rho^2)}\left[\frac{(m_{x_t}-\bar{x}_t)^2+\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{(m_{y_t}-\bar{y}_t)^2+\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho\{(m_{x_t}-\bar{x}_t)(m_{y_t}-\bar{y}_t)+r_t\sigma_{x_t}\sigma_{y_t}\}}{\Sigma_{x_t}\Sigma_{y_t}}\right]}\prod_{a=1}^{a=n_t}(\delta x_{t_a}\delta y_{t_a}) \quad \text{......(19).}$$

We have a similar value for each of the u sets of samples. Now we suppose the u sets of samples to be independent and further nothing known about the constants of the u-parent populations except that they have the same ρ by hypothesis. Accordingly we have to make the product of u expressions like the above, a maximum by choice of the $4u+1$ variates $\bar{x}_t$, $\bar{y}_t$, Σ_{x_t}, Σ_{y_t} $(t=1, 2, \dots u)$ and ρ. As they are independent, we can differentiate the single values like (18) to determine the values for the first four types of variates, but we must differentiate the combined product to obtain the value of ρ. We have at once $m_{x_t}=\bar{x}_t$, $m_{y_t}=\bar{y}_t$. Inserting these we have to maximise the expression

$$\frac{1}{(1-\rho^2)^{\frac{1}{2}n_t}}\left(\frac{1}{\Sigma_{x_t}\Sigma_{y_t}}\right)^{n_t}e^{-\frac{n_t}{2(1-\rho^2)}\left[\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right]}\prod_{a=1}^{a=n_t}(\delta x_{t_a}\delta y_{t_a}) \quad \text{...(20)}$$

to find the proper values of Σ_{x_t} and Σ_{y_t}. If we take the logarithm of this and differentiate with regard to Σ_{x_t} and Σ_{y_t} we find, after dividing out a factor,

$$0=-1+\frac{1}{1-\rho^2}\left(\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}-\frac{\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right),$$
$$0=-1+\frac{1}{1-\rho^2}\left(\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right),$$

and accordingly by subtraction $\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}=\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}$; or, since the standard deviations must be positive,

$$\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}=\frac{1-\rho^2}{1-\rho r_t}=\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}} \quad \text{..............................(21).}$$

We have now to differentiate the product of u expressions like (20) with regard to ρ, where after differentiation we can make use of (21). The required expression as far as ρ is concerned is

$$\frac{1}{(1-\rho^2)^{\frac{1}{2}N}}e^{-\frac{1}{2(1-\rho^2)}\overset{t=u}{\underset{t=1}{S}}n_t\left[\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right]}\prod_{t=1}^{t=u}\prod_{a=1}^{a=n_t}(\delta x_{t_a}\delta y_{t_a}).$$

Assuming that it is allowable to consider the double product independent of Σ_{x_t}, Σ_{y_t} and ρ we have, by taking a logarithmic differential with regard to ρ,

$$0=\frac{N\rho}{(1-\rho^2)^2}-\frac{\rho}{(1-\rho^2)^2}\overset{t=u}{\underset{t=1}{S}}n_t\left[\frac{\sigma^2_{x_t}}{\Sigma^2_{x_t}}+\frac{\sigma^2_{y_t}}{\Sigma^2_{y_t}}-\frac{2\rho r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}\right]+\frac{1}{1-\rho^2}\overset{t=u}{\underset{t=1}{S}}\frac{n_t r_t\sigma_{x_t}\sigma_{y_t}}{\Sigma_{x_t}\Sigma_{y_t}}.$$

Now making use of (21) we find, since $\overset{t=u}{\underset{t=1}{S}} n_t = N$,

$$\frac{N\rho}{1-\rho^2} = \overset{t=u}{\underset{t=1}{S}} \left(\frac{n_t r_t}{1-\rho r_t}\right) \quad \text{.............................(22).}$$

Our (21) agrees with the (B) equation of Mr Brandner's (11)* and our (22) agrees with his (11) (C), if we take the simple case of $u = 2$, when $N = n_1 + n_2$.

Returning to (22), let us write

$$\mu_v = \overset{t=u}{\underset{t=1}{S}} \frac{n_t r_t^v}{N} \quad \text{..............................(23),}$$

or, μ_v is the weight mean of the vth powers of the r_t's. Accordingly we may write our equation for the most probable value of ρ in the form

$$\rho = \mu_1 + \rho(\mu_2 - \rho^2) + \rho^2(\mu_3 - \rho^3) + \rho^3(\mu_4 - \rho^4) + \rho^4(\mu_5 - \rho^5) + \dots \quad \text{...(24).}$$

This equation allows us to make rapid approximations to the value of ρ, according as to what power of the correlation coefficients may for a practical purpose be considered negligible. Thus

$\rho_1 = \mu_1 =$ mean of the r_t's,

$\rho_2 = \mu_1 + \rho_1(\mu_2 - \rho_1^2)$,

$\rho_3 = \mu_1 + \rho_2(\mu_2 - \rho_2^2) + \rho_1^2(\mu_3 - \rho_1^3)$,

$\rho_4 = \mu_1 + \rho_3(\mu_2 - \rho_3^2) + \rho_2^2(\mu_3 - \rho_2^3) + \rho_1^3(\mu_4 - \rho_1^4)$,

and so on ....(25).

The following example is taken from the paper by E. S. Pearson and S. S. Wilks in the current issue†:

It is assumed that the frequency of Head Length and Head Breadth in skulls follows a normal distribution. Samples of 20 skulls were taken from 30 different races and their correlations calculated. The dimensions of the skulls and their

TABLE VI.

Racial Correlation Coefficients for equal small Samples of Thirty Races.

Race	r_t	Race	r_t	Race	r_t	Race	r_t	Race	r_t
1	+·097	7	−·037	13	+·319	19	+·018	25	+·245
2	+·198	8	+·667	14	+·310	20	+·160	26	+·360
3	+·576	9	+·014	15	+·019	21	+·178	27	+·592
4	−·015	10	−·112	16	+·445	22	+·763	28	−·515
5	+·173	11	+·219	17	+·410	23	+·101	29	+·023
6	+·764	12	−·152	18	+·946	24	+·449	30	+·254

* *Biometrika*, Vol. xxv. p. 104.

† *Biometrika*, Vol. xxv. pp. 372—374.

standard deviations for the 30 parent populations are supposed unknown, but undoubtedly differ significantly. The problem is whether the correlations based on these small samples can be considered random samples from parent populations having a common correlation coefficient*. It is unnecessary to repeat the names of the races here, as if the problem were to be considered in earnest, we should not take all the n_t's equal to the small number 20, but use all the skulls available and weight the r_t's, using formulae (22) and (23).

The first four moments obtained by adding the powers given in *Barlow's Tables* (Edition, Comrie) are as follows, due attention being paid to the sign of r_t:

$$\mu_1 = \cdot 2489{,}6667, \quad \mu_2 = \cdot 1568{,}5010, \quad \mu_3 = \cdot 0904{,}4030, \quad \mu_4 = \cdot 0677{,}0455.$$

Hence $$\rho_1 = \text{mean}, \quad r_t = \cdot 2489{,}6667.$$

Thus we have

$$\rho_1^2 = \cdot 0619{,}8440, \quad \rho_1^3 = \cdot 0154{,}3205, \quad \rho_1^4 = \cdot 0038{,}4207.$$

$$\rho_2 = \cdot 2489{,}6667 + \cdot 2489{,}6667\,(\cdot 1568{,}5010 - \cdot 0619{,}8440)$$
$$= \cdot 2725{,}8507,$$

whence $$\rho_2^2 = \cdot 0743{,}0262, \quad \rho_2^3 = \cdot 0202{,}5378,$$

$$\rho_3 = \cdot 2489{,}6667 + \cdot 2725{,}8507\,(\cdot 1568{,}5010 - \cdot 0743{,}0262)$$
$$+ \cdot 0619{,}8440\,(\cdot 0904{,}4030 - \cdot 0154{,}3205)$$
$$= \cdot 2761{,}1503,$$

whence $$\rho_3^2 = \cdot 0762{,}3951,$$

$$\rho_4 = \cdot 2489{,}6667 + \cdot 2761{,}1503\,(\cdot 1568{,}5010 - \cdot 0762{,}3951)$$
$$+ \cdot 0743{,}0262\,(\cdot 0904{,}4030 - \cdot 0202{,}5378)$$
$$+ \cdot 0154{,}3205\,(\cdot 0677{,}0455 - \cdot 0038{,}4207)$$
$$= \cdot 2774{,}2504.$$

We may therefore take as the most probable value of the correlation coefficient in the series of u parent populations to three decimal places, $\rho = \cdot 277$.

Starting with this value of ρ we have now to find for the 30 values of r_t given above, their probability integrals. Miss David's Probability Integral Table† for the Correlation Coefficient will enable such values to be determined. Meanwhile she has provided the values for $\rho = \cdot 277$. The probability integral table for this value of ρ runs as follows:

Thus, if the samples were all derived from populations having the same correlation we should only in less than 4 °/。 of cases get a more improbable result than that observed. It is thus unlikely that the thirty races have the same correlation coefficient between Head Length and Head Breadth.

Messrs E. S. Pearson and S. S. Wilks, by applying the approximate method of Fisher, obtain a $\chi^2 = 96{\cdot}01$ and a probability $P_{\chi^2} < \cdot 000{,}030$ that the correlation

* *A priori* the hypothesis of common correlation is extremely unlikely, for if these races were produced by selection from a common stock, that selection would modify the correlations.

† Manuscript Table shortly to be published.

TABLE VII.

Ordinates and Probability Integrals of samples of 20 from a Parent Population of correlation $\rho = 0{\cdot}277$.

	Ordinates	Prob. Int. correct to 5 figs.	δ^2	δ^4
− 1·00	—	—	—	—
− ·95	—	—	—	—
− ·90	—	—	—	—
− ·85	—	—	—	—
− ·80	·01	—	—	1
− ·75	·03	—	1	− 2
− ·70	·13	·00001	—	4
− ·65	·45 *	·00002	3	—
− ·60	1·27	·00006	6	6
− ·55	3·17	·00016	15	3
− ·50	7·10	·00041	27	12
− ·45	14.56 −	·00093	51	11
− ·40	27·72	·001,96	86	17
− ·35	49·59	·003,85 −	138	23
− ·30	83·92	·007,12	213	20
− ·25	135·13	·012,52	308	24
− ·20	207·94	·021,00	427	21
− ·15	306·81	·033,76 −	567	12
− ·10	435·13	·052,17	719	− 3
− ·05	594·15 −	·077,78	868	− 19
·00	781·89	·112,07	998	− 43
·05	992·07	·156,34	1085	− 70
·10	1213·39	·211,46	1102	− 96
·15	1429·44	·277,60	1023	− 105
·20	1619·53	·353,97	839	− 117
·25	1760·79	·438,73	538	− 98
·30	1831·32	·528,87	139	− 60
·35	1814·42	·620,40	− 320	− 6
·40	1703·00	·708,73	− 784	67
·45	1503·12	·789,22	− 1181	138
·50	1235·43	·857,90	− 1440	184
·55	933·13	·912,18	− 1516	195
·60	635·96	·951,31	− 1395	162
·65	381·09	·976,49	− 1113	76
·70	193·23	·990,54	− 756	− 22
·75	78·10	·997,04	− 419	− 91
·80	22·75	·999,35 +	− 174	− 120
·85	3·95	·999,92	− 49	− 84
·90	·33	1·000,00	− 8	− 33
·95	·00	1·000,00	—	− 8
1·0	·00	1·000,00	—	—

δ^6 is negligible.

Probability Integrals and their logarithms for observed values of r.

r_1	P. Int. correct to 4 figs.	Logarithm of P. Int.
+ ·097	·2065 −	$\bar{1}$·310,1061
+ ·198	·3507	$\bar{1}$·544,9358
+ ·570	·9418	$\bar{1}$·973,9587
− ·015	·1008	$\bar{1}$·003,4605
+ ·173	·3115 *	$\bar{1}$·493,4581
+ ·704	·9980	$\bar{1}$·999,1305
− ·037	·0858	$\bar{2}$·933,4873
+ ·067	·9823	$\bar{1}$·992,2441
+ ·014	·1234	$\bar{1}$·091,3152
− ·112	·0472	$\bar{2}$·673,9420
+ ·219	·3853	$\bar{1}$·585,7980
− ·152	·0331	$\bar{2}$·519,8280
+ ·319	·5637	$\bar{1}$·751,0480
+ ·310	·5473	$\bar{1}$·738,2254
+ ·019	·1277	$\bar{1}$·106,1909
+ ·445	·7810	$\bar{1}$·892,9840
+ ·410	·7256	$\bar{1}$·860,6973
+ ·946	1·0000 −	·000,0000
+ ·018	·1268	$\bar{1}$·103,1193
+ ·160	·2921	$\bar{1}$·465,5316
+ ·178	·3100	$\bar{1}$·491,3617
+ ·703	·9979	$\bar{1}$·999,0870
+ ·101	·2127	$\bar{1}$·327,7675
+ ·449	·7877	$\bar{1}$·896,3008
+ ·245	·4300	$\bar{1}$·633,4685
+ ·360	·6385	$\bar{1}$·805,1609
+ ·592	·9460	$\bar{1}$·975,8911
− ·515	·0003	$\bar{4}$·477,1213
+ ·023	·1311	$\bar{1}$·117,6027
+ ·259	·4458	$\bar{1}$·649,1401

$$\log \lambda_n = -17\cdot 678{,}5760$$

$$\sqrt{30} \times \cdot 434{,}2945 = 2\cdot 378{,}7289^4,$$

or $$u = 7\cdot 389{,}902.$$

Accordingly

$$Q\lambda_n = 1 - I\,(29,\ 7\cdot 389{,}902)$$
$$= 1 - \cdot 963 = \cdot 037.$$

coefficients between head length and head breadth are not the same for all the thirty races. I am not prepared to state whether the extreme difference in this case is due to the test applied being really more stringent, or to the fact that Fisher's approximate z-test for r can give exaggeratedly improbable values in the

case of outlying values of r such as those of $+\cdot946$ and $-\cdot592$ attributed to the Guanche and the Turk.

Illustration 6. Use of the Incomplete Γ-Function Table to find the Probability Integrals. We proceed first to indicate the Γ-function expression for the Probability Integral p_n of the Standard Deviation s of a random sample of n drawn from a normal curve of mean M and standard deviation Σ.

The equation for its distribution is

$$y = y_0 \left(\frac{s}{\Sigma/\sqrt{2n}}\right)^{n-2} e^{-\frac{1}{2}\left(\frac{s}{\Sigma/\sqrt{2n}}\right)^2} \quad [ds] \quad \text{...........(26).}$$

Take $x = \frac{1}{2}\left(\frac{s}{\Sigma/\sqrt{2n}}\right)^2$, and we have

$$y = y_0' \, x^{\frac{n-3}{2}} e^{-x} \, dx \quad \text{.............................(27).}$$

Therefore the probability integral in the form ready for entering the incomplete Γ-function table is

$$p_n = I\left(\tfrac{1}{2}(n-3),\ \frac{ns^2}{\sqrt{2(n-1)}\,\Sigma^2}\right).$$

If we suppose the means and standard deviations are due to random sampling from a common normal population, then we have, by (12) and (13),

$$M = \frac{\overset{t=v}{\underset{t=1}{S}}(n_t m_t)}{N}, \quad \text{and} \quad \Sigma^2 = \frac{\overset{t=v}{\underset{t=1}{S}} n_t\left(s_t^2 + (m_t - M)^2\right)}{N}.$$

If we apply these results to the data in Table I we have $\Sigma^2 = 289{,}749$ and $n_t = 5$ for all values of t. Hence

$$p_{n_t} = I\left(1,\ \frac{s_t^2}{163907}\right)$$

Taking the values from column (f) of Table II we find

TABLE VIII.

t	$u = s_t^2/163{,}907$	p_{n_t}	$\log p_{n_t}$	t	$u = s_t^2/163{,}907$	p_{n_t}	$\log p_{n_t}$
1	1·18116	·4973	$\bar{1}$·696,6185	9	·90432	·3656	$\bar{1}$·563,0062
2	1·15447	·4863	$\bar{1}$·686,0103	10	1·29098	·5447	$\bar{1}$·736,1574
3	2·05238	·7855	$\bar{1}$·895,1462	11	1·91328	·7523	$\bar{1}$·876,3911
4	·72617	·2741	$\bar{1}$·437,9090	12	·47832	·1480	$\bar{1}$·170,2617
5	·51310	·1649	$\bar{1}$·217,2207	13	1·31919	·5562	$\bar{1}$·745,2310
6	1·58688	·6559	$\bar{1}$·816,8376	14	1·55591	·6452	$\bar{1}$·809,6944
7	1·20815	·5093	$\bar{1}$·706,9737	15	·88099	·3538	$\bar{1}$·560,8627
8	·60537	·2115	$\bar{1}$·325,3104	—	—	—	—

$S(\log p_{n_t}) = \log \lambda_n = -5\cdot756{,}3691$, $\quad \sqrt{n}\log_{10} e = 1\cdot682{,}0153$, $\quad u = 3\cdot42233$.

Thus: $I(14, 3\cdot42233) = \cdot3512$, or some 65 °/。 of series of 15 samples of 5 lamps would have a more improbable set of standard deviations.

Illustration 7. In *Illustration* 1 we have obtained the probability integrals of 15 tests of means and in the last illustration of 15 tests of standard deviations. As we have seen, probability integrals may be combined, and this is possible here because there is no correlation in samples from a normal population between mean and standard deviation.

The combined $\log \lambda_n = -8{\cdot}547{,}8434 - 5{\cdot}756{,}3691$,

$$= -14{\cdot}304{,}2125,$$

and $\sqrt{n} \log_{10} e = \sqrt{30} \times {\cdot}434{,}2945 = 2{\cdot}3787{,}2895$.

Accordingly: $u = 6{\cdot}01338$, and

$$I(n-1, u) = I(29, 6{\cdot}01338) = {\cdot}7189.$$

and $1 - I(29, 6{\cdot}01338) = {\cdot}281$, or there would be samples of 15 drawn from a normal surface some 28 % with more improbable sets of means and standard deviations than occur in this set of 15 samples of five lamps. Thus from whatever standpoint we regard the problem, we have not succeeded in condemning the hypothesis that the 15 samples of five lamps each may have been drawn from the *same* normal population.

But there is a considerable difference in the combined z-test and the combined λ_n-test (i.e. $Q_{\lambda_{15}} = {\cdot}070$ and $Q_{\lambda_{30}} = {\cdot}281$). We have already pointed out that the z-test only involves the populations from which the individual z's are obtained having the same mean and being normal; these populations may have *different* standard deviations, and if the selections from each of these distinct populations be random, their probability integrals for z will all follow a rectangular distribution and may be combined. On the other hand our combined test assumes normal parent populations with the same mean and the *same* standard deviation. We should thus expect it to be more stringent than the z-test; actually it is less. But this may be accounted for by another factor which arises here: the z-test associates every difference of mean with a definite standard deviation.

In the present illustration the 15 standard deviations, s_t, of the samples might have been associated with any one of the 15 differences of mean, $m_t - M$. In this respect the combined test of this illustration seems to be less stringent than the z-test of *Illustration* 2.

Accordingly we ought to be very careful in considering a test to state the hypothesis we are testing on as wide a basis as it admits in regard to the methods employed.

Thus in *Illustration* 2 we are not really testing whether the 15 means come from a single normal population with the same mean and standard deviation. We are testing whether the 15 means come from 15 normal populations with the same mean and possibly different standard deviations. If the result were such that we rejected the latter hypothesis, the former must be rejected for it is involved in the latter. But if the latter be reasonable, it does not follow that the former also may be.

In *Illustrations* 1 and 5 we have made—the hypothesis that all the 15 samples were drawn from the same normal population, and calculated its mean and standard deviation as the most probable values on the basis of the data provided by the 15 samples. But we have not specified that the 15 standard deviations are to be associated with special values of the differences of the means. Had the result come out highly improbable we should have rejected the wider hypothesis, and accordingly the narrower, which is included in it.

One point may be noted. The probability of a worse result for the means $= \cdot 6488$ and for a worse result for the standard deviations $= \cdot 1179$. The combined improbability of a worse result for both, as these are independent, $= \cdot 0765$, which is of the same order of probability as the $\cdot 0696$ provided by the z-test in *Illustration* 2. The correspondence is possibly of no significance.

If we take the 30 probability integrals provided by *Illustrations* 1 and 5, and distribute them in five subranges of $\cdot 2$, we find

	0·0—0·2	0·2—0·4	0·4—0·6	0·6—0·8	0·8—1·0
Expected	6	6	6	6	6
Observed	6	6	7	8	3

The $\chi^2 = 2 \cdot 3333$, and the corresponding P_{χ^2} for five groups $= \cdot 676$.

This may be compared with the $P_{\lambda_n} = \cdot 281$, as indicating the weakness of the assumptions on which the χ^2, P method depends for a small number of groups and a small total frequency.

Illustration 8. *Application to Linearity of Regressions.* In some recent work by Professor H. Ruger the correlation tables for Weight and Vital Capacity in 28 age groups were reached and the values of r, $\eta_{x.y}$, $\eta_{y.x}$ calculated for each of these groups. From these values the functions $\zeta_{x.y} = (\eta^2_{x.y} - r^2)/(1 - r^2)$ and $\zeta_{y.x} = (\eta^2_{y.x} - r^2)/(1 - r^2)$ were tabled. The distribution of ζ on the assumptions made by Fisher of linearity of regression, with homoscedastic normal arrays, leads to the curve

$$y = y_0 \zeta^{\frac{1}{2}(a-4)} (1 - \zeta)^{\frac{1}{2}(n-a-2)} \qquad \text{.........................(28)},$$

where n is the size of the sample and a the number of arrays; and accordingly the probability integral of ζ could be found from the *Tables of the Incomplete* B-*Function* as long as $n - a - 2 \eqslantless 100$.

Unfortunately in the data we have referred to this condition is only satisfied in one case; in the others $N - a - 2$ is in excess and often considerably in excess of 100. We therefore replace in (28) the second factor by an exponential term, and we have, if $\xi = \frac{1}{2}(n - a - 2)\zeta$, the distribution

$$y = y_0' \cdot \xi^{\frac{1}{2}(a-4)} e^{-\xi} \qquad \text{............................(29)}.$$

The probability integral will accordingly be $I(\frac{1}{2}(a-4), u)$, where

$$u = \frac{\frac{1}{2}(n-a-2)}{\sqrt{\frac{1}{2}(a-2)}}\zeta,$$

and $I(p, u)$ is the incomplete Γ-function ratio to be sought for in the *Tables of the Incomplete Γ-Function*. This is feasible up to 104 arrays, a number unlikely to be required.

The reader must bear in mind that a is the number of arrays on which η is computed, and we must distinguish $a_{x.y}$ corresponding to $\eta_{x.y}$ from $a_{y.x}$ corresponding to $\eta_{y.x}$.

The problem before us is the following: we have 28 tables and 28 values of r and say $\eta_{y.x}$, i.e. Weight on Vital Capacity. May the 28 regression lines of Weight on Capacity be considered as a system, which does not differ from straight lines

TABLE IX.

Regression of Weight on Vital Capacity.

of ple	Age Group	Size n_t	$a_{y.x}$	$\frac{1}{2}(a_{y.x}-4)$	$n_t-a_{y.x}-2$	$\zeta_{y.x}$	$\sqrt{2(a_{x.y}-2)}$	u	p_t	$\log p_t$
1	6—12	105	13	$p=5·5$, $q=46$	—	·247,5080	—	—	·004	$\bar{3}$·602,0600
2	13—15	331	15	5·5	314	·067,7070	5·099,0195	4·16943	·068	$\bar{2}$·832,5089
3	16	241	15	5·5	224	·068,6530	5·099,0195	3·01593	·285	$\bar{1}$·454,8449
4	17	288	14	5	272	·031,4678	4·898,9795	1·74715	·022	$\bar{2}$·342,4227
5	18	320	15	5·5	303	·040,2568	5·099,0195	2·39219	·512	$\bar{1}$·709,2700
6	19	310	18	7	290	·055,5070	5·656,8542	2·84558	·446	$\bar{1}$·649,3349
7	20	389	17	6·5	370	·013,1104	5·477,2256	0·88564	·993	$\bar{1}$·996,9492
8	21	321	16	6	303	·057,4679	5·291,5026	3·29070[5]	·235	$\bar{1}$·371,0679
9	22	289	16	6	271	·009,2677	5·291,5026	0·47464	·999(64)	$\bar{1}$·999,8436
10	23	316	17	6·5	297	·063,0081	5·477,2256	3·41658	·227	$\bar{1}$·356,0259
11	24	276	14	5	260	·067,7565	4·898,9795	3·59599	·128	$\bar{1}$·107,2100
12	25	223	17	6·5	204	·063,3770	5·477,2256	2·36048	·608	$\bar{1}$·783,9036
13	26	224	16	6	226	·049,5475	5·291,5026	2·11617	·670	$\bar{1}$·826,0748
14	27	195	15	5·5	178	·075,0919	5·099,0195	2·62136	·430	$\bar{1}$·633,4685
15	28	188	16	6	170	·085,6810	5·291,5026	2·75267	·409	$\bar{1}$·611,7233
16	29	186	15	5·5	169	·096,6815	5·099,0195	3·20436	·231	$\bar{1}$·363,6120
17	30	198	14	5	182	·068,2614	4·898,9795	2·53595	·412[5]	$\bar{1}$·615,4240
18	31—32	305	17	6·5	286	·081,0745	5·477,2256	4·23360	·080	$\bar{2}$·903,0900
19	33—34	280	16	6	262	·036,9507	5·291,5026	1·82955	·785	$\bar{1}$·894,8697
20	35—36	290	16	6	272	·056,4152	5·291,5026	2·89932	·355	$\bar{1}$·550,2284
21	37—38	266	16	6	248	·036,2694	5·291,5026	1·69986	·831	$\bar{1}$·919,6010
22	39—41	319	16	6	301	·040,0175	5·291,5026	2·26878	·606	$\bar{1}$·782,4726
23	42—44	267	14	5	251	·022,8722	4·898,9795	1·17186	·282	$\bar{1}$·450,2491
24	45—47	196	16	6	178	·055,8809	5·291,5026	1·87977	·766	$\bar{1}$·884,2288
25	48—51	222	16	6	204	·023,4922	5·291,5026	0·90568	·884	$\bar{1}$·946,4523
26	52—55	147	15	5·5	130	·097,3283	5·099,0195	2·48139	·475	$\bar{1}$·676,6936
27	56—61	186	14	5	170	·055,0076	4·898,9795	1·90882	·673	$\bar{1}$·828,0151
28	62—81	163	17	6·5	144	·150,1917	5·477,2256	3·94864	·118	$\bar{1}$·071,8820

p_t gives the probability of a higher $\zeta_{y.x}$ occurring than that observed.

$g_1\ \lambda_n = -14·836,4732$, $\sqrt{\pi}\log_{10} e = 2·298,0705$. Hence $u = 6·45606$.

more than would reasonably be the result of random sampling? In other words are the 28 ξ's as based on the 28 ζ's a random sample from the distribution curve (29)?

For the tth sample, we require n_t the size of the sample, $a_{y.x}$, the number of arrays of Weight on Vital Capacity, the $\zeta_{y.x}$, $\sqrt{2(a_{y.x}-4)}$, $n_t - a_{y.x} - 2$ and the $p = \frac{1}{2}(a_{y.x}-4)$ of the incomplete Γ-function, and lastly the value of

$$u = \frac{n_t - a_{y.x} - 2}{\sqrt{2(a_{y.x}-4)}} \zeta_{y.x}.$$

Hence from p and u we find the probability integral p_t of the sample and record its logarithm. *Their sum gives the required* $\log_{10}\lambda_n$.

Thus we find $I(27, 6{\cdot}45606) = {\cdot}8748$ and sets of 28 tables between Weight and Vital Capacity for age groups like the above, if obtained as random samples from parent populations with a linear regression of Weight on Vital Capacity would give a less probable set in 12·5 % of cases. Our results therefore taken as a whole do not provide strong evidence of non-linear regression in the case of Weight on Vital Capacity.

Illustration 9. *Random Samples of Correlation Coefficient from a Normal Population of zero Coefficient.* The following system of racial correlation coefficients for Cephalic Index (100 B/L) and Upper Face Index is given by Tippett*. Applying an approximate method he concludes as follows·

"Thus the combined experience of Table XLIII [this corresponds to our Table X below] lends no support to the view that the two characters are associated even after making allowance for the possibility of racial differences," p. 143.

It seems worth while investigating whether the methods of the present paper confirm Tippett's conclusions.

TABLE X.

Correlation Coefficients of Cephalic Index with Upper Face Index for thirteen Races†.

Index No.	Race	Size of Sample n_t	Correlation Coefficient r_t	Index No.	Race	Size of Sample n_t	Correlation Coefficient r_t
1	Australians ...	66	+0·089	7	Polynesians ...	44	+0·002
2	Negroes ...	77	+0·182	8	Alfourons ...	19	−0·302
3	Duke of York Islanders	53	−0·093	9	Micronesians ...	32	−0·251
				10	Copts	34	−0·147
4	Malays ...	60	−0·185	11	Etruscans ...	47	−0·021
5	Fijians ...	32	+0·217	12	Europeans ...	80	−0·198
6	Papuans ...	39	−0·255	13	Ancient Thebans	152	−0·067

* *The Methods of Statistics*, p. 142, 1931.

† Several of the groups are scarcely anthropological unities, but the series will serve as an example of method.

We will first find the most probable value of ρ, the correlation of the parent populations, if they all had the same coefficient. Applying the method of p. 394 above we find, if $N = S(n_t)$,

$$\frac{S(n_t r_t)}{N} = -\cdot 060{,}9878, \quad \frac{S(x_t r_t^2)}{N} = \cdot 024{,}4218,$$

$$\frac{S(n_t r_t^3)}{N} = -\cdot 002{,}7694, \quad \frac{S(n_t r_t^4)}{N} = \cdot 001{,}1779.$$

Hence we deduce

$$\rho_1 = -\cdot 060{,}9878, \quad \rho_2 = -\cdot 062{,}2504, \quad \rho_3 = -\cdot 062{,}2763, \text{ and } \rho_4 = -\cdot 062{,}2772.$$

Accordingly we find as the most probable value for ρ

$$\rho = -\cdot 06228.$$

It cannot therefore be asserted that the most probable value of ρ as indicated by the data is zero. It looks as if there existed a small negative correlation between the First Cephalic and the Upper Face Indices.

Let us first try what the series leads to when we replace this most probable value of ρ by $\rho = 0$. In this case the frequency distribution of r as a selection from a population having zero correlation coefficient, $\rho = 0$, is given by

$$y = y_0 (1 - r^2)^{\frac{n-4}{2}} \quad \text{..............................(30)},$$

and accordingly the probability integral is given by

$$p_{n,t} = \int_{-1}^{r} (1 - r^2)^{\frac{1}{2}(n-4)}\, dr \Big/ \int_{-1}^{+1} (1 - r^2)^{\frac{1}{2}(n-4)}\, dr \quad \text{...........(31)}.$$

This can be reduced to the incomplete B-function ratio by one or other of two transformations.

(i) Take $r^2 = x$, and we find

$$p_{n,r} = 1 - \tfrac{1}{2} I_{1-r^2}\left(\tfrac{1}{2}(n-2), \tfrac{1}{2}\right), \text{ if } r \text{ be positive},$$
$$= \tfrac{1}{2} I_{1-r^2}\left(\tfrac{1}{2}(n-2), \tfrac{1}{2}\right), \text{ if } r \text{ be negative}.$$

(ii) Put $\tfrac{1}{2}(1 + r) = x$, and we reach

$$p_{n,r} = I_{\frac{1}{2}(1+r)}\left(\tfrac{1}{2}(n-2), \tfrac{1}{2}(n-2)\right), \text{ for both signs of } r.$$

We shall make use of the second transformation as more readily lending itself to interpolation* into the *Incomplete* B-*Function Tables*.

* I have considered it adequate for present purposes to interpolate linearly into the *Tables of the Incomplete* B-*Function*, but even with this simplification the determination of $I_{x+\theta d}(p+\frac{1}{2}, p+\frac{1}{2})$ causes some little trouble, when we are using that part of the tables wherein p is only given to the unit; further we have to remember that $I_x(p, q)$ is only tabled for $p > q$, so that we must find when q is $> p$, $I_x(p, q)$ from the relation $I_x(p, q) = 1 - I_{1-x}(q, p)$. The requisite formula is the following, where $\phi = 1 - \theta$ and d, the tabulating interval of x, is here unity:

$$I_{x+\theta}(p+\tfrac{1}{2}, p+\tfrac{1}{2}) = \tfrac{1}{4}\phi\,\{I_x(p, p) + 1 - I_{1-x}(p+1, p) + I_x(p+1, p+1) + I_x(p+1, p)\}$$
$$+ \tfrac{1}{4}\theta\,\{I_{x+d}(p, p) + 1 - I_{1-x-d}(p+1, p) + I_{x+d}(p+1, p+1) + I_{x+d}(p+1, p)\}.$$

Here all the incomplete B-function ratios will be found in the tables, as p is an integer.

TABLE XI.

P_λ Test for Tippett's Data.

Index Number t	Size of Sample n_t	Correlation r_t	Value of p_{n_t, r_t}		Logarithm of p_{n_t, r_t}
			β-function Value	Numerical Value	
1	66	$+\cdot 089$	$I_{\cdot 6443}(32, 32)$	$\cdot 761$	$\bar{1}\cdot 881,3847$
2	77	$+\cdot 182$	$I_{\cdot 5910}(37\cdot 5, 37\cdot 5)$	$\cdot 946$	$\bar{1}\cdot 975,8911$
3	53	$-\cdot 093$	$I_{\cdot 4535}(25\cdot 5, 25\cdot 5)$	$\cdot 255$	$\bar{1}\cdot 406,5402$
4	60	$-\cdot 185$	$I_{\cdot 4075}(29, 29)$	$\cdot 079$	$\bar{2}\cdot 897,6271$
5	32	$+\cdot 217$	$I_{\cdot 0085}(15, 15)$	$\cdot 117$	$\bar{1}\cdot 068,1859$
6	39	$-\cdot 255$	$I_{\cdot 3725}(18\cdot 5, 18\cdot 5)$	$\cdot 060$	$\bar{1}\cdot 778,1513$
7	44	$+\cdot 002$	$I_{\cdot 5010}(21, 21)$	$\cdot 505$	$\bar{1}\cdot 703,2914$
8	19	$-\cdot 302$	$I_{\cdot 3490}(8\cdot 5, 8\cdot 5)$	$\cdot 104^5$	$\bar{1}\cdot 019,1163$
9	32	$-\cdot 251$	$I_{\cdot 3745}(15, 15)$	$\cdot 083$	$\bar{2}\cdot 919,0781$
10	34	$-\cdot 147$	$I_{\cdot 4265}(16, 16)$	$\cdot 203$	$\bar{1}\cdot 307,4960$
11	47	$-\cdot 021$	$I_{\cdot 4895}(22\cdot 5, 22\cdot 5)$	$\cdot 445$	$\bar{1}\cdot 648,3600$
12	80	$-\cdot 198$	$I_{\cdot 4010}(39, 39)$	$\cdot 039$	$\bar{2}\cdot 591,0646$
13	152	$-\cdot 067$	$I_{\cdot 4665}(75, 75)$*	$\cdot 207^5$	$\bar{1}\cdot 317,0181$

$\log \lambda_n = \text{sum} \log p_{n_t, r_t} = -8\cdot 486,7952$, $\sqrt{13} \log_{10} e = 1\cdot 565,8711$, $u = 5\cdot 41986$.

Chance of more improbable sets $= 1 - I(12, 5\cdot 41986) = \cdot 048$.

Tippett applying a normal curve of standard deviation $\dfrac{1}{\sqrt{n-3}}$ instead of (30) finds a χ^2 for 13 lying between 17 and 18 (17·26) which leads to a $P = \cdot 141$, or between the ·1 and ·2 levels. It would thus appear that our test is more stringent than that applied by Tippett, or the difference may be due to the approximate nature of the method used by him. Not being a very enthusiastic advocate of ·02 as a fit measure for rejection of randomness, I am inclined to doubt whether ·048 is to be taken as sufficient evidence that the series of correlation coefficients are random samples of populations with zero coefficients of correlation.

Illustration 10. *Comparison of two Hypotheses.* We have already noted that, apart from the replacement of binomials by normal curves, the χ^2 test suffers under the disadvantage that it gives the same resulting probability wherever the individuals may be in the same set of subranges. The $(\lambda_n, P_{\lambda_n})$ test allows for this, and accordingly is far better suited for answering the problem of whether a Hypothesis A is or is not more probable than a Hypothesis B.

We will illustrate such a comparison by Tippett's data in *Illustration* 9. In that illustration we have taken Tippett's hypothesis that the data are random samples from parent populations having zero correlations. This shall be Hypothesis A. We have seen that the most likely value of the correlation is not zero, but a correlation measured by $-\cdot 06228$. We will ask what is the probability that the thirteen samples were drawn from parent populations with the correlation of the variates measured by the coefficient $-\cdot 06228$. This is our Hypothesis B.

* Obtained approximately from corresponding normal curve.

To answer the problem requires us to determine the probability integrals for $\rho = -\cdot06228$ of thirteen samples ranging from size 19 to size 152. The tables of the probability integral of r as sampled from normal distributions are not yet sufficiently advanced to cover this wide field. Only the sample of 19 falls within the present range of those tables. It seemed best to adopt a uniform process for all cases, and accordingly the following method was used. It is known that from about $n = 20$ onwards an excellent fit to the distribution curve of r is obtained by aid of a Pearson curve* having the same first four moment coefficients† as the r-distribution. The resulting curves belong to Type I:

$$y = y_0 (a_1 + x)^{m_1'-1} (a_2 - x)^{m_2'-1} \text{ ....(32)},$$

and accordingly the probability integrals may all be found from the *Tables of the Incomplete* B-*Function*. In order to reduce these curves we need to find: (i) $m_1' < m_2'$, these are found from the β_1 and β_2; (ii) the range b, this is found from μ_2, β_1 and β_2 and (iii) the distance d of the observed correlation coefficient from the start of the Pearson curve. This must be reduced to d' by dividing by the range b, for entry into the B-function table.

In determining the incomplete B-function ratio we must remember that as m_2' is always greater than m_1' we must look up in the tables not $I_{d'}(m_1', m_2')$ but its equivalent $1 - I_{1-d'}(m_2', m_1') = p_n$. The probability integral thus obtained is the probability of the occurrence of samples with a greater improbability than the observed sample. Table XII gives the values of the constants of the curves, the corresponding p_n's and their logarithms and finally the value of u with which we enter the *Incomplete* Γ-*Function Table*. This gives by linear interpolation:

$$P_{\lambda_n} = 1 - I(12, 3\cdot674{,}963) = 1 - \cdot5640 = \cdot4360.$$

Thus by the hypothesis that Cephalic Index and Upper Face Index have a correlation coefficient equal to the most probable value provided by the whole set of experiences we obtain a probability rather more than nine times as great as that provided by the hypothesis that the correlation coefficient is really zero. To those who have had experience of the correlation between cranial characters in *long* series, the fact that small correlations between such may be significant is familar. There is accordingly no ground for assuming that because we have four positive and nine negative small correlation coefficients it is a reasonable hypothesis that the correlation coefficient between these two characters is zero.

But here I reach my main criticism of the method now frequently adopted for testing hypotheses. Arbitrary values like $P = \cdot01$ or $P = \cdot02$ are taken to indicate the improbability of a hypothesis. A hypothesis is then found to have a $P = \cdot04$ or $\cdot06$, and it is stated to be thus shown to be reasonable. A conclusion is then drawn from the hypothesis, which is taken as a physical principle, for examples that no difference exists between two populations, or that a correlation coefficient is zero. No attention is paid to the fact that another hypothesis may prove more

* *Biometrika*, Vol. XI. pp. 332—336, or *Tables for Statisticians*, Part II. pp. clvii—clxi.

† *Biometrika*, Vol. XI. pp. 387—388, or *Tables for Statisticians*, Part II. p. clxii.

TABLE XII. *Probability Integrals for Tippett's Series on the Hypothesis that* $\rho = -\cdot06228$.

Index Number and Size of Sample	Observed r	Mean r for Size of Sample $=\bar{r}$	Variance μ_2 for Size of Sample	β_1 for Size of Sample	β_2 for Size of Sample	Range of fitted Type I Curve $=b$	Value of r at Mode $=\tilde{r}$	Distance of Mode from Start $=a_1$	Distance of Start from Observed r $=d$ (d')	m_1', m_2'	Probability Integral $1-I_{1-x}(m_2', m_1')$ $=p_n$	$\log p_n$
(8) 19	−·302	−·0605,8054	·0551,8370	·006,004	2·710,552	2·0078,3827	−·0717,8237	·9079,5440	·6777,3677 (·3375,465)	7·843,568 9·290,208	·161	$\bar{1}$·206,8259
(5) 32	+·217	−·0612,8732	·0320,2889	·003,854	2·825,322	2·0098,9723	−·0676,2820	−·9089,4567	1·1935,7387 (·5938,482)	13·789,469 16·491,116	·936	$\bar{1}$·971,2758
(9) 32	−·251	−·0612,8732	·0320,2889	·003,854	2·825,322	2·0098,9723	−·0676,2820	−·9089,4567	·7255,7387 (·3609,995)	13·789,469 16·491,116	·150	$\bar{1}$·176,0913
(10) 34	−·147	−·0613,4702	·0300,8665	·003,654	2·835,371	2·0100,5458	−·0672,8157	−·9091,7394	·8294,5551 (·4126,532)	14·706,676 17·596,840	·319	$\bar{1}$·503,7907
(6) 39	−·255	−·0614,6897	·0261,2591	·003,234	2·856,071	2·0103,5996	−·0665,8482	·9096,1436	·7211,9918 (·3587,413)	16·999,381 20·361,225	·145	$\bar{1}$·161,3680
(7) 44	+·002	−·0615,6273	·0230,8665	·002,900	2·872,149	2·0106,1194	−·0660,5880	·9099,4799	·9780,0679 (·4864,225)	19·292,236 23·126,105	·662	$\bar{1}$·820,8580
(11) 47	−·021	−·0616,0925	·0215,8037	·002,731	2·880,179	2·0107,2473	−·0658,0109	·9100,9783	·9548,9892 (·4749,029)	20·667,489 24·784,880	·610	$\bar{1}$·785,5298
(3) 53	−·093	−·0616,8628	·0190,8939	·002,445	2·893,553	2·0109,4016	−·0653,7809	·9103,6524	·8827,4333 (·4389,705)	23·418,940 28·103,103	·515	$\bar{1}$·711,8072
(4) 60	−·185	−·0617,5642	·0168,2379	·002,179	2·906,005	2·0111,4740	−·0649,9769	·9105,9828	·7905,9597 (·3931,069)	26·628,856 31·975,036	·175	$\bar{1}$·243,0380
(1) 66	+·089	−·0618,0457	·0152,7035	·001,992	2·914,286	2·0113,0239	−·0647,3855	·9107,7362	1·0645,1217 (·5292,651)	29·380,999 35·294,039	·886	$\bar{1}$·947,4337
(2) 77	+·182	−·0618,7316	·0130,5958	·001,723	2·926,410	2·0114,7477	−·0643,7317	·9109,8490	1·1573,5807 (·5753,779)	34·424,315 41·377,311	·982	$\bar{1}$·992,1115
(12) 80	−·198	−·0618,8856	·0125,6352	·001,661	2·929,144	2·0115,0943	−·0642,9167	·9110,2893	·7673,2060 (·3814,651)	35·799,574 43·036,264	·097	$\bar{2}$·986,7717
(13) 152*	−·067	−·0620,9490	·0065,7216	·000,895	2·962,547	2·0121,6079	−·0633,3995	·9250,9024	·9214,3019 (·4579,307)	68·820,716 82·871,001	·548	$\bar{1}$·738,7806

$\sqrt{n}\,\log_{10} e = 1\cdot565,8711$, $u = 3\cdot674,963$. $S(\log p_n) = \log \lambda_n = -5\cdot754,5178$.

* The frequency is too great to bring this case into the range of the *Incomplete B-Function Table*, and accordingly the normal curve was used.

stringent* and indicate a difference between two populations, or show that it is more reasonable to suppose the correlation coefficient not zero.

Illustration 9. *Comparison of two Hypotheses.* I will take a further illustration of such a comparison which will also cast more light on the difficulties I feel with regard to small samples.

The following thirty observations given in column (*b*) of Table XIII are a random sample and we take as our Hypothesis *A* that the parent population had

TABLE XIII.

Test for Normality in Parent Population from Sample.

(*a*) Index No.	(*b*) Observation x_i	(*c*) Deviation from Mean x_i'	(*d*) x_i'/Σ	(*e*) p_i†	(*f*) $\log_{10} p_i$
1	25	−419·5	−1·485	·0688	$\bar{2}$·837,5884
2	550	+105·5	+ ·373	·6454	$\bar{1}$·809,8290
3	517	+ 72·5	+ ·257	·6014	$\bar{1}$·779,1634
4	33	−411·5	−1·457	·0726	$\bar{2}$·860,9380
5	210	−234·5	− ·830	·2033	$\bar{1}$·308,1374
6	641	+196·5	+ ·696	·7568	$\bar{1}$·878,9811
7	477	+ 32·5	+ ·115	·5458	$\bar{1}$·737,0335
8	318	−126·5	− ·448	·3271	$\bar{1}$·514,6805
9	418	− 26·5	− ·094	·4626	$\bar{1}$·665,2056
10	277	−167·5	− ·593	·3766	$\bar{1}$·575,8803
11	532	+ 87·5	+ ·310	·6217	$\bar{1}$·793,5809
12	466	+ 21·5	+ ·076	·5303	$\bar{1}$·724,5216
13	671	+226·5	+ ·802	·7887	$\bar{1}$·896,9118
14	595	+150·5	+ ·533	·7030	$\bar{1}$·846,9553
15	152	−292·5	−1·035	·1503	$\bar{1}$·176,9590
16	988	+543·5	+1·924	·9726	$\bar{1}$·988,0236
17	420	− 24·5	− ·087	·4653	$\bar{1}$·667,7331
18	625	+180·5	+ ·639	·7380	$\bar{1}$·868,4093
19	389	− 55·5	− ·196	·4333	$\bar{1}$·636,7887
20	171	−273·5	− ·968	·1665	$\bar{1}$·221,4142
21	968	+523·5	+1·853	·9681	$\bar{1}$·985,9202
22	728	+283·5	+1·004	·8423	$\bar{1}$·925,4668
23	949	+504·5	+1·786	·9629	$\bar{1}$·983,5812
24	178	−266·5	− ·943	·1728	$\bar{1}$·237,5437
25	120	−324·5	−1·149	·1253	$\bar{1}$·097,9511
26	144	−300·5	−1·064	·1437	$\bar{1}$·157,4568
27	37	−407·5	−1·443	·0745	$\bar{2}$·872,1563
28	944	+499·5	+1·768	·9615	$\bar{1}$·982,9493
29	289	−155·5	− ·550	·2912	$\bar{1}$·464,1914
30	503	+ 58·5	+ ·207	·5820	$\bar{1}$·764,9230
Mean = 444·5		$\Sigma = 282\cdot49328$		$\log_{10}\lambda_n = -12\cdot739{,}7252$	

$$\frac{1}{\sqrt{30}\,\log_{10} e} = \cdot4203{,}9259{,}67, \quad I(n-1,\, u) = I(29,\, 5\cdot3556{,}8616) = \cdot47551.$$

* For illustrations in degrees of stringency of different tests, see *Biometrika*, Vol. XXIV. pp. 305 *et seq.*

† Found by linear interpolation from Sheppard's Tables.

a normal distribution. The mean M of the observations is $M = 444{\cdot}5$ and their standard deviation $\Sigma = 282{\cdot}49328$. We will adopt these as probable values in the parent population. Column (c) of Table XIII gives the deviations from the mean; column (d) expresses these in terms of the standard deviation, column (e) gives the probability integrals p_t, and column (f) their logarithms the sum of which is $-12{\cdot}739{,}7252$. Accordingly since $1/\sqrt{n}\log_{10}e = {\cdot}4203{,}9259{,}67$, we require to find $I(29, 5{\cdot}3556{,}8616)$ from the *Table of the Incomplete* Γ-*Function*. Interpolating by means of δ^2 (i.e. to third difference accuracy), we have

$$P_{\lambda_n} = I(29,\ 5{\cdot}3556{,}8616) = {\cdot}47551, \quad \text{and} \quad Q_{\lambda_n} = {\cdot}52449.$$

Thus between 52 °/。 and 53 °/。 of samples from the above mentioned normal population would have a less degree of probability than the observed sample. Shall we therefore assume it "reasonable" to suppose the sample drawn from a normal population? I fear many statisticians will say that it is, and not hesitate to draw any inferences that may be based on such an assumption. I doubt, however, whether the use of the word "reasonable" is proper in a case of this kind. There are possibly far more probable hypotheses as to the nature of the parent population, which might lead us to base very different conclusions on the nature of the sample, for example that the range of possible observations was narrowly limited, or that the frequency of the parent population was not such that small or large values of the observations occurred with relatively small frequency. We will take a second hypothesis, Hypothesis B, that the sample has been drawn from a rectangular population. The maximum range as shown by the sample $= 988 - 25 = 963$. This is most probably the modal range of samples from the parent population $= (n-2)\,b/(n-1)^* = \frac{28}{29}\,b$, where b is the range of the parent population. Hence $b = \frac{29}{28} \times 963 = 997{\cdot}39286$. This seems a good value to take for the range of the parent population. Table XIV, column (b) gives the observations; (c) the probability integrals; (d) their logarithms, with their sum, leading to the incomplete Γ-function $P_{\lambda_n} = I(29,\ 6{\cdot}2168{,}5177)$, the evaluation of which by δ^2 interpolation from the *Incomplete* Γ-*Function Tables* is equal to ·77905, and $Q_{\lambda_n} = {\cdot}22095$.

Accordingly almost 22 °/。 of samples of sets of observations from a rectangular parent of the above range would be more improbable than the observed set, and the Hypothesis A is seen to be much more probable than Hypothesis B, although if Hypothesis B had been first tried and its probability, ·22095, computed, many statisticians would have been content with its "reasonableness," and not have proceeded further. Now the strange fact is that the observations were actually taken as the first three figures of the first six sets of five on sheet XXIV of Tippett's *Random Sampling Numbers*, and may therefore be supposed to form a random sample of 30 from a rectangular parent population of range 1000. Our new method should enable us fairly readily to compare the probability of different hypotheses. But the main point to be noted is that because one hypothesis has a

* *Biometrika*, Vol. XXIII. p. 394.

TABLE XIV.

Test for Rectangularity of Parent Population from Sample.

(a) Index No.	(b) Observation	(c) $p_t = (b) \div 997 \cdot 89286$	(d) $\log_{10} p_t$	(a) Index No.	(b) Observation	(c) $p_t = (b) \div 997 \cdot 89286$	(d) $\log_{10} p_t$
1	25	·0251	$\bar{2}$·399,0737	16	988	·9906	$\bar{1}$·995,8983
2	550	·5514	$\bar{1}$·741,4668	17	420	·4211	$\bar{1}$·624,3852
3	517	·5184	$\bar{1}$·714,6650	18	625	·6266	$\bar{1}$·796,9904
4	33	·0331	$\bar{2}$·519,8280	19	389	·3900	$\bar{1}$·591,0646
5	210	·2105	$\bar{1}$·323,2521	20	171	·1714	$\bar{1}$·234,0108
6	641	·6427	$\bar{1}$·808,0083	21	968	·9705	$\bar{1}$·986,9955
7	477	·4782	$\bar{1}$·679,6096	22	728	·7299	$\bar{1}$·863,2634
8	318	·3188	$\bar{1}$·503,5183	23	949	·9515	$\bar{1}$·978,4088
9	418	·4191	$\bar{1}$·622,3177	24	178	·1785	$\bar{1}$·251,6382
10	277	·2777	$\bar{1}$·443,5759	25	120	·1203	$\bar{1}$·080,2656
11	532	·5334	$\bar{1}$·727,0530	26	144	·1444	$\bar{1}$·159,5672
12	466	·4672	$\bar{1}$·669,5028	27	37	·0371	$\bar{2}$·569,3739
13	671	·6728	$\bar{1}$·827,8860	28	944	·9466	$\bar{1}$·976,1206
14	595	·5966	$\bar{1}$·775,6832	29	289	·2898	$\bar{1}$·462,0984
15	152	·1524	$\bar{1}$·182,9850	30	503	·5043	$\bar{1}$·702,6890
p_t values retained only to four places						$\log_{10} \lambda_n = -14 \cdot 788,2047$	
$P_{\lambda_n} = I(n-1,\ 6 \cdot 2168,5177) = \cdot 77905$ *						$\dfrac{-\log_{10} \lambda_n}{\sqrt{n} \log_{10} e} = 6 \cdot 2168,5177$	

very considerable probability and a higher probability than a second, it does not follow that it is reasonable to suppose that hypothesis to hold and another to be false, and hence draw conclusions from the former holding†.

The effect of using *small* samples is to render it quite probable (probability = ·50 about!) that a sample was drawn from a population differing very widely from the population out of which it was actually extracted. In short the fact that a hypothesis is even very probable in the case of a small sample by no means demonstrates that it is a "reasonable" hypothesis, and that accordingly inferences may be drawn from it‡. All in fact that the present test and other tests besides

* If b be found from $\Sigma^2 = \frac{1}{12} b^2$, it equals 978·592,856 and $P_{\lambda_n} = \cdot 75012$. But this b is less than the observation 988 and gives *one* probability value $= 1 \cdot 0096$, i.e. > 1!

† The greater probability of the normal hypothesis is here explicable because we have *two* constants to dispose of, while in the case of the rectangle, we have only made use of the range, measuring it from the zero of observation. It is not very easy, short of approximations of considerable length, to determine the best "centre" and the best range of a rectangular population from a given sample, and clearly any attempt to do so moves us away from the true parent population.

‡ I have elsewhere (*Biometrika*, Vol. xxiv. p. 371) indicated that given two parent populations as divergent in distribution as the normal and rectangular, it is not possible to deny the truth of one or other hypothesis unless the sample approaches 100 to 150 in magnitude.

can achieve if several hypotheses are found to have considerable probability is to test their *relative* reasonableness, and even this may deceive us, as just exemplified.

Conclusions.

(i) A very general test, the P_{λ_n} test, has been discussed which seems to the writer to involve fewer approximations and assumptions than the P_{χ^2} test. He would emphasise its advantages in this respect in the case of small samples, where it appears to him that the application of the P_{χ^2} test may well lead to erroneous conclusions, for it fails in stringency.

(ii) The P_{λ_n} is not claimed to be a test of maximum stringency, but as having a very wide field of application, especially when the constants of unknown parent populations are given their most probable values. The method which proceeds from these values seems to him as effective as attempting to find tests, which involve only sample values.

(iii) The P_{λ_n} test involves determining probability integrals, but tables of such integrals are now largely available and more will shortly be published.

(iv) It appeals first to the principle of independent probabilities, to ascertain the probability of more improbable *individual* occurrences, and then starting from this probability measures the probability of all sets of occurrences,—not necessarily greater in each individual variate but more improbable as a whole set.

(v) A number of illustrations are provided to indicate the breadth of the method, and in particular its value in the comparison of hypotheses.

(vi) The writer endeavours to emphasise a point which has, he thinks, not always been sufficiently regarded, namely that because a set of occurrences is found on a selected hypothesis H_1 not to be very improbable by test A, it does not follow that that hypothesis may be safely regarded as applying to the occurrences. A more stringent test B may show H_1 to be very improbable, or either or both tests, A and B, may show another hypothesis H_2 to be far more probable. In other words a test may suffice to allow us reasonably to reject a hypothesis, but only rarely (and generally in the cases where there is large previous experience) justifies us in accepting the hypothesis as a rule of conduct, or as a mode of extracting further information from our data.

(vii) Lastly, emphasis should be laid on the point that while probability integrals for a given investigation should all be measured in one direction, that direction may initially be *either* direction. In other words, a very high P_{λ_n} is calculated to arouse our suspicion as well as a very low P_{λ_n}. Really this warning needs to be borne in mind with nearly all tests, in particular with the P, χ^2 test. P_{χ^2} is a probability integral of the χ^2 curve measured in a particular direction, but there is no more valid reason for initially measuring it in one and not the opposite direction, than in the case of the normal curve probability integral, and, when the

size of the sample is not too small, χ^2 *approaching zero, or* P_{χ^2} approaching unity is as definite a warning as χ^2 very large and P_{χ^2} approaching zero. These considerations are equally valid when we consider P_{λ_n}, which should approach the value ·5 with increasing size of sample if the probability integrals are truly random, but marked deviation *either* way from this value is a warning that something is improbable either in the sampling or in the hypothesis from which the probability integrals have been deduced.

The present paper would have been impossible without the use of the *Incomplete* B- *and* Γ-*Function Tables.* The author has gratefully to acknowledge the aid in computing work of Miss F. N. David under a grant from the Department of Scientific and Industrial Research.

NOTE, *added December* 6, 1933.

After this paper had been set up Dr Egon S. Pearson drew my attention to Section 21·1 in the Fourth Edition of Professor R. A. Fisher's *Statistical Methods for Research Workers*, 1932. Professor Fisher is brief, but his method is essentially what I had thought to be novel. He uses, however, a χ^2 method, not my incomplete Γ-function solution; this explains the relation referred to in the footnote on p. 383 of my paper. As my paper was already set up and illustrates, more amply than Professor Fisher's two pages, some of the advantages and some of the difficulties of the new method, which may be helpful to students, I have allowed it to stand.

K. P.

MISCELLANEA.

(i) The Distribution of β_2 in samples of 4 from a Normal Universe.

BY A. T. McKAY, M.Sc.

The estimated value of $\frac{1}{4}\beta_2$ is the statistic

$$\theta = \frac{\sum_1^4 (x_r - \bar{x})^4}{\left\{\sum_1^4 (x_r - \bar{x})^2\right\}^2} \quad \text{...(1).}$$

Let us employ the orthogonal transformations

$$\left.\begin{aligned} 2x_1 &= y_1 + y_2 + y_3 + y_4 \\ 2x_2 &= -y_1 + y_2 - y_3 + y_4 \\ 2x_3 &= -y_1 - y_2 + y_3 + y_4 \\ 2x_4 &= y_1 - y_2 - y_3 + y_4 \end{aligned}\right\} \quad \text{......................................(2),}$$

noting that $\sum_1^4 x_r^2 = \sum_1^4 y_r^2$, $\sum_1^4 (x_r - \bar{x})^2 = \sum_1^3 y_r^2$, $\bar{x} = \frac{1}{2} y_4$, $2(x_1 - \bar{x}) = \sum_1^3 y_r$, etc., then

$$16 \sum_1^4 (x_r - \bar{x})^4 = \{(y_1 + y_2 + y_3)^4 + (y_1 + y_3 - y_2)^4 + (y_1 + y_2 - y_3)^4 + (y_2 + y_3 - y_1)^4\} \quad \text{.........(3).}$$

Since the expression on the right-hand side of (3) is cyclical and unaltered by a change of sign of any variable, we may infer that

$$16 \sum_1^4 (x_r - \bar{x})^4 = A(y_1^2 + y_2^2 + y_3^2)^2 + B(y_1^2 y_2^2 + y_2^2 y_3^2 + y_3^2 y_1^2) \quad \text{..................(4).}$$

Giving suitable values to the variables, we can readily find $A = 4$, $B = 16$, whence

$$\theta_1 = (\theta - \tfrac{1}{4}) = \frac{y_1^2 y_2^2 + y_2^2 y_3^2 + y_3^2 y_1^2}{(y_1^2 + y_2^2 + y_3^2)^2} \quad \text{..................................(5).}$$

Now since the expression on the right-hand side of (1) has four variables but only three degrees of freedom, we should be able to ascertain the range of fluctuation of β_2 by considering the expression (5). Write $y_1^2 = X$, $y_2^2 = Y$, $y_3^2 = Z$; then

$$\theta_1 = \frac{XY + YZ + ZX}{(X + Y + Z)^2} \quad \text{..(6).}$$

Differentiating partially with respect to each of the three variables in turn and equating to zero, we derive

$$\left.\begin{aligned} Y(Y - X) + Z(Z - X) &= 0 \\ X(X - Y) + Z(Z - Y) &= 0 \\ X(X - Z) + Y(Y - Z) &= 0 \end{aligned}\right\} \quad \text{....................................(7),}$$

from which it follows that there is a turning value when $X = Y = Z$. Further, since

$$\frac{\partial^2 \theta_1}{\partial X^2} = \frac{\partial^2 \theta_1}{\partial Y^2} = \frac{\partial^2 \theta_1}{\partial Z^2} = \frac{-2}{27a^2} \quad \text{......................................(8),}$$

when $X=Y=Z=a>0$, we can conclude that

$$\left.\begin{array}{l} 0 \leqslant \beta_1 \leqslant \frac{1}{3} \\ 1 \leqslant \beta_2 \leqslant \frac{7}{3} \end{array}\right\} \quad \text{...(9)},$$

irrespective of the character of the parent universe.

Let us now return to our main problem. We see from (5) that it is necessary to integrate

$$\frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}\sum_1^3 y_r^2} dy_1 dy_2 dy_3 \quad \text{...............................(10)}$$

over a triply infinite field conditioned by

$$w < \frac{y_1^2 y_2^2 + y_2^2 y_3^2 + y_3^2 y_1^2}{(y_1^2+y_2^2+y_3^2)^2} < w + \delta w \quad \text{...............................(11)}.$$

Transform to polar coordinates by writing $y_1 = r\cos\theta$, $y_2 = r\sin\theta \,.\, \cos\phi$, $y_3 = r\sin\theta \,.\, \sin\phi$, hence equation (10) becomes

$$\frac{\sin\theta}{(2\pi)^{\frac{3}{2}}} d\theta \,.\, d\phi \,.\, r^2 e^{-r^2/2} \,.\, dr \quad \text{.......................................(12)},$$

and condition (11) reduces to

$$w < \sin^2\theta \,.\, \cos^2\theta + \sin^4\theta \,.\, \sin^2\phi \,.\, \cos^2\phi < w + \delta w \quad \text{...............(13)},$$

where $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$.

Integrating out the r-term in (12) and writing $\cos\theta = -x$, $2\phi = \Phi$ in (12) and (13), we may derive $(2/\pi)\, dx \,.\, d\Phi$ in place of (12) and

$$w < x^2(1-x^2) + \tfrac{1}{4}(1-x^2)^2 \sin^2\Phi < w + \delta w \quad \text{...........................(14)},$$

where now $0 \leqslant x \leqslant 1$ and $0 \leqslant \Phi \leqslant \pi/2$ indicate the limits of the field of integration which is to be conditioned by the last inequality. The moments of the distribution of w about the origin are thus given by

$$\nu_k = \frac{2}{\pi}\int_0^{\pi/2}\int_0^1 \{x^2(1-x^2) + \tfrac{1}{4}(1-x^2)^2 \sin^2\Phi\}^k \, dx\, d\Phi \quad \text{.....................(15)}.$$

By expanding the integrand by the binomial theorem and performing the integration term by term, we find

$$\nu_k = \frac{1}{2\Gamma\left(\frac{4k+3}{2}\right)}\left\{\Gamma(k+1)\,\Gamma(k+\tfrac{1}{2}) + \frac{1}{2}\,\frac{\Gamma(k-\frac{1}{2})\, k\Gamma(k+2)}{4\,(1\,!)^2} + \frac{1}{2}\cdot\frac{3}{2}\,\frac{\Gamma(k-\frac{3}{2})\, k(k-1)\,\Gamma(k+3)}{4^2\,(2\,!)^2} + \dots\right\} \quad \text{..................(16)}.$$

Hence $\nu_1 = \frac{1}{5}$, $\nu_2 = \frac{1}{21}$, $\nu_3 = \frac{61}{5005}$, $\nu_4 = \frac{377}{85085}$, so that if μ refers to the moments of β_2 about the mean, $\mu_2 = 0{\cdot}12100$, $\mu_3 = -0{\cdot}02466$, $\mu_4 = 0{\cdot}034276$, $B_1 = 0{\cdot}33273$, $B_2 = 2{\cdot}30645$.

Returning now to (15) and the remarks which precede it, we conclude that if $\phi(w)$ is the distribution which is being sought, then

$$\phi(w) = \frac{2}{\pi}\int \frac{\partial\Phi}{\partial w}\, dx = \frac{1}{\pi}\int \frac{dx}{\{(w - x^2 + x^4)\,(\frac{1}{4}(1-x^2)^2 - w + x^2 - x^4)\}^{\frac{1}{2}}} \quad \text{...............(17)},$$

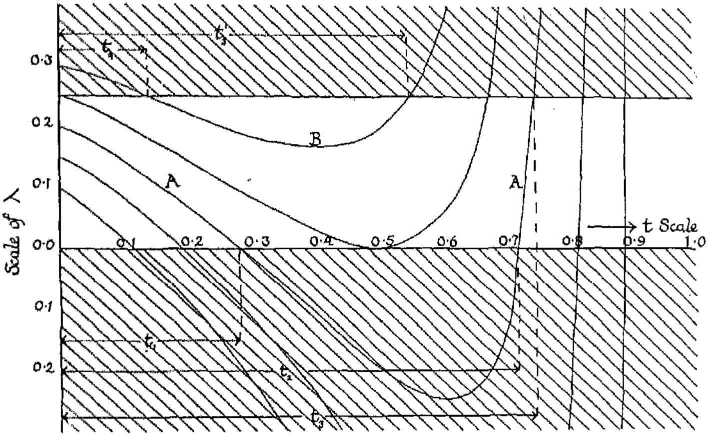
where the limits of the integral are such that x runs from 0 to 1 subject to the condition

$$0 \leqslant 4\,\frac{(w - x^2 + x^4)}{(1-x^2)^2} \leqslant 1 \quad \text{.......................................(18)}.$$

In order to find the appropriate limits, we consider the family of curves defined by

$$\lambda = \frac{w - t + t^2}{(1-t)^2} \quad \text{...(19)},$$

and shown in the following diagram. We note from Equation (18) that only values of λ in the unshaded part of the diagram are consistent with our requirements.



There are plainly two cases to consider.

Case (i). $0 \leqslant w \leqslant \frac{1}{4}$.

Let the curve marked A be regarded as a typical curve for this case. Then there are two independent ranges which satisfy our conditions, i.e.

$$(1)\ 0 \leqslant t \leqslant t_1,$$

and

$$(2)\ t_2 \leqslant t \leqslant t_3.$$

Case (ii). $\frac{1}{4} \leqslant w \leqslant \frac{1}{3}$.

Here we take B as a typical curve, which shows that there is only one possible range, viz.

$$(3)\ t_4 \leqslant t \leqslant t_3'.$$

By writing $\lambda = 0$ and $\lambda = \frac{1}{4}$ in Equation (19), we readily find

$$\left.\begin{aligned} t_1 &= \tfrac{1}{2}(1 - \sqrt{1-4w}) \\ t_2 &= \tfrac{1}{2}(1 + \sqrt{1-4w}) \\ t_3 = t_3' &= \tfrac{1}{3}(1 + 2\sqrt{1-3w}) \\ t_4 &= \tfrac{1}{3}(1 - 2\sqrt{1-3w}) \end{aligned}\right\} \quad \text{.................(20).}$$

If now in (17) we change the variable by writing $x^2 = t$, we conclude

$$\phi(w) = \frac{1}{2\pi}\left(\int_0^{t_1} + \int_{t_2}^{t_3}\right) F(t)\, dt \qquad 0 \leqslant w \leqslant \tfrac{1}{4} \quad \text{.........(21)}$$

$$= \frac{1}{2\pi}\int_{t_4}^{t_3} F(t)\, dt \qquad \tfrac{1}{4} \leqslant w \leqslant \tfrac{1}{3} \quad \text{.........(22),}$$

where

$$F(t) = \{t(w - t + t^2)(\tfrac{1}{4}(1-t)^2 - w + t - t^2)\}^{-\frac{1}{2}} \quad \text{.........(23).}$$

Let us now make the substitution

$$t=\tfrac{1}{3}\{1+2\sqrt{1-3w}\sin\theta\}\quad\text{.......................................(24)},$$

then

$$F(t)\frac{dt}{d\theta}=6\{(9w-2)-2(1-3w)^{\frac{3}{2}}\sin 3\theta\}^{-\frac{1}{2}}\quad\text{..............................(25)}.$$

Hence writing $\frac{1}{2}(1-3w)^{-\frac{1}{2}}=\sin\alpha$ when $0\leqslant w\leqslant\frac{1}{4}$, we find

$$\phi(w)=\frac{1}{2\pi}\left(\int_{-\alpha}^{\alpha-\pi/3}+\int_{2\pi/3-\alpha}^{\pi/2}\right)F(t)\frac{dt}{d\theta}\,d\theta\qquad\text{when } 0\leqslant w\leqslant\tfrac{1}{4}\text{......(26)}$$

$$=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}F(t)\frac{dt}{d\theta}\,d\theta\qquad\text{when } \tfrac{1}{4}\leqslant w\leqslant\tfrac{1}{3}\text{......(27)}.$$

Substituting $\theta=\frac{1}{3}\xi-\pi/6$, $3\alpha=\pi/2+\beta$, and noting that $\sin 3\alpha=\frac{1}{2}(2-9w)/(1-3w)^{\frac{3}{2}}$, we deduce

$$\phi(w)=\frac{3}{\pi(1-3w)^{\frac{1}{4}}}\int_0^{\beta}\frac{d\xi}{(2\cos\xi-2\cos\beta)^{\frac{1}{2}}}\qquad 0\leqslant w\leqslant\tfrac{1}{4}\text{............(28)}$$

$$=\frac{1}{\pi}\int_0^{3\pi}\frac{d\xi}{\{(9w-2)+2(1-3w)^{\frac{3}{2}}\cos\xi\}^{\frac{1}{2}}}\qquad \tfrac{1}{4}\leqslant w\leqslant\tfrac{1}{3}\text{............(29)}.$$

The first of these integrals (28) is a Mehler Integral, hence

$$\phi(w)=\frac{3}{2(1-3w)^{\frac{1}{4}}}\left\{P_{-\frac{1}{2}}(\cos\beta)=P_{-\frac{1}{2}}(-\gamma)=F\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{1+\gamma}{2}\right)\right\}\quad\text{............(30)}$$

when $0\leqslant w\leqslant\frac{1}{4}$ and $\gamma=(9w-2)/2(1-3w)^{\frac{3}{2}}$.

The second integral (29) may be written

$$\phi(w)=\frac{1}{\pi(9w-2)^{\frac{1}{2}}}\int_0^{3\pi}\frac{d\xi}{(1+(\cos\xi)/\gamma)^{\frac{1}{2}}}\quad\text{..............................(31)},$$

the integrand of which can be expanded by the binomial theorem and integrated term by term to yield

$$\phi(w)=\frac{3}{(9w-2)^{\frac{1}{2}}}F\left(\tfrac{1}{4},\tfrac{3}{4};1;\frac{1}{\gamma^2}\right)\qquad\text{when } \tfrac{1}{4}\leqslant w\leqslant\tfrac{1}{3}\quad\text{......(32)}.$$

We note in passing that this last equation may also be written

$$\phi(w)=\frac{3}{\pi(1-3w)^{\frac{1}{4}}}Q_{-\frac{1}{2}}(-\gamma)\quad\text{....................................(33)},$$

where Q represents the Legendre Function of the second kind.

By writing $w=\frac{1}{4}(x-1)$ in equations (30) and (32), we finally find that the distribution of β_2 in samples of four from a normal universe is given by $f(x)$, where

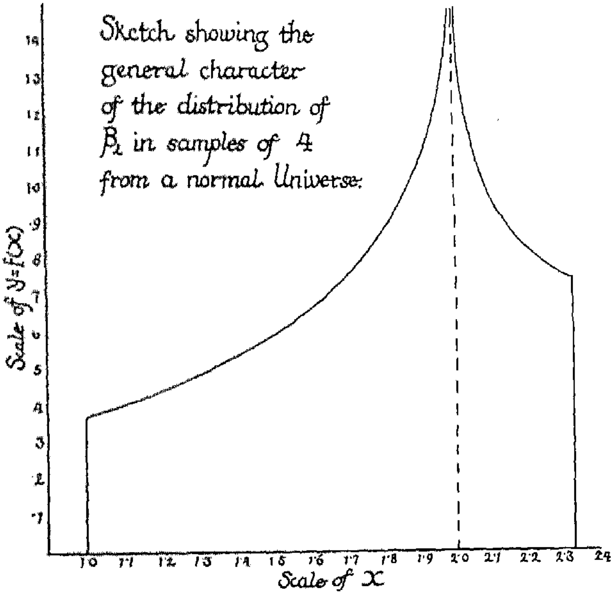
$$f(x)=\frac{3}{2^{\frac{1}{2}}(7-3x)^{\frac{1}{4}}}F\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{1+g}{2}\right)\qquad\text{when } 1\leqslant x\leqslant 2\text{......(34)}$$

$$=\frac{3}{2(9x-17)^{\frac{1}{2}}}F\left(\tfrac{1}{4},\tfrac{3}{4};1;\frac{1}{g^2}\right)\qquad\text{when } 2\leqslant x\leqslant\tfrac{7}{3}\text{......(35)},$$

where $g=(9x-17)/(7-3x)^{\frac{3}{2}}$.

The chief characteristics of this distribution are shown in the accompanying diagram. Owing to the extreme difficulty of calculating the values of the hyper-geometric functions this had to be omitted, so that the two curves have merely been sketched in. Should, however, it prove desirable to have an exact knowledge of the form of these curves, the procedure indicated in the previous paper* would most likely prove of considerable assistance.

* *Biometrika*, Vol. XXV. Parts I and II (1933) (Miscellanea).



(ii) A Note on the Distribution of Range in Samples of n.

By A. T. McKAY, M.Sc., and E. S. PEARSON, D.Sc.

In a recent paper* one of the present writers has provided a table giving certain percentage limits for the distribution of range in samples from a normal population. These limits were obtained on the assumption that the distribution could be adequately represented by Pearson curves having the appropriate moment coefficients. The theoretical treatment in Section (1) below, while following the general method of approach previously employed, leads to certain new results regarding the *form* of the range curve at the terminals, and also provides the exact distribution of range in the case of samples of 3 from a normal population. In this latter case, therefore, it makes possible a check on the accuracy of the published table.

(1) *Theoretical Treatment.*

Let $x_1, x_2, \ldots, x_n$ be a random sample of n from a universe defined in the interval $(-b, a)$ for x by the probability function

$$y=f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

* E. S. Pearson, *Biometrika*, Vol. XXIV. p. 416.

To find the distribution of the range, i.e. the numerical value of the difference between the greatest and least observations in a random sample of n, we require to integrate

$$f(x_1)\,f(x_2)\ldots f(x_n)\,dx_1\,dx_2\ldots dx_n \quad\ldots\ldots\ldots\ldots(2)$$

over an appropriate field.

Consider any pair of values x_1 and x_2 selected from the group of n. Then the compound probability that these two values are the extremes of the group, x_1 being the least and x_2 the greatest, and at the same time give a value of the range lying between w and $w+\delta w$, is found by integrating the expression (2) over all possible values of x_1 and x_2, subject to the conditions

$$\left.\begin{array}{ll} x_1 < x_r < x_2 & r=3,\,4,\,\ldots\ldots\,n \\ \text{and}\quad w < |x_2 - x_1| < w + \delta w & \end{array}\right\} \quad\ldots\ldots\ldots\ldots(3).$$

Since, however, the pair of values x_1 and x_2 can be selected in $n(n-1)$ different ways, the total probability required is to be derived by multiplying the results of the integration by $n(n-1)$.

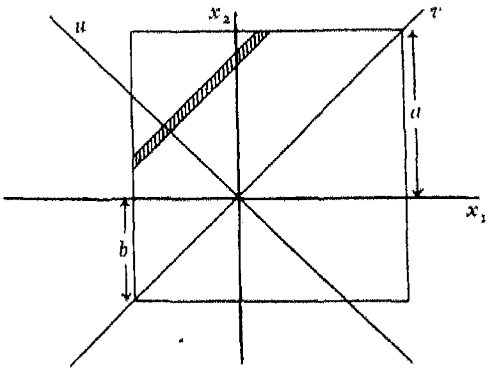
Thus if the required distribution function is $\phi(w)$, we find

$$\phi(w)\,\delta w = n(n-1)\iint f(x_1)\,f(x_2)\left(\int_{x_1}^{x_2} f(x)\,dx\right)^{n-2} dx_1\,dx_2 \quad\ldots\ldots\ldots\ldots(4),$$

where the limits of the integrals for x_1 and x_2 are chosen to satisfy the second condition of Equation (3).

Let us now transform the variables by the substitutions

$$(x_2 - x_1) = u\sqrt{2},$$
$$(x_2 + x_1) = v\sqrt{2}.$$



Then condition (3) requires that we integrate throughout the shaded strip in the diagram. The limits for u are therefore u and $u+\delta u$, while those for v are $(u - b\sqrt{2})$ and $(a\sqrt{2} - u)$. Hence,

$$\phi(w)\,dw = n(n-1)\int_{u-b\sqrt{2}}^{a\sqrt{2}-u}\int_{u}^{u+\delta u} f\left(\frac{u+v}{\sqrt{2}}\right) f\left(\frac{v-u}{\sqrt{2}}\right)\left(\int_{(v-u)/\sqrt{2}}^{(v+u)/\sqrt{2}} f(x)\,dx\right)^{n-2} du\,dv \ldots\ldots(5),$$

or substituting $u = w\sqrt{2}$ and $v = \sqrt{2}\,t$,

$$\phi(w) = n(n-1)\int_{\frac{1}{2}w-b}^{a-\frac{1}{2}w} f(t+\tfrac{1}{2}w)\,f(t-\tfrac{1}{2}w)\left(\int_{t-\frac{1}{2}w}^{t+\frac{1}{2}w} f(x)\,dx\right)^{n-2} dt \quad\ldots\ldots\ldots\ldots(6),$$

where the range for w is from 0 to $(a+b)$.

Example 1. Distribution of the Range in samples of n from a Rectangular Universe*.

In this case $f(x)=\frac{1}{2}$, $a=b=1$, whence

$$\phi(w)=\frac{n(n-1)}{2^n}w^{n-2}(2-w)\quad\text{.....................(7).}$$

Example 2. Distribution of the Range in samples of n from a Straight Line Universe.

Take
$$y=f(x)=2(1-x)\qquad 0\leqslant x\leqslant 1,$$
$$=0\qquad\qquad x<0 \text{ and } >1,$$

so that $b=0$ and $a=1$, then after a simple integration and reduction we find

$$\phi(w)=\frac{n(n-1)}{2}w^{n-2}\left\{\frac{2w^{n+1}}{(n^2-1)}+\frac{(2-w)^{n+1}}{(n+1)}-\frac{w^2(2-w)^{n-1}}{(n-1)}\right\}\quad\text{.....................(8).}$$

Example 3. The distribution of the Range in samples of 3 from a Normal Universe.

Write $n=3$, $f(x)=\frac{e^{-x^2/2}}{\sqrt{2\pi}}$ and $a=b=\infty$ in Equation (6), whence

$$\phi(w)=\frac{6e^{-w^2/4}}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}e^{-t^2}\int_{t-w/2}^{t+w/2}e^{-x^2/2}\,dx\,dt\quad\text{.....................(9).}$$

Putting $x=(y+t)$ and changing the order of integration, we get

$$\phi(w)=\frac{6e^{-w^2/4}}{(2\pi)^{3/2}}\int_{-w/2}^{w/2}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(3t^2+2yt+y^2)}\,dt\,dy\quad\text{.....................(10)}$$
$$=\frac{6e^{-w^2/4}}{\pi\sqrt{3}}\int_0^{w/2}e^{-y^2/3}\,dy$$
$$=\frac{6e^{-w^2/4}}{\pi\sqrt{2}}\int_0^{w/\sqrt{6}}e^{-u^2/2}\,du\qquad\left(\text{writing } u=\sqrt{\tfrac{2}{3}}y\right)\quad\text{.....................(11).}$$

The kth moment, μ_k, of this distribution about the origin is

$$\mu_k=\frac{6}{\pi\sqrt{2}}\int_0^{\infty}w^k e^{-w^2/4}\int_0^{w/\sqrt{6}}e^{-u^2/2}\,du\,dw=\frac{6}{\pi\sqrt{2}}\int_0^{\infty}\int_0^{1/\sqrt{6}}w^{k+1}e^{-(1+2v^2)w^2/4}\,dv\,dw\quad\text{......(12).}$$

Writing $v=u/w$, changing the order of integration and integrating for w in the last equation, we find

$$\mu_k=\frac{2^{k+1}\times 6\,.\,\Gamma\left(\frac{k}{2}+1\right)}{\pi\sqrt{2}}\int_0^{1/\sqrt{6}}\frac{dv}{(1+2v^2)^{\frac{1}{2}(k+2)}}\quad\text{.....................(13).}$$

Substituting $v=(1/\sqrt{2})\tan\theta$ in the latter integral, we finally derive

$$\mu_k=\frac{3}{\pi}\times 2^{k+1}\,\Gamma\left(\frac{k}{2}+1\right)\int_0^{\pi/6}\cos^k\theta\,d\theta\quad\text{.....................(14).}$$

To obtain some idea of the character of the range curve in the neighbourhood of its terminals, *for the case in which the parent universe extends from* $-\infty$ *to* $+\infty$, we can approximate as follows.

Supposing w is very large, the term

$$\left(\int_{t-w/2}^{t+w/2}f(x)\,dx\right)^{n-2}$$

* This has already been given by J. Neyman and E. S. Pearson, *Biometrika*, Vol. xx^{A}, p. 210.

of Equation (6) tends to unity, thus we infer that

$$\phi(w)=n(n-1)\int_{-\infty}^{\infty} f\left(t+\frac{w}{2}\right) f\left(t-\frac{w}{2}\right) dt \qquad (15),$$

provided w is large enough. On the other hand, when w is very small, we may write

$$\int_{t-w/2}^{t+w/2} f(x)\, dx = wf(t),$$

whence we infer that

$$\phi(w)=n(n-1)\, w^{n-2}\int_{-\infty}^{\infty} f\left(t+\frac{w}{2}\right) f\left(t-\frac{w}{2}\right) [f(t)]^{n-2} dt \qquad (16),$$

provided w is small enough.

There does not appear to be any ready means of determining the accuracy of the approximations (15) and (16), but it is possible that they may be of assistance in selecting a suitable approximation curve, when proceeding to find a range distribution by indirect methods.

(2) *Application in the Case of Samples of 3 from a Normal Population.*

Using Equation (11) the ordinates of the frequency curve, $\phi(w)$, were obtained with the help of tables of the normal curve. These ordinates have been compared in Table I with those previously calculated from a Pearson Type I curve*. It will be seen that the greatest relative difference occurs at the start of the curves. From the practical point of view, we are concerned

TABLE I.

Comparison of ordinates of {(A) *true theoretical range curve* ($n=3$). (B) *fitted Pearson curve.*}

Range	Ordinates A.	Ordinates B.	Range	Ordinates A.	Ordinates B.	Range	Ordinates A.	Ordinates B.
·1	5·5	3·6	1·7	42·1	41·2	3·3	9·1	9·6
·2	10·9	9·2	1·8	40·5	39·4	3·4	7·8	8·3
·3	16·1	15·2	1·9	38·6	37·5	3·5	6·7	7·1
·4	21·1	21·1	2·0	36·5	35·3	3·6	5·7	6·1
·5	25·7	26·5	2·1	34·2	33·1	3·7	4·8	5·2
·6	29·9	31·3	2·2	31·8	30·9	3·8	4·0	4·3
·7	33·7	35·4	2·3	29·4	28·6	3·9	3·4	3·6
·8	36·9	38·7	2·4	27·0	26·3	4·0	2·8	3·0
·9	39·6	41·4	2·5	24·6	24·1	4·1	2·3	2·5
1·0	41·8	43·3	2·6	22·2	21·9	4·2	1·9	2·0
1·1	43·3	44·5	2·7	20·0	19·8	4·3	1·5	1·6
1·2	44·4	45·1	2·8	17·8	17·8	4·4	1·2	1·3
1·3	44·9	45·2	2·9	15·8	15·9	4·5	1·0	1·0
1·4	44·8	44·8	3·0	13·9	14·1	4·6	0·8	0·8
1·5	44·3	43·9	3·1	12·2	12·5	4·7	0·6	0·6
1·6	43·4	42·7	3·2	10·6	11·0	4·8	0·5	0·5

N.B.—The unit for range is the population standard deviation, and the curves are calculated so that the area under each is 100. The tails of the curves extend of course beyond $w=4·8$.

* *Loc. cit.* This curve was made to start at $w=0$, and given the correct first three moment coefficients.

with the extent of error in position of the percentage limits calculated from the approximate curve. This may be seen in Table II, where the limits are given for:

A. The true curve of Equation (11), obtained by quadrature and backward interpolation.

B. The Pearson curve; the limits are taken from the row, $n=3$, of the published table*.

C. A Normal curve having correct mean and standard deviation, namely $w=1{\cdot}6926$, $\sigma=0{\cdot}8884$.

TABLE II.

Percentage limits for range ($n=3$) calculated by various methods.

Curve used	Lower Limits				Upper Limits			
	0·5 °/₀	1 °/₀	5 °/₀	10 °/₀	10 °/₀	5 °/₀	1 °/₀	0·5 °/₀
A. True curve	·13	·19	·43	·62	2·90	3·31	4·12	4·43
B. Pearson curve	·17	·22	·45	·63	2·92	3·34	4·10	4·36
C. Normal curve	− ·60	− ·37	·23	·55	2·83	3·15	3·76	3·98

The largest difference between A and B (of ·07 °/₀) occurs at the upper 0·5 °/₀ limit, the second largest (of ·04) at the lower 0·5 °/₀ limit; otherwise the differences are ·03 or less. Since the approximate method might be expected to give least satisfactory agreement for this case of $n=3$ (where the distribution curve for range has greatest skewness), these results seem to confirm the opinion previously given when publishing the tables: "the addition of a 3rd decimal place in the limits would clearly be meaningless, but the retention of the 2nd decimal appears worth while †."

Limits calculated by using a normal curve (C) have been shown in Table II to emphasise the fact that while the Pearson curve (B) may not lead to mathematically exact limits, it provides a far more accurate and useful approximation than can be obtained by the crude method (C).

(3) *Limiting Form of* $\phi(w)$.

Finally it seemed of interest to make a trial of the terminal formula (15). When the population sampled is normal, we have

$$\phi(w)=\frac{n(n-1)}{2\pi}\int_{-\infty}^{+\infty}e^{-(w^2/4+t^2)}\,dt=\frac{n(n-1)}{2\sqrt{\pi}}e^{-w^2/4} \quad\text{.....................(17).}$$

Whence we obtain

$$A_w=\int_w^\infty\phi(w)\,dw=n(n-1)\int_x^{+\infty}\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}\,dx=n(n-1)\times\tfrac{1}{2}(1-\alpha_x) \quad\text{...........(18),}$$

where $x=\frac{1}{\sqrt{2}}w$, and α_x is in the notation of Sheppard's Tables.

* *Loc. cit.* p. 416. † *Loc. cit.* p. 405.

For $n=3$, we obtain from (18),

A_n	·05 (5 %, limit)	·01 (1 %)	·005 (0·5 %)
w	3·39	4·15	4·46

the last two of which are in very close agreement with the true limits given in Table II.

For $n=50$ and $n=100$, the following comparisons are obtained:

	A_n	·05 (5 %)	·01 (1 %)	·005 (0·5 %)
$n=50$	Equation (18)	5·80	6·31	6·52
	Table A of previous paper	5·64	6·23	6·45
$n=100$	Equation (18)	6·25	6·72	6·92
	Table A of previous paper	6·08	6·63	6·85

In these cases the true position of the limits is not of course known. It seems, however, likely that for $A_n \leqslant {\cdot}01$, the equation (18) may lead to an approximation of considerable practical value when limits are required beyond the range of those tabled, e.g. either for $A_n < {\cdot}005$, or $n > 100$.

(iii) On a Recurrence Relation connected with the Double Bessel Functions $\mathcal{K}_{\tau_1,\tau_2}(x)$ and $T_{\tau_1,\tau_2}(x)$.

By CONSTANCE M. RIGBY, Ph.D.

A recurrence relation is required for

$$I_{\tau_1,\tau_2}=\int_0^x e^{-px}\,T_{\tau_1,\tau_2}(x)\,dx,$$

where $$T_{\tau_1,\tau_2}(x)=\frac{1}{\sqrt{\pi}}\,\frac{1}{2^\nu}\,\frac{1}{\Gamma(\nu+\frac{1}{2})}\,x^\nu\mathcal{K}_{\tau_1,\tau_2}(x)\quad\ldots\ldots\ldots\ldots(\text{xxxvii}),$$

and $$\mathcal{K}_{\tau_1,\tau_2}(x)=\frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+\frac{1}{2})}\,x^\nu\int_1^\infty e^{-xt}(t-1)^{\tau_1}(t+1)^{\tau_2}\,dt\quad\ldots\ldots\ldots(\text{xxix}),$$

where $$\nu=\tfrac{1}{2}(\tau_1+\tau_2+1),$$

the numbers (xxxvii) and (xxix) referring to the definition of $T_{\tau_1,\tau_2}(x)$ and $\mathcal{K}_{\tau_1,\tau_2}(x)$ in Professor Karl Pearson's paper in *Biometrika*, Vol. xxv. pp. 158—178.

We have by (xliii) of the same paper

$$\frac{d}{dx}\left(T_{\tau_1,\tau_2}(x)\right)=\frac{-x}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)T_{\tau_1-1,\tau_2-1}(x)-\frac{\tau_1-\tau_2}{2\nu-1}\,T_{\tau_1,\tau_2}(x).$$

Also $$e^{-px}\frac{d}{dx}\,T_{\tau_1,\tau_2}(x)=\frac{d}{dx}\left(e^{-px}T_{\tau_1,\tau_2}(x)\right)+pe^{-px}\,T_{\tau_1,\tau_2}(x).$$

Write $$e^{-px}\,T_{\tau_1,\tau_2}(x)=\mathcal{T}_{\tau_1,\tau_2}(x),$$

then $$\frac{d}{dx}\left(\mathcal{T}_{\tau_1,\tau_2}(x)\right)=-\frac{x}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)\mathcal{T}_{\tau_1-1,\tau_2-1}(x)-\left(p+\frac{\tau_1-\tau_2}{2\nu-1}\right)\mathcal{T}_{\tau_1,\tau_2}(x).$$

Integrating,

$$\mathcal{T}_{\tau_1,\tau_2}(x)-\mathcal{T}_{\tau_1,\tau_2}(0)=-\frac{1}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)\int_0^x x\,\mathcal{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$
$$-\left(p+\frac{\tau_1-\tau_2}{2\nu-1}\right)I_{\tau_1,\tau_2}\quad\ldots\ldots\ldots(1).$$

Multiplying equation (xliii *bis*) by $e^{-\rho x}$ we have

$$\tilde{T}_{\tau_1,\tau_2}(x) = \left\{\frac{2\nu-2}{2\nu-1} + \frac{\tau_1-\tau_2}{(2\nu-1)^2}x\right\}\tilde{T}_{\tau_1-1,\tau_2-1}(x) - \frac{x}{2\nu-1}e^{-\rho x}\frac{d}{dx}T_{\tau_1-1,\tau_2-1}(x)$$

$$= \left\{\frac{2\nu-2}{2\nu-1} + \left(\frac{\tau_1-\tau_2}{(2\nu-1)^2} - \frac{\rho}{2\nu-1}\right)x\right\}\tilde{T}_{\tau_1-1,\tau_2-1}(x) - \frac{x}{2\nu-1}\frac{d}{dx}\{\tilde{T}_{\tau_1-1,\tau_2-1}(x)\}.$$

Integrating

$$I_{\tau_1,\tau_2} = \frac{2\nu-2}{2\nu-1}I_{\tau_1-1,\tau_2-1} + \left(\frac{\tau_1-\tau_2}{(2\nu-1)^2} - \frac{\rho}{2\nu-1}\right)\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$-\frac{1}{2\nu-1}\int_0^x x\frac{d}{dx}\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$= \frac{2\nu-2}{2\nu-1}I_{\tau_1-1,\tau_2-1} + \left(\frac{\tau_1-\tau_2}{(2\nu-1)^2} - \frac{\rho}{2\nu-1}\right)\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$-\frac{1}{2\nu-1}x\tilde{T}_{\tau_1-1,\tau_2-1}(x) + \frac{1}{2\nu-1}I_{\tau_1-1,\tau_2-1}$$

$$= I_{\tau_1-1,\tau_2-1} + \left(\frac{\tau_1-\tau_2}{(2\nu-1)^2} - \frac{\rho}{2\nu-1}\right)\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$$

$$-\frac{1}{2\nu-1}x\tilde{T}_{\tau_1-1,\tau_2-1}(x) \quad \text{.......................(2).}$$

Using (1) to eliminate $\int_0^x x\tilde{T}_{\tau_1-1,\tau_2-1}(x)\,dx$ we have

$$I_{\tau_1,\tau_2} = I_{\tau_1-1,\tau_2-1} + \frac{\frac{1}{2\nu-1}\left(\frac{\tau_1-\tau_2}{2\nu-1}-\rho\right)}{-\frac{1}{2\nu-1}\left(1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2\right)}\left\{\tilde{T}_{\tau_1,\tau_2}(x) - \tilde{T}_{\tau_1,\tau_2}(0) + \left(\rho+\frac{\tau_1-\tau_2}{2\nu-1}\right)I_{\tau_1,\tau_2}\right\}$$

$$-\frac{1}{2\nu-1}x\tilde{T}_{\tau_1,\tau_2}(x),$$

$$I_{\tau_1,\tau_2} = I_{\tau_1-1,\tau_2-1} + \frac{\rho-\frac{\tau_1-\tau_2}{2\nu-1}}{1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2}\{\tilde{T}_{\tau_1,\tau_2}(x) - \tilde{T}_{\tau_1,\tau_2}(0)\} + \frac{\rho^2-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2}{1-\left(\frac{\tau_1-\tau_2}{2\nu-1}\right)^2}I_{\tau_1,\tau_2}$$

$$-\frac{x}{2\nu-1}\tilde{T}_{\tau_1,\tau_2}(x).$$

Writing
$$\frac{\tau_1-\tau_2}{\tau_1+\tau_2} = \frac{\tau_1-\tau_2}{2\nu-1} = \kappa,$$

$$\frac{1-\rho^2}{1-\kappa^2}I_{\tau_1,\tau_2} = I_{\tau_1-1,\tau_2-1} + \frac{\rho-\kappa}{1-\kappa^2}\{\tilde{T}_{\tau_1,\tau_2}(x) - \tilde{T}_{\tau_1,\tau_2}(0)\} - \frac{x}{2\nu-1}\tilde{T}_{\tau_1,\tau_2}(x),$$

i.e.
$$I_{\tau_1,\tau_2} = \frac{1-\kappa^2}{1-\rho^2}I_{\tau_1-1,\tau_2-1} + \frac{\rho-\kappa}{1-\rho^2}\{\tilde{T}_{\tau_1,\tau_2}(x) - \tilde{T}_{\tau_1,\tau_2}(0)\} - \frac{1-\kappa^2}{1-\rho^2}\frac{x}{2\nu-1}\tilde{T}_{\tau_1,\tau_2}(x),$$

which is the required recurrence formula.